I=2 S-wave Pion Scattering Phase Shift
with
Two Flavor Dynamical Quark Effect

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Reference CP-PACS Collaboration, hep-lat/0309155

Winter School and Workshop on Lattice Gauge Theories
"Non-perturbative improvement and renormalization"
@CCP February 1, 2004

1. Introduction

Isospin $I = 2$ S-wave $\pi\pi$ scattering phase shift $\delta_0(p)$

Scattering amplitude $E = 2\sqrt{m_\pi^2 + p^2}$

$$T(p) = \frac{16\pi E}{p} \sum_l (2l+1) P_l(\cos\theta) \frac{1}{2i} \left(e^{2i\delta_l(p)} - 1 \right)$$

Motivation

- Understanding of hadron dynamics based on QCD

Hadronic effect included in cross section, etc. cannot be estimated by perturbaion theory.
Especially $\pi\pi$ scattering case, many works employed effective theory such as chiral perturbation theory (ChPT).

- 'Dynamical' physical quantity

Most of the lattice studies have focused on 'static' physical quantities, e.g. hadron spectrum.
Scatterings and decays of hadrons are important study beyond the static quantities.

- First step toward decays of hadrons

$I = 1$ Channel	$\rho \to \pi\pi$
$I = 0$ Channel	$\sigma \to \pi\pi$
$I = 0,2$ Channel	$K \to \pi\pi$

Previous work of $I = 2$ $\pi\pi$ scattering system

scattering length $a_0 = \lim_{p \to 0} \frac{\delta_0(p)}{p}$

- Sharpe, Gupta, and Kilcup, 1992
- Gupta, Patel, and Sharpe, 1993
- Kuramashi *et al.*, 1993
- JLQCD Collaboration, 1999
- Lin, Zhang, Chen, and Ma, 2001
- BGR Collaboration, 2003

scattering phase shift $\delta(p)$

more difficult than a_0,
so that only pioneering study has been reported.

- Fiebig, Rabitsch, Markum, and Mihály, 2000
  $\pi\pi$ potential
- CP-PACS Collaboration, 2002
  finite volume method in center of mass system
- Kim, 2003
  finite volume method with anti-periodic boundary condition

They employed quenched approximation, and did not take the continuum limit.
Unphysical divergence appears in scattering length in quenched theory due to lack of unitarity.

In the calculation of phase shift $\delta(p)$

1. dynamical u, d quark effect ($N_f = 2$ full QCD)
   get rid of systematic error of unitarity violation
2. three different lattice spacings
   take the continuum limit
3. center of mass and two laboratory systems
   add little numerical costs, more dense sampling of energy states

The aim of this work
obtain $\delta(p)$ with two flavor full QCD in the continuum limit

Plan of talk
1. Introduction √
2. Methods
   - Finite volume method
   - Diagonalization method
3. Parameters
4. Results
   - Four-point function
   - Effect of diagonalization
   - Scattering length
   - Scattering phase shift
5. Conclusions

2. Methods

2.1 Finite volume method

finite volume L^3 with periodic boundary condition

(i) Center of mass system

Lüscher, Commmun. Math. Phys. 105, 153(1986)
Nucl. Phys. B354, 531(1991)

finite volume formula for S-wave ($l = 0$)

$$\tan\delta(\overline{p}) = \frac{\pi^{3/2}\sqrt{\overline{n}}}{Z_{00}(1;\overline{n})},$$

$$\overline{E} = 2\sqrt{m_\pi^2 + \overline{p}^2},\ \overline{p}^2 = \left(\frac{2\pi}{L}\right)^2 \overline{n},\ \overline{n} \notin Z,$$

$$Z_{00}(1;\overline{n}) = \frac{1}{\sqrt{4\pi}} \sum_{\mathbf{n}\in\mathbf{Z}^3} \frac{1}{\mathbf{n}^2 - \overline{n}}$$

(ii) Laboratory system $\pi(\mathbf{p}_1) \longrightarrow \leftarrow \pi(\mathbf{p}_2)$ $(\mathbf{p}_1+\mathbf{p}_2=\mathbf{P}\neq 0)$

Rummukainen and Gottlieb, Nucl. Phys. B450, 397(1995)

General finite volume formula for S-wave ($l = 0$)

$$\tan\delta(\overline{p}) = \frac{\gamma\pi^{3/2}\sqrt{\overline{n}}}{Z_{00}^P(1;\overline{n})},$$

$$\overline{p}^2 = \left(\frac{\overline{E}^2}{4} - m_\pi^2\right) = \left(\frac{2\pi}{L}\right)^2 \overline{n},\ \overline{E} = \gamma^{-1}\overline{E}_L,\ \gamma = \frac{\overline{E}_L}{\sqrt{\overline{E}_L^2 - \mathbf{P}^2}}$$

$$Z_{00}^P(1;\overline{n}) = \frac{1}{\sqrt{4\pi}} \sum_{\mathbf{r}\in\mathbf{P_d}} \frac{1}{\mathbf{r}^2 - \overline{n}},\ \mathbf{P_d} = \{\mathbf{r} = \gamma^{-1}(\mathbf{n} + \mathbf{d}/2)\},\ \mathbf{d} = \frac{L\mathbf{P}}{2\pi}$$

$\delta(\overline{p})$ is obtained from $\overline{E}$ or $\overline{E}_L$.

$\overline{p}$ is discrete value due to finite volume.

Problem on lattice (center of mass system)

pion four-point function

$$\begin{aligned} G_{nn}(t) &= \langle 0|\Omega_n^\dagger(t)\Omega_n(0)|0\rangle, \quad \Omega_n = \pi(\mathbf{p})\pi(-\mathbf{p}) \\ &= \sum_l |V_{ln}|^2 e^{-\overline{E}_l t}, \quad V_{ln}=\langle\overline{\Omega}_l|\Omega_n|0\rangle \\ &\to |V_{0p}|^2 e^{-\overline{E}_0 t}, \quad t\to\infty \end{aligned}$$

where, Ω_n : two-pion operator ($p^2 = (2\pi/L)^2 \cdot n$),
$\langle\overline{\Omega}_l|$: the l-th two-pion state

Four-point function behaves a multiexponential form.

We cannot obtain $\overline{E}_l (l \neq 0)$ by single exponential fit.

Diagonalization of four-point function matrix

Lüscher and Wolff, Nucl. Phys. B339 222(1990)

pion four-point function matrix

$$G_{nm}(t) = \langle 0|\Omega_n^\dagger(t)\Omega_m(0)|0\rangle$$

$$\begin{aligned} M(t,t_0)w_\nu &= \lambda_\nu(t,t_0)w_\nu \\ &\quad \lambda_\nu(t,t_0) = \exp\left(-\overline{E}_\nu(t-t_0)\right) \end{aligned}$$

$M(t,t_0) = G^{-1/2}(t_0)G(t)G^{-1/2}(t_0)$, t_0:reference point

We can extract $\overline{E}_\nu$ from $\lambda_\nu(t,t_0)$.

In actual calculation we need the state cut-off N.

i.e. $G(t)$: $N\times N$ matrix

Although same problem exists in laboratory systems, it is possible to be obtained $\overline{E}_\nu$ by diagonalization method.

3. Parameters

$N_f = 2$ full QCD configuration

CP-PACS Collaboration, Phys. Rev. D65, 054505(2002)

- Improved action

  gauge : Iwasaki action
  fermion : clover action

- Simulation parameters $La \sim 2.5$[fm]

β	c_{SW}	a^{-1}[GeV]	a[fm]	$L^3 \cdot T$
1.80	1.60	0.9176(93)	0.2250(22)	$12^3 \cdot 24$
1.95	1.53	1.268(13)	0.1555(17)	$16^3 \cdot 32$
2.10	1.47	1.833(22)	0.1076(13)	$24^3 \cdot 48$

a is determined by m_ρ

- Pion mass and number of configurations

$\beta = 1.80$				
m_π [GeV]	1.06	0.90	0.76	0.49
# of conf.	645	520	725	405
$\beta = 1.95$				
m_π [GeV]	1.13	0.93	0.76	0.54
# of conf.	595	690	685	495
$\beta = 2.10$				
m_π [GeV]	1.15	0.95	0.78	0.54
# of conf.	395	390	380	640

- Periodic boundary condition in spatial directions

  Dirichlet boundary condition in temporal direction

Systems $(\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2,\ \mathbf{P} : \frac{2\pi}{L}$ unit)

- Center of mass system CM
- Laboratory system 1 L1
- Laboratory system 2 L2

	$\mathbf{P}$	size of $G(t)$	state
CM	$(0,0,0)$	3×3 $(N = 3)$	0, 1, 2
L1	$(1,0,0)$	4×4 $(N = 4)$	0, 1, 2, 3
L2	$(1,1,0)$	4×4 $(N = 4)$	0, 1, 2, 3

$G(t)$: pion four-point function matrix

N : state cut-off

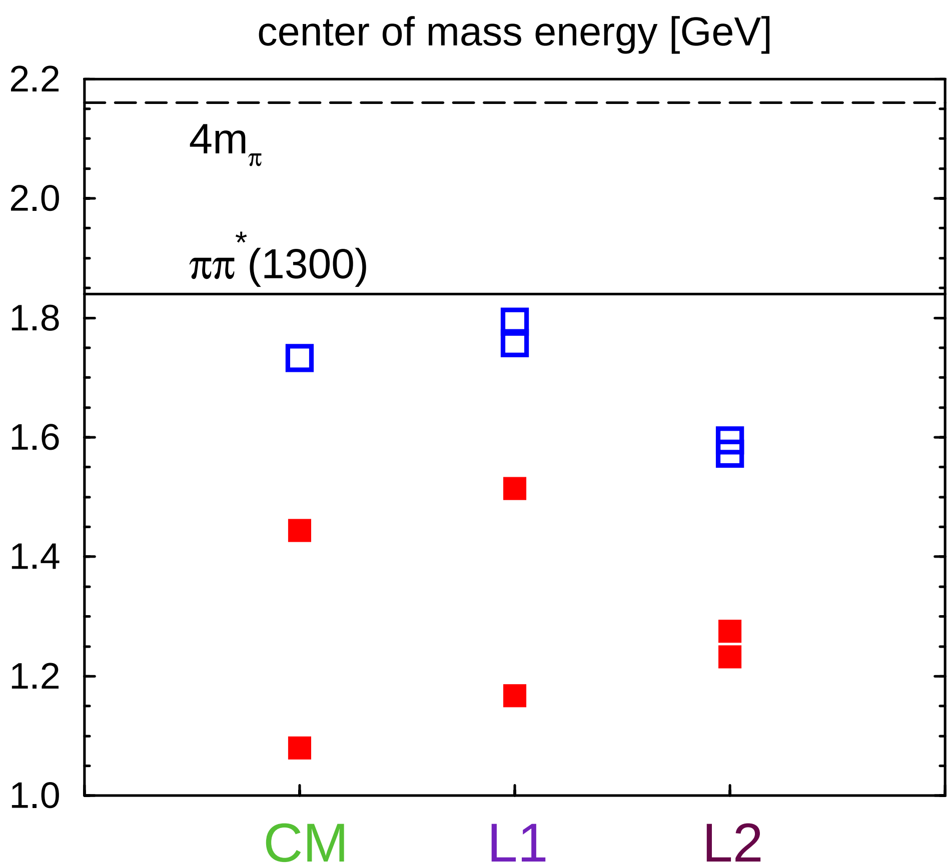
Estimated energy states without interaction

at $\beta = 2.10$ and $m_\pi = 0.54$ GeV

$$E = \sqrt{E_L^2 - \mathbf{P}^2},\ E_L = \sqrt{m_\pi^2 + \mathbf{p}_1^2} + \sqrt{m_\pi^2 + \mathbf{p}_2^2}$$

where E_L is laboratory system energy, and

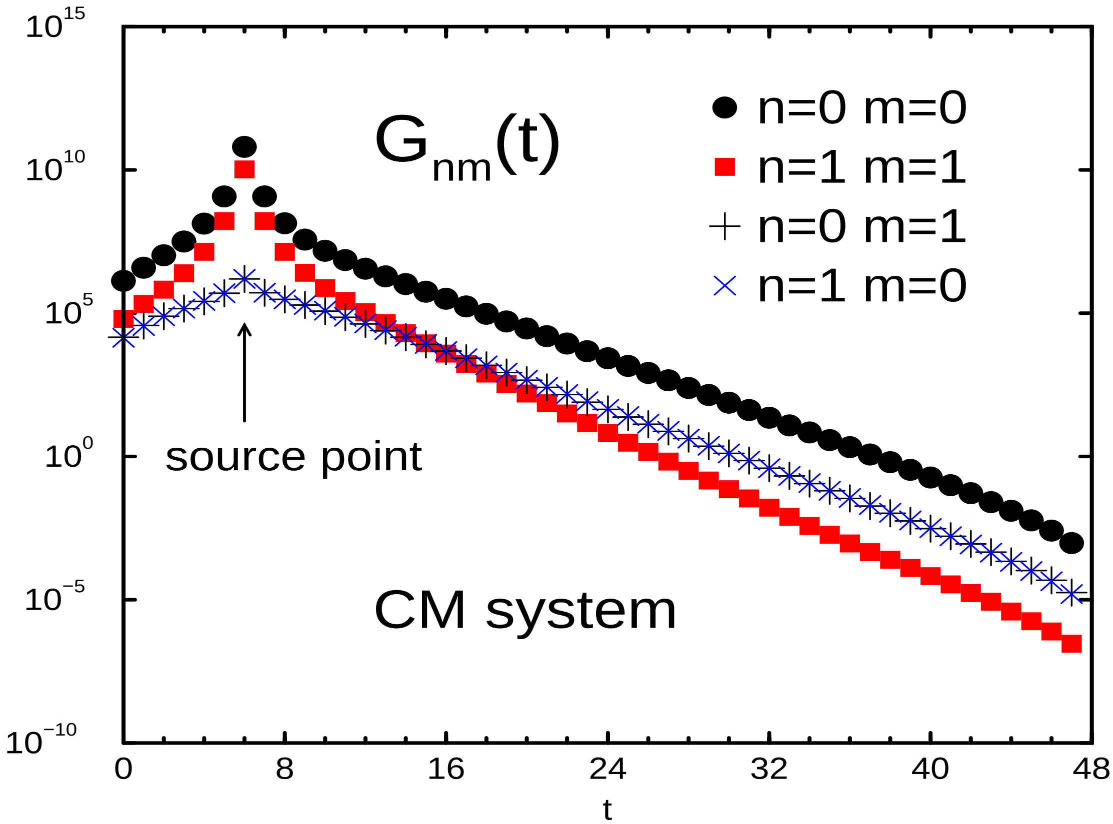
$$\mathbf{p}_i^2 = (2\pi/L)^2 \cdot n, n \in Z$$



4. Results

4.1 four-point function matrix $G_{nm}(t)$

Center of mass system ($\beta = 2.10$, $m_\pi = 0.54$[GeV])



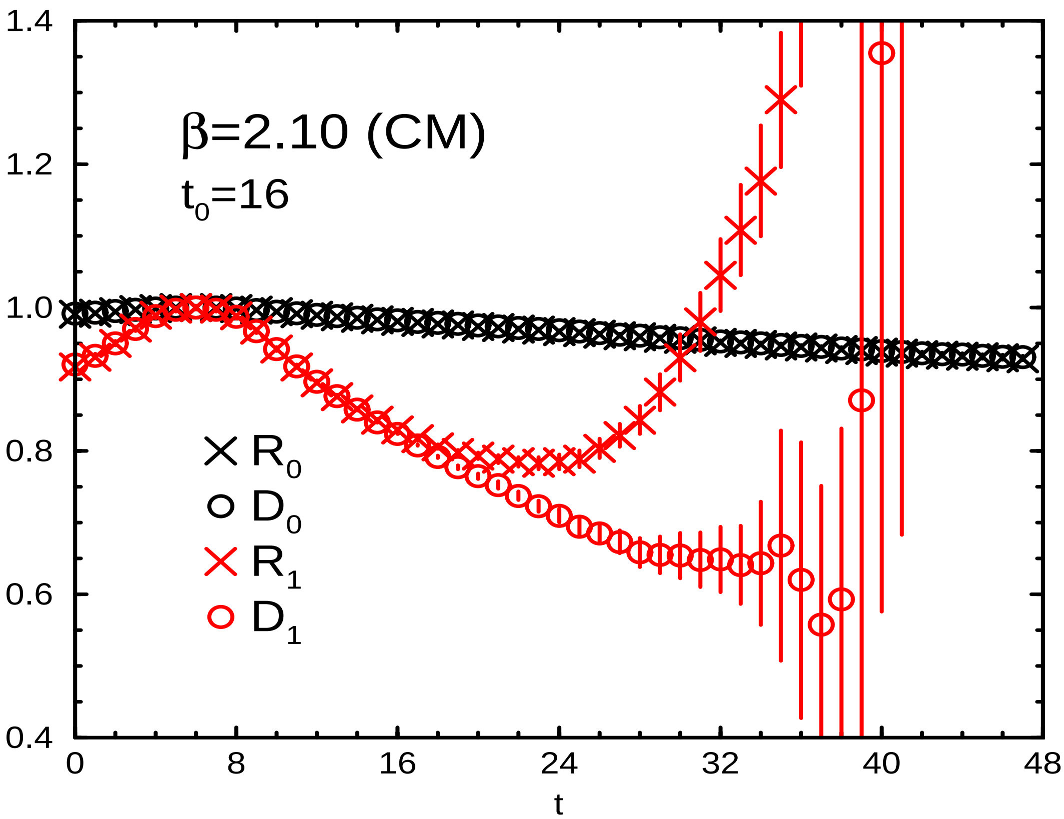
Cross symbols denote negative value.

Clear signal is obtained.

Off-diagonal parts are finite.

4.2 diagonalization

(i) Center of mass system ($\beta = 2.10,\ m_\pi = 0.54$[GeV])



ratio before diagonalization (cross symbol)

$$R_n(t) = \frac{\langle 0|\Omega_n^\dagger(t)\Omega_n(0)|0\rangle}{\langle 0|\pi_n^\dagger(t)\pi_n(0)|0\rangle^2} \to e^{-(\overline{E}_0 - E_n)t} \quad (t\to\infty)$$

exponentially increase ($n \neq 0$)

ratio after diagonalization (open symbol)

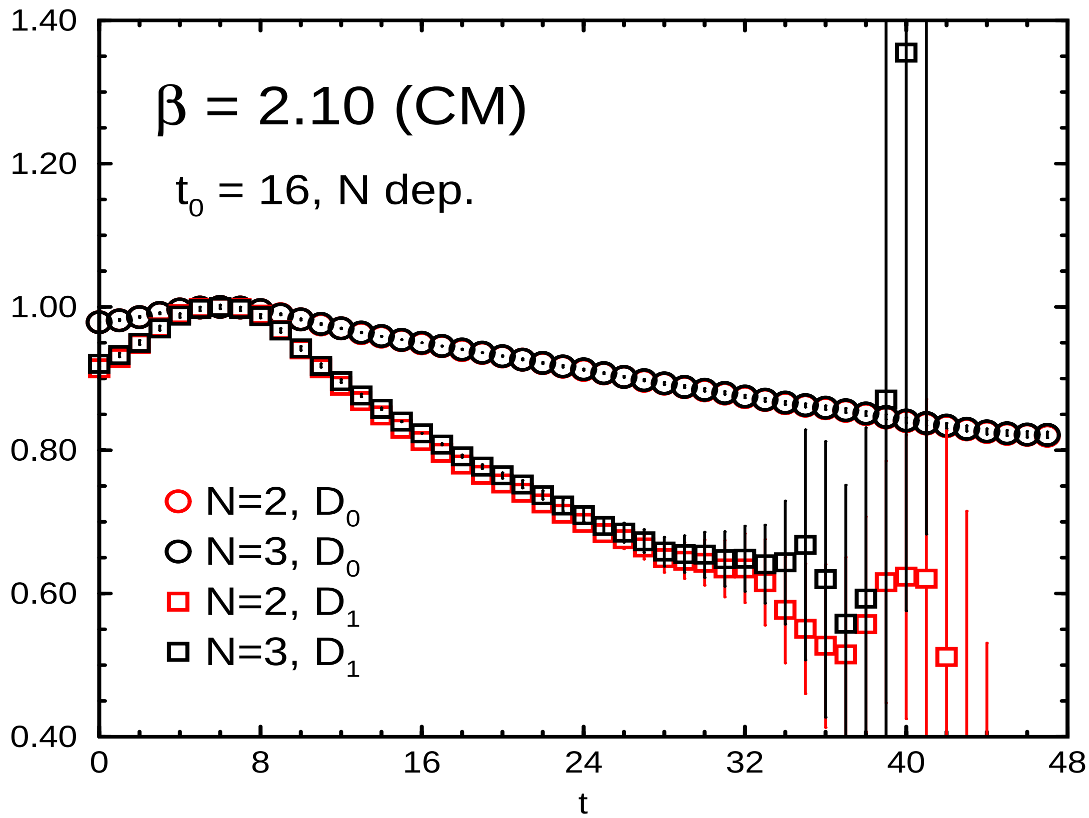
$$D_n(t) = \frac{\lambda_n(t)}{\langle 0|\pi_n^\dagger(t)\pi_n(0)|0\rangle^2} \propto e^{-(\overline{E}_n - E_n)t}$$

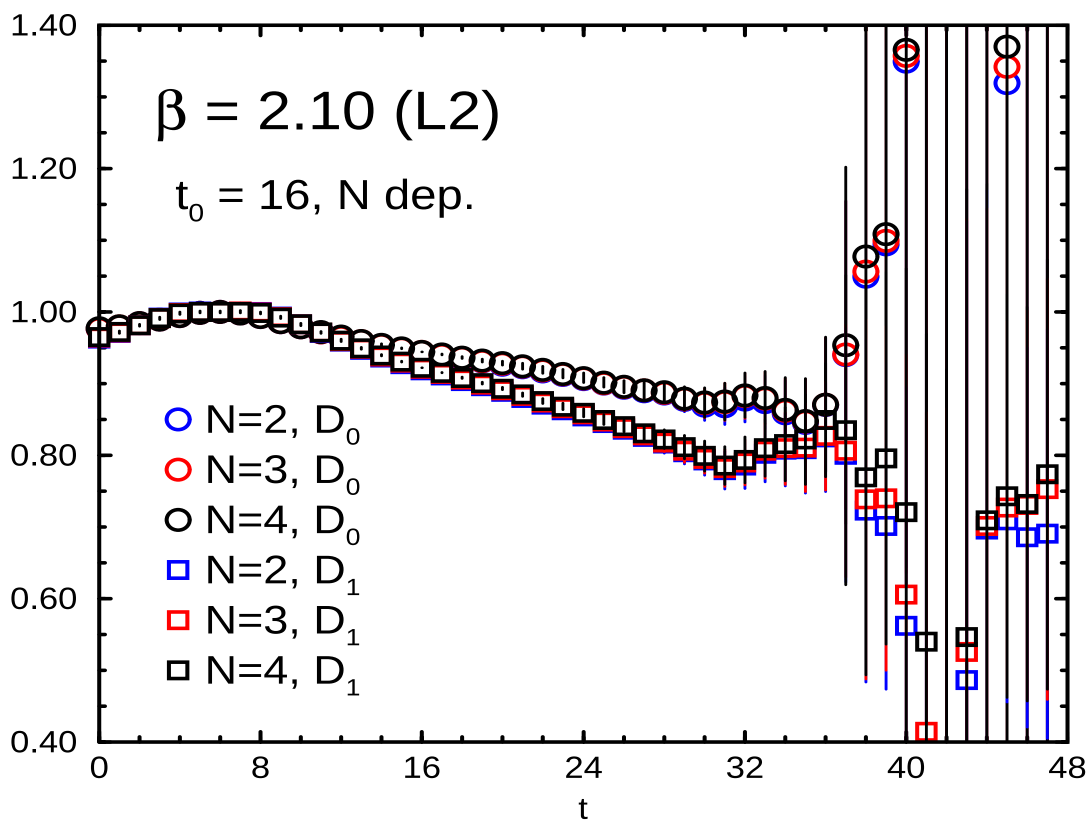
normalized by $D_n(t_S)$

Ω_n : two-pion operator
$\overline{E_n}$: two-pion energy of n-th state with interaction
E_n : two-pion energy of n-th state without interaction
$\lambda_n(t)$: n-th state eigenvalue of $M(t, t_0)$

We can extract $\overline{E}_1$ from $D_1(t)$.

(iii) Cut-off N dependence ($\beta = 2.10$, $m_\pi = 0.54$[GeV])



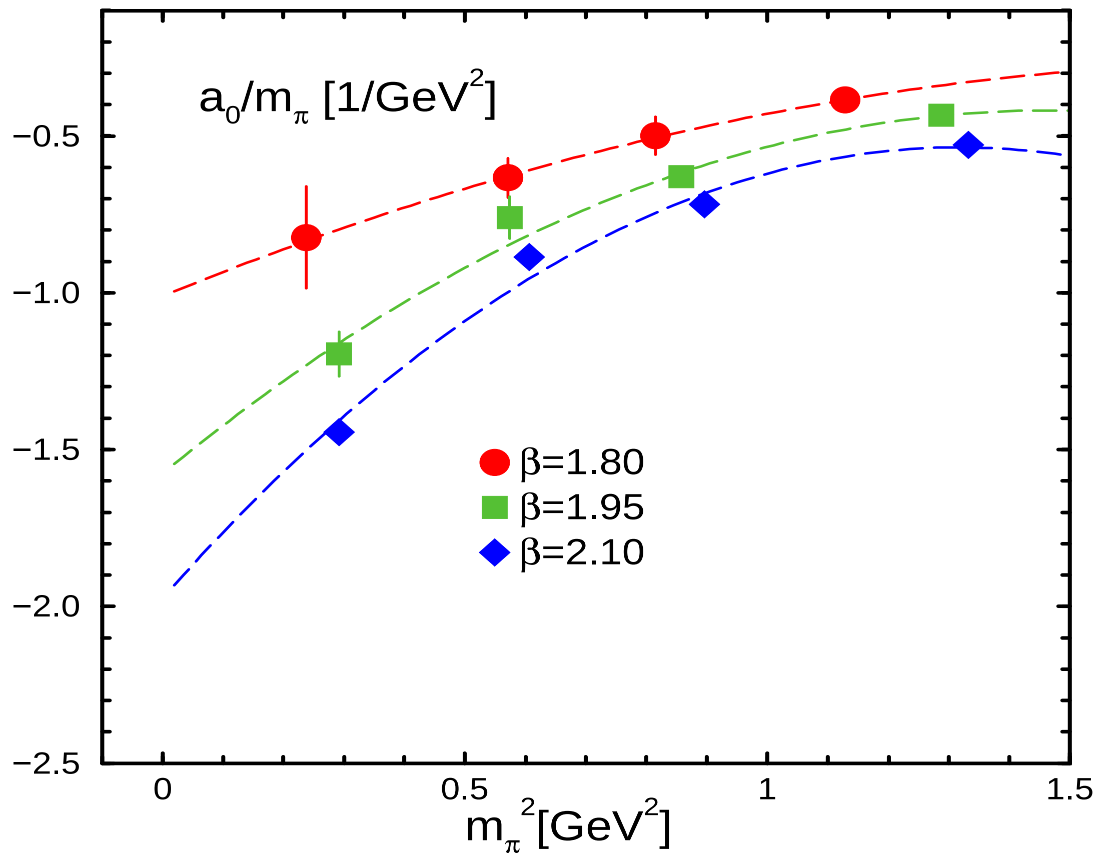


Results are reasonably independent of N.

4.3 scattering length $a_0 = \lim_{\overline{p}\to 0} \frac{\cdots(\overline{p})}{\overline{p}}$

(i) Chiral extrapolations

In CM system for $n = 0$ state $\overline{p} \approx 0$,
we obtain the scattering length.



Results cannot be compared with prediction of ChPT, due to large $O(a)$ effect and m_π.

Curvature causes difficulty of chiral extrapolation.
polynomial assumption fitting

$$a_0/m_\pi = A_{10} + A_{20} m_\pi^2 + A_{30} m_\pi^4$$

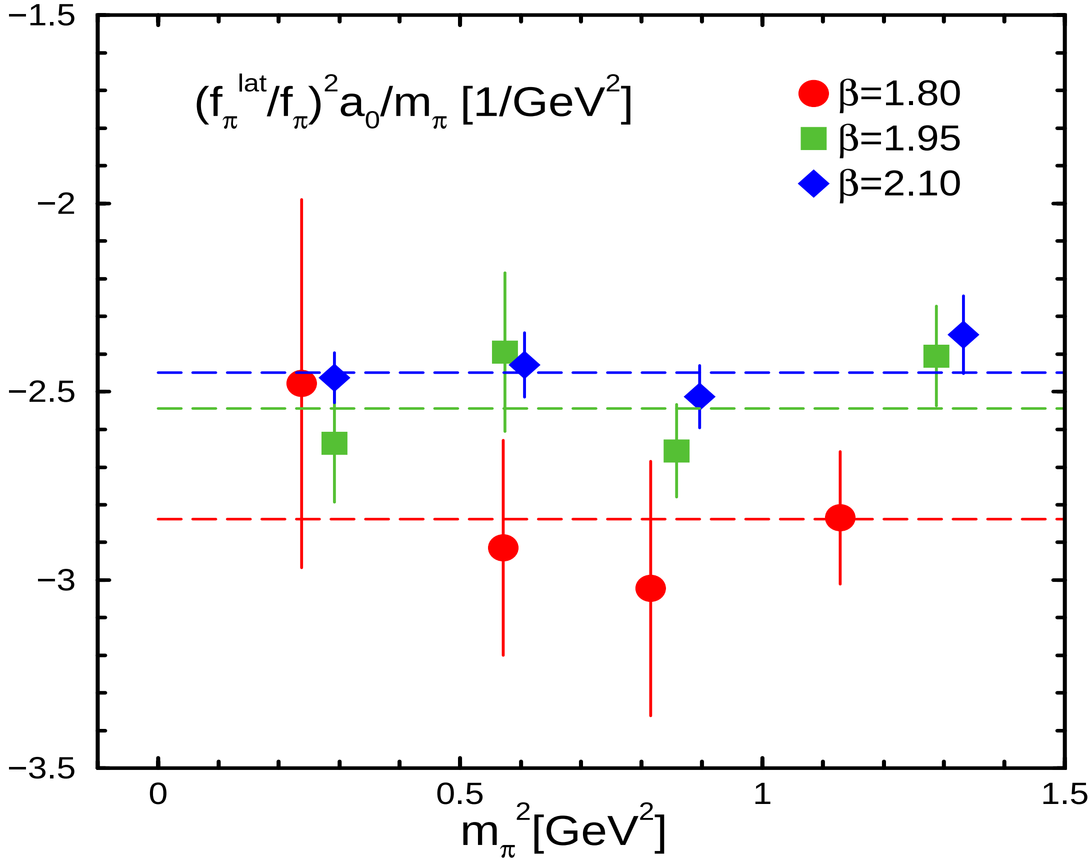
β	1.80	1.95	2.10
χ^2/d.o.f.	0.02	2.1	8.2

χ^2/d.o.f. is very large at $\beta = 2.10$.

$(f_\pi^{lat}/f_\pi)^2 a_0/m_\pi[\text{GeV}^{-2}]$

Sharpe *et al.*, Nucl. Phys. B383, 309(1992)
Guputa *et al.*, Phys. Rev. D48, 388(1993)
Kuramashi *et al.*, Phys. Rev. D52, 3003(1995)

f_π^{lat} : measured pseudoscalar decay constant
f_π : physical pseudoscalar decay constant = 93 [MeV]



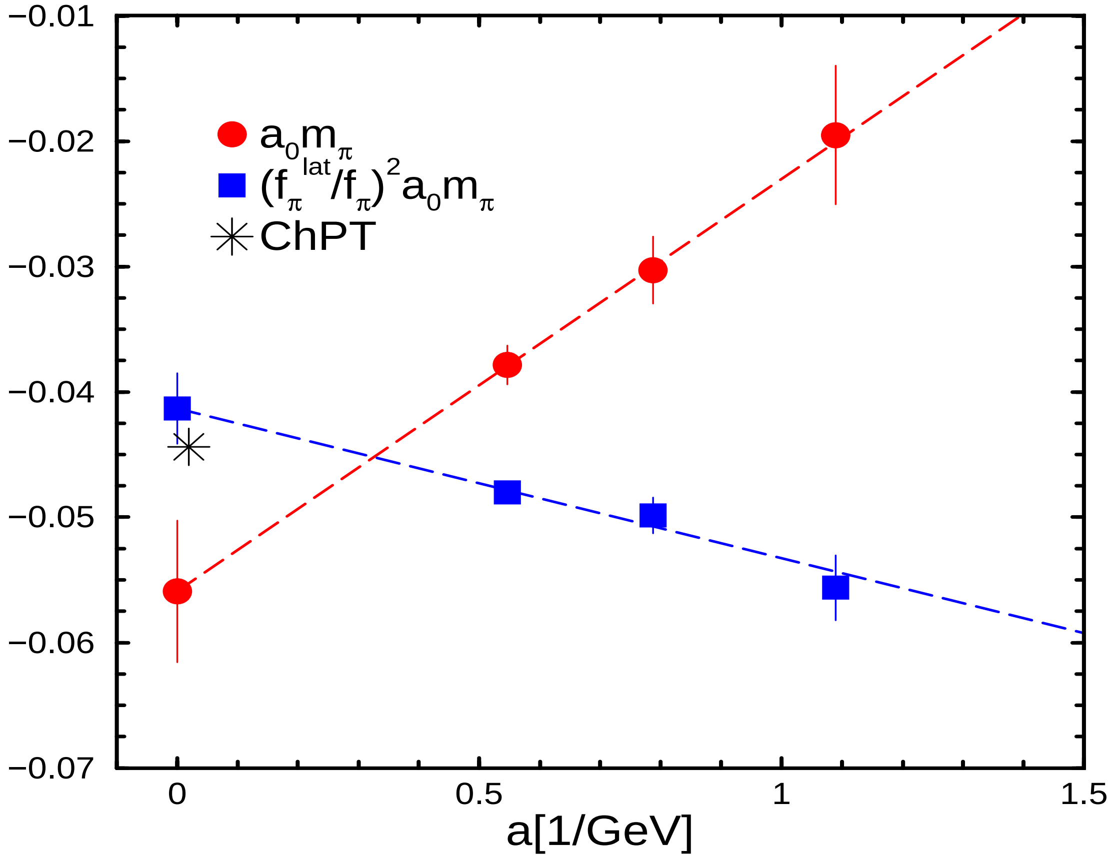
The curvatures disappear.

β	1.80	1.95	2.10
χ^2/d.o.f.	0.27	0.94	0.54

χ^2/d.o.f. is reasonable small.

The m_π^2 dependence of a_0/m_π is strongly correlated with one of $(1/f_\pi^{lat})^2$.

(ii) Continuum extrapolations at $m_\pi = 0.14$ GeV



Large $O(a)$ dependence exists.

In the continuum limit f_π^{lat} has a problem: f_π^{lat} in $a \to 0$ seems to be smaller than $f_\pi = 93$ MeV.
We cannot extract a reliable a_0 from $(f_\pi^{lat}/f_\pi)^2 a_0 m_\pi$.

We obtain scattering lengths in the continuum limit

$$\begin{aligned} a_0 m_\pi &= -0.0558(56) \\ (f_\pi^{lat}/f_\pi)^2 a_0 m_\pi &= -0.0413(28) \\ a_0 m_\pi(\text{ChPT}) &= -0.0444(10) \end{aligned}$$

4.3 Scattering phase shift $\delta(\overline{p})$

(i) Chiral extrapolations

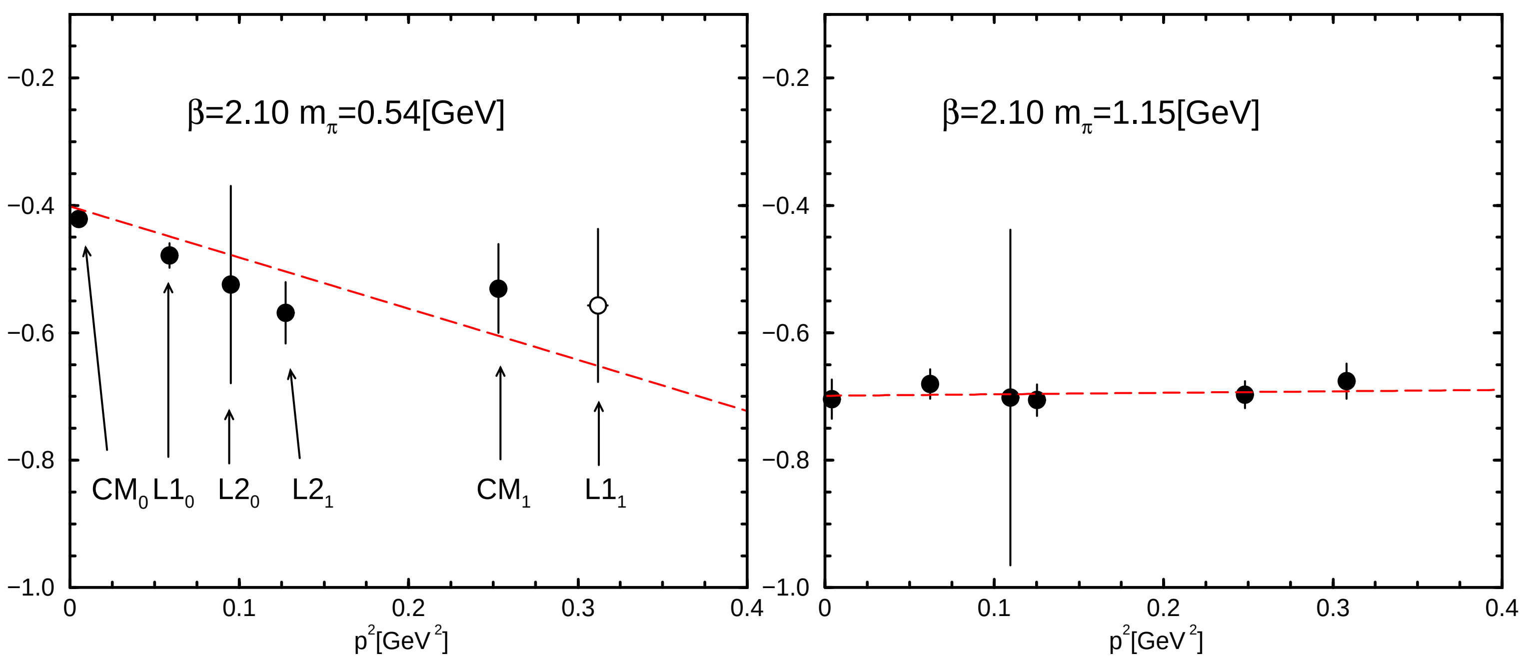
'Scattering amplitude' $A(\overline{p}, m_\pi)$

$$A(\overline{p}, m_\pi) = \frac{\tan\delta(\overline{p})}{\overline{p}} \cdot \frac{\overline{E}}{2},$$

$$\lim_{\overline{p}\to 0} A(\overline{p}, m_\pi) = a_0 m_\pi$$

global fit for m_π^2 and $\overline{p}^2$ at each β

$$A(\overline{p}, m_\pi) = A_{10} m_\pi^2 + A_{20} m_\pi^4 + A_{30} m_\pi^6 + A_{01}\overline{p}^2 + A_{11} m_\pi^2 \overline{p}^2 + A_{21} m_\pi^4 \overline{p}^2$$



The reasonable fits are possible with the amplitudes obtained from different systems.

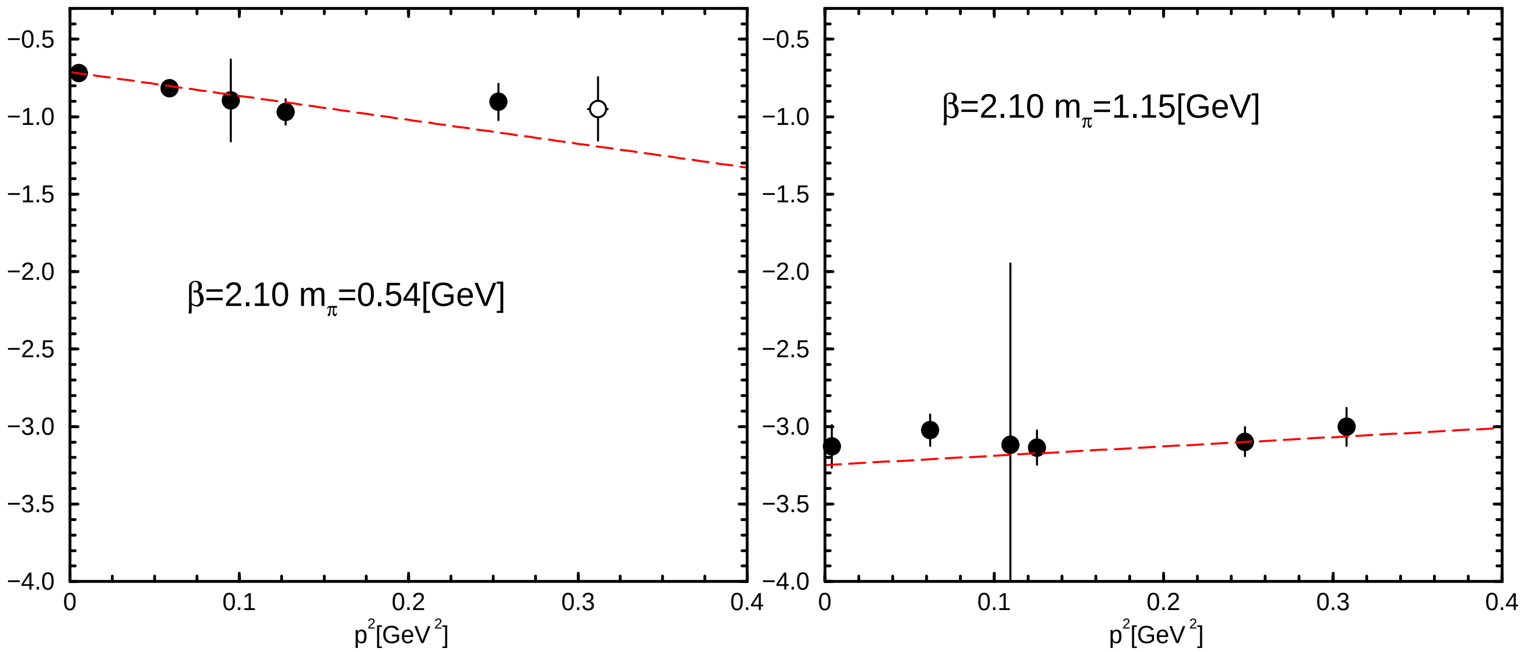
β	1.80	1.95	2.10
χ^2/d.o.f.	0.90	0.64	1.33

Normalized amplitude $A^f(\overline{p}, m_\pi)$

$$A^f(\overline{p}, m_\pi) = (f_\pi^{lat}/f_\pi)^2 A(\overline{p}, m_\pi)$$

global fit for m_π^2 and $\overline{p}^2$ at each β

$$A^f(\overline{p}, m_\pi) = A^f_{10} m_\pi^2 + A^f_{01}\overline{p}^2 + A^f_{11} m_\pi^2 \overline{p}^2$$

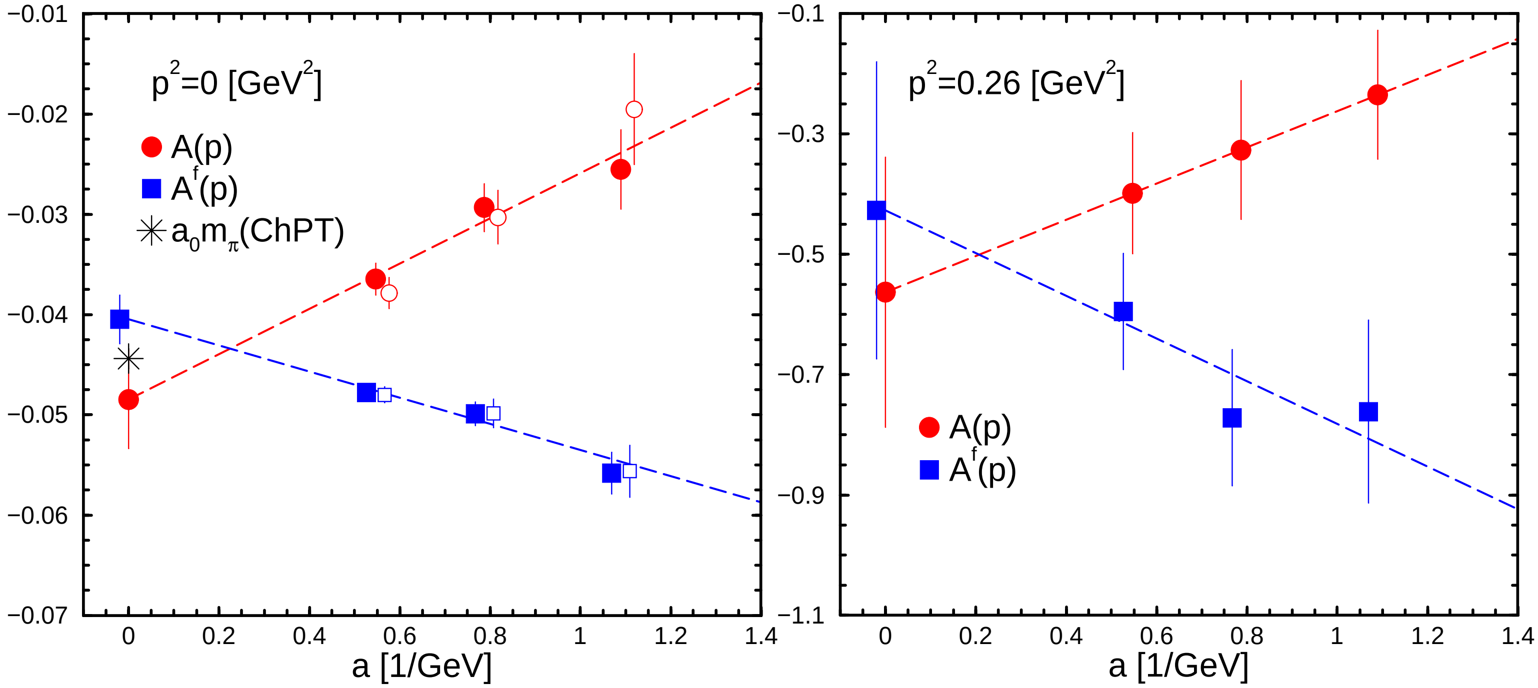


The fits are also reasonable with the amplitudes obtained from different systems.

β	1.80	1.95	2.10
χ^2/d.o.f.	0.77	0.50	0.75

(ii) Continuum extrapolations at $m_\pi = 0.14$[GeV]

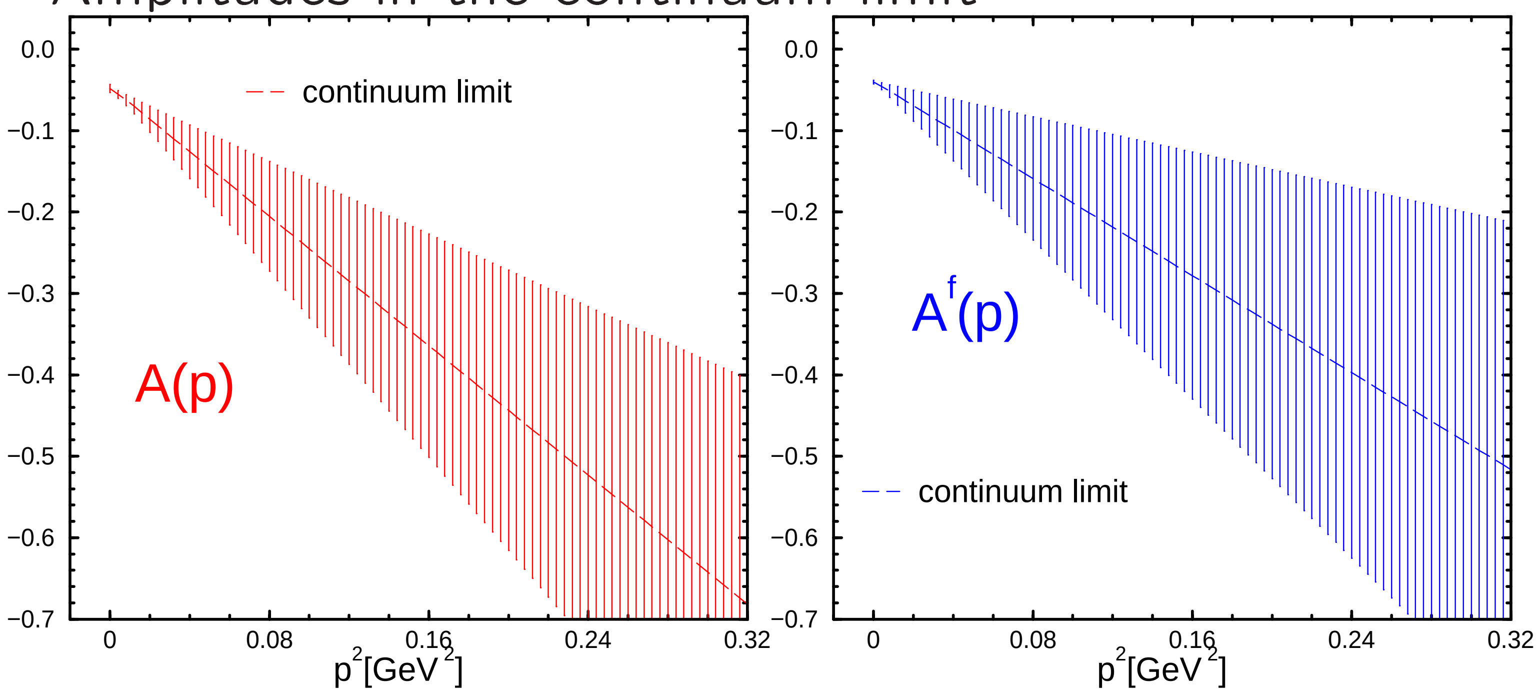
linear fit at fixed $\overline{p}$: $A(\overline{p}) + aA^{(a)}(\overline{p})$



$$A(0) = -0.0484(49), \quad A^f(0) = -0.0404(24)$$

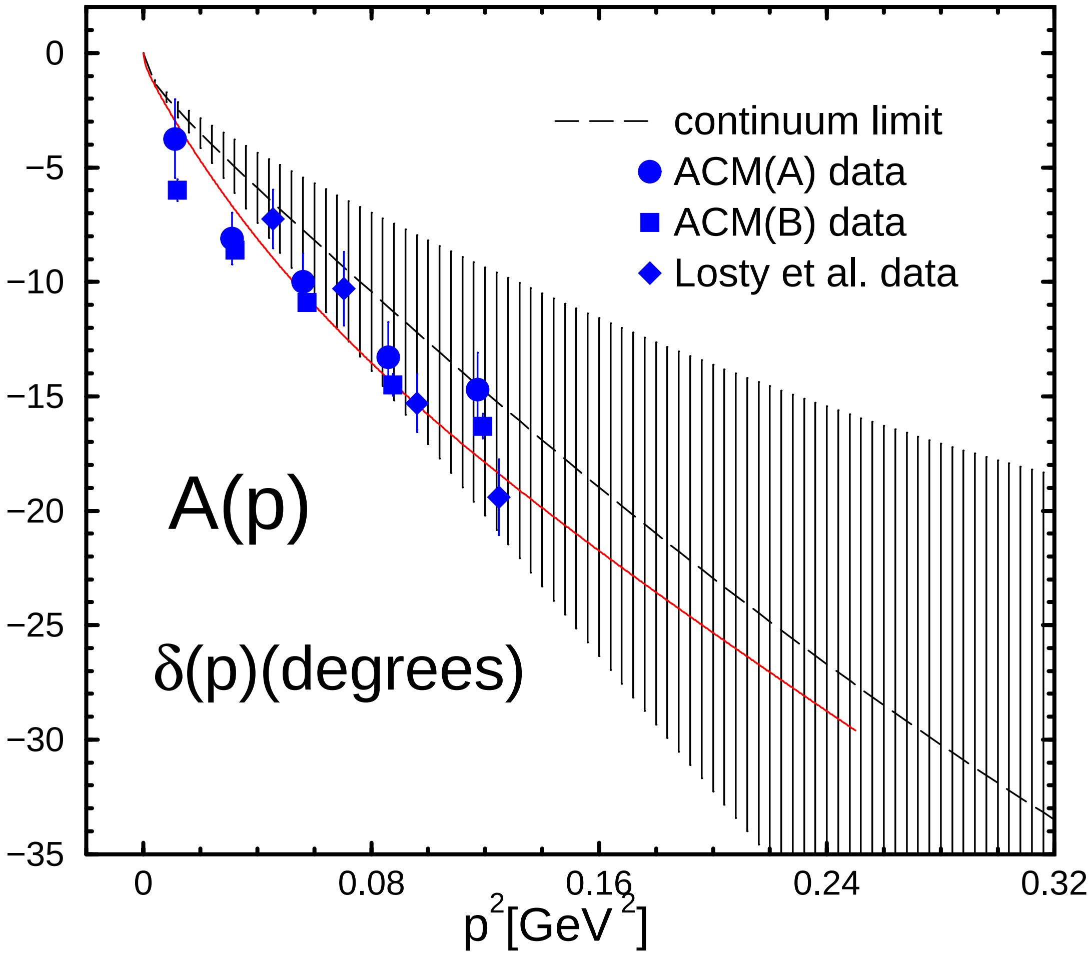
$$a_0 m_\pi(\text{ChPT}) = -0.0444(10)$$

Amplitudes in the continuum limit



Results are consistent with each other.

$\delta(p)$ (degrees) in $a = 0$ at $m_\pi = 0.14$[GeV]



The symbols are the experimental data.

Hoogland *et al.*, Nucl. Phys. B126, 109(1977)

Losty*et al.*, Nucl. Phys. B69, 185(1974)

The solid line is parametrized by experimental input.

Colangelo *et al.*, Nucl. Phys. B603, 125(2001)

Result is consistent with experimental data.

5. Conclusions

We calculate $I = 2$ S-wave $\pi\pi$ scattering phase shift.

- $N_f = 2$ full QCD
- Continuum limit
- Center of mass system and two laboratory systems

Scattering length a_0

- Curvature causes difficulty of chiral extrapolation.
- Large $O(a)$ effect exists.

$$\begin{aligned} a_0 m_\pi &= -0.0558(56) \\ (f_\pi^{lat}/f_\pi)^2 a_0 m_\pi &= -0.0413(28) \end{aligned}$$

Scattering phase shift $\delta(p)$

- Large $O(a)$ effect exists.
- The result in the continuum limit is consistent with the experimental data.

$$\begin{aligned} a_0 m_\pi &= -0.0484(49) \\ (f_\pi^{lat}/f_\pi)^2 a_0 m_\pi &= -0.0404(24) \end{aligned}$$

future work

1. Calculation at smaller m_π and a
   - Chiral and continuum extrapolations
   - Low energy constant in ChPT
2. Volume dependence
   method with $\pi\pi$ wave function
3. $I = 1$ $\pi\pi$ resonance scattering phase shift
   $\rho \to \pi\pi$ decay
4. $I = 0$ $\pi\pi$ scattering phase shift
5. $K \to \pi\pi$ decay