Non-perturbative renormalization and O(a) improvement

ALPHA
Collaboration

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Why non–perturbative renormalization of QCD ?

1) QCD has two extremal regimes:

- at high energies quarks are essentially free (asymptotic freedom)
- at low energies quarks are confined into hadrons

Can the same Lagrangian describe those limits at the same time ?
Lattice QCD can provide a first principle computation to study this question $\Rightarrow$ NP renormalization of QCD parameters.

This requires exploring distances r s.t.

asymptotic freedom $\quad 0.01 \text{ fm} \leq r \leq 1 \text{ fm} \quad$ confinement

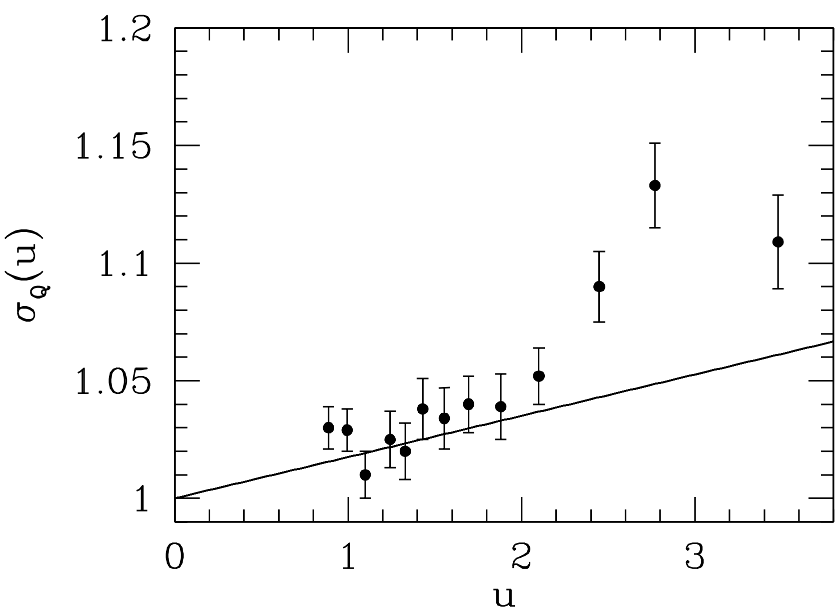
leading to a multiple scale problem.

2) In addition to the fundamental parameters, composite operators must be renormalized. Where are they from ? Effective theories.

For example, weak interactions can be treated as perturbations to strong interactions $\Rightarrow$ expectation values of composite operators among QCD states.

"Effective Weak Hamiltonians" obtained *integrating out* heavy particles (W,t) from the Standard Model Lagrangian and *expanding* the resulting non–local theory in $1/m_{\rm W}$.

E.g. NP evolution of 4-fermion operators:



bare operators renormalized at $u \simeq 3 \ldots$

3) ... a more practical reason.

Lattice discretizations of QCD preserving chiral symmetry at finite a have been introduced but are still very expensive. For Wilson fermions chiral symmetry is recovered in the c.l. $\Rightarrow$ at finite a operators belonging to different (continuum) chiral multiplets can mix among themselves !! (lattice symmetries used to rule out mixings !!)

It's important to compute the mixing coeff. in a NP way (e.g. JLQCD '97)

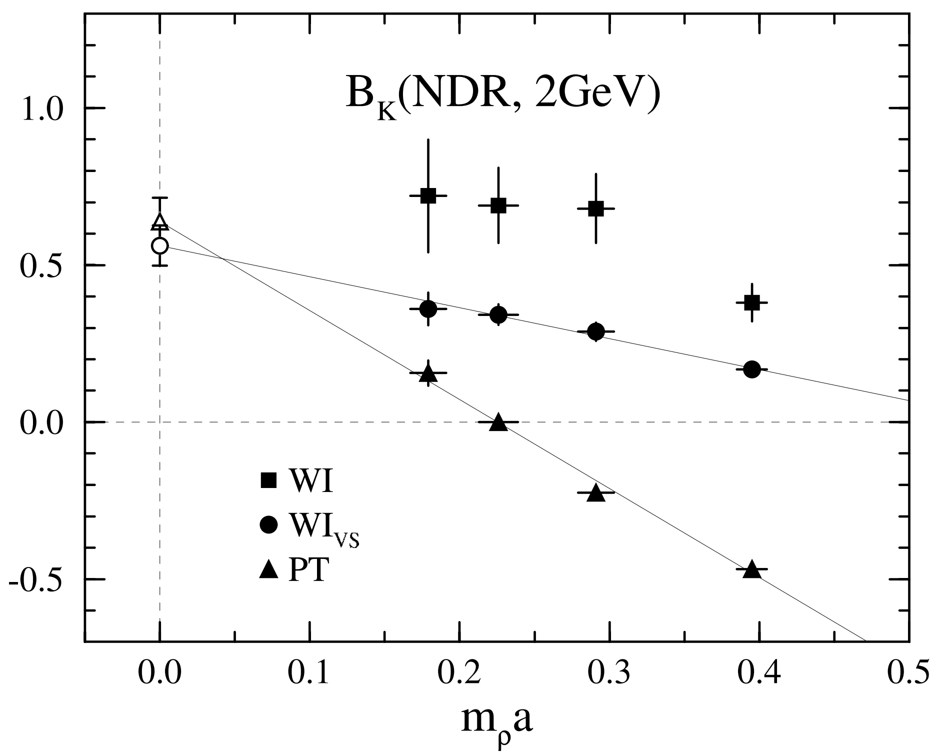


Fig. 4

NP renormalization

- Scale dependent quantities:

  $\bar{g}^2,\ \bar{m} \leftrightarrow \hat{P}^a$

  how ? RI-MOM, finite volume recursive scheme (Schrödinger functional)

- Scale independent quantities:

  mixing coeff. for B_{K}, $\hat{V}^a,\ \hat{A}^a$

  how ? Enforcing continuum symmetries at finite lattice spacing (up to O(a) or O(a^2)) through Ward identities.

Scale dependent renormalization

coupling: in a renormalization scheme, a coupling $g_{\rm R}(\mu)$ can be defined by constraining the value of a set of renormalized correlation functions at a momentum scale μ.

Bare correlation functions are scale independent $\Rightarrow$ μ dependence of the coupling by RGE

$$\mu \frac{\partial g_{\rm R}}{\partial \mu} = \beta(g_{\rm R}) \ ,$$

$\beta-$function has a perturbative expansion

$$\beta(g_{\rm R}) = -g_{\rm R}^3 (b_0 + g_{\rm R}^2 b_1 + \ldots) \ ,$$

solving the RGE equation

$$\mu = \exp\left(\int^{g_{\rm R}} \frac{dg}{\beta(g)} \right) \ ,$$

$$\mu^{-1} = \Lambda^{-1} e^{-1/2b_0 g_{\rm R}^2} (b_0 g_{\rm R}^2)^{-b_1/2b_0^2} (1 + O(g_{\rm R}^2))$$

where we have introduced as integration constant the RGI quantity Λ

$$\Lambda = \lim_{g_{\rm R} \to 0} \mu e^{-1/2b_0 g_{\rm R}^2} (b_0 g_{\rm R}^2)^{-b_1/2b_0^2}$$

concerning operators . . .

One introduces an intermediate NP renormalization scheme s.t.

$$\langle \beta | O_{\rm inter}(\mu) | \alpha \rangle = \langle \beta | O_{\rm bare} | \alpha \rangle_{\rm tree\ level}$$

with $\mu^2 = p^2$ for RI-MOM, $\mu = 1/L$ for SF.

This allows to define

$$O_{\rm inter}(\mu) = Z_{\rm inter}(g_0, \mu) O_{\rm bare}$$

$$\Phi_{\rm inter}(\mu) \equiv \langle f | O_{\rm inter}(\mu) | i \rangle$$

$\Phi_{\rm inter}$ has an explicit dependence on μ and an implicit one through the coupling

$$\mu \frac{d\Phi_{\rm inter}(\mu)}{d\mu} = \gamma(g_{\rm R}^2(\mu)) \Phi_{\rm inter}(\mu)$$

$$\gamma(g_{\rm R}^2) = -(\gamma_0 g_{\rm R}^2 + \gamma_1 g_{\rm R}^4 + \ldots) \ ,$$

again, a RGI quantity can be defined

$$\Phi_{\rm RGI} = \lim_{\mu \to \infty} \Phi_{\rm inter}(\mu) (2 b_0 g_{\rm R}^2(\mu))^{-\gamma_0/2b_0^2} \ .$$

on the lattice:

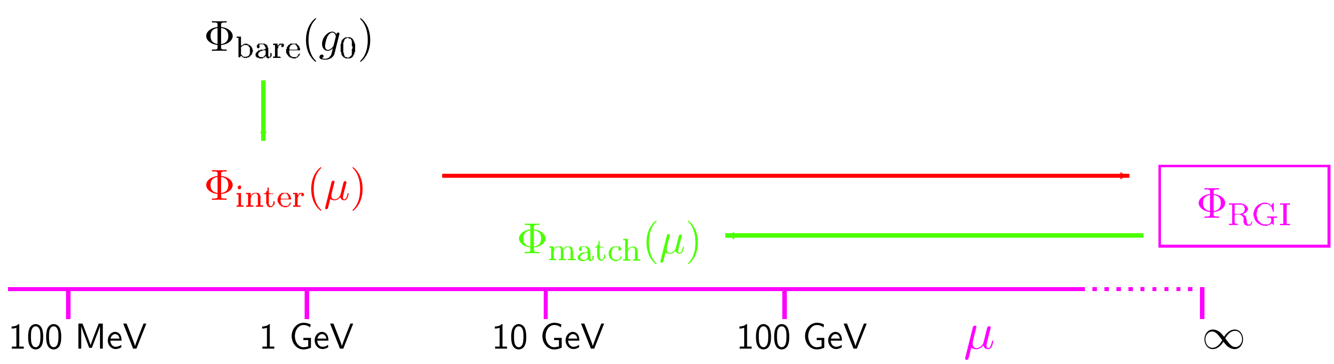
$$\begin{aligned}\Phi_{\mathrm{RGI}} &= Z_{\mathrm{RGI}} \times \Phi_{\mathrm{bare}} \\ &= \underbrace{\Phi_{\mathrm{RGI}}/\Phi_{\mathrm{inter}}(\mu)}_{\text{universal, continuum}} \times \underbrace{\Phi_{\mathrm{inter}}(\mu)/\Phi_{\mathrm{bare}}(g_0)}_{\text{regularization dep.}} \times \underbrace{\Phi_{\mathrm{bare}}(g_0)}_{\text{bare matrix element}}\end{aligned}$$

The *physical* matrix element (typically in the $\overline{\mathrm{MS}}$ scheme) may be written

$$\Phi_{\overline{\mathrm{MS}}}(\mu) = \Phi_{\overline{\mathrm{MS}}}(\mu)/\Phi_{\mathrm{RGI}} \times \Phi_{\mathrm{RGI}} \quad \text{e.g. at } \mu = m_{\mathrm{W}}$$

where $\Phi_{\overline{\mathrm{MS}}}(\mu)/\Phi_{\mathrm{RGI}}$ is a universal, continuum quantity computable in PT.

Summarising



Schrödinger functional scheme

finite volume scheme, identifying $\mu^{-1} \equiv L$.

$$\mathcal{Z}(C, C')$$

is the propagation kernel for going from field conf. C at time $x_0 = 0$ to a conf C' at $x_0 = T$.

We choose particular **boundary fields**

$$C = \frac{i}{L}\mathrm{diag}(\phi_1, \phi_2, \phi_3) \text{ and } C' = \frac{i}{L}\mathrm{diag}(\phi'_1, \phi'_2, \phi'_3)$$

$$\begin{array}{rclcrcl} \phi_1 &=& \eta - \pi/3, && \phi'_1 &=& -\phi_1 - \frac{4\pi}{3} \\ \phi_2 &=& -\frac{1}{2}\eta, && \phi'_2 &=& -\phi_3 + \frac{2\pi}{3} \\ \phi_3 &=& -\frac{1}{2}\eta + \frac{\pi}{3}, && \phi'_3 &=& -\phi_2 + \frac{2\pi}{3} \end{array}$$

these are Abelian classical fields
(the particular choice results in a theorem that the classical minimum of the action is unique)

Boundary fields induce a **background field**

$$B_0 = 0, \quad B_k = \left[x_0 C' + (L - x_0) C\right]/L, \quad k = 1, 2, 3$$

at the tree level order

$$\Gamma_0[B] = \frac{3L^4}{g_0^2} \sum_\alpha \left\{ \frac{2}{a^2} \sin \left[\frac{a^2}{2L^2} (\phi'_\alpha - \phi_\alpha) \right] \right\}$$

the effective action depends explicitly on L (and on L only if we scale all other dimensionful quantities in proportion to L)

$\Rightarrow$ we are led to a renormalized coupling constant computed at a scale $\mu = L^{-1}$

$$\boxed{\left. \frac{\partial \Gamma[B]}{\partial \eta} \right|_{\eta=0} = \frac{k}{\bar{g}^2(L)}}$$

in general, we can compute the derivative of the effective action as an expectation value

$$\frac{\partial \Gamma}{\partial \eta} = -\frac{\partial}{\partial \eta} \ln \int \mathcal{D}U e^{-S} = \langle \frac{\partial S}{\partial \eta} \rangle$$

- definition is equally applicable on the lattice
- results in a local (plaquette like) observable
- well suitable for numerical calculations

non–perturbative running

We define the step scaling function $\sigma(u)$

$$\sigma(u) = \bar{g}^2(2L)|_{\bar{g}^2(L)=u, m_i=0}$$

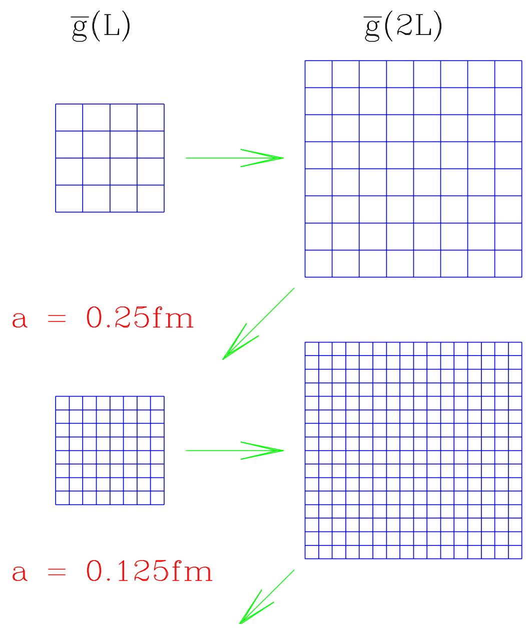
which describes a finite jump in the scale evolution (e.g. by a factor 2)

$\Rightarrow$ integrated form of β-function.

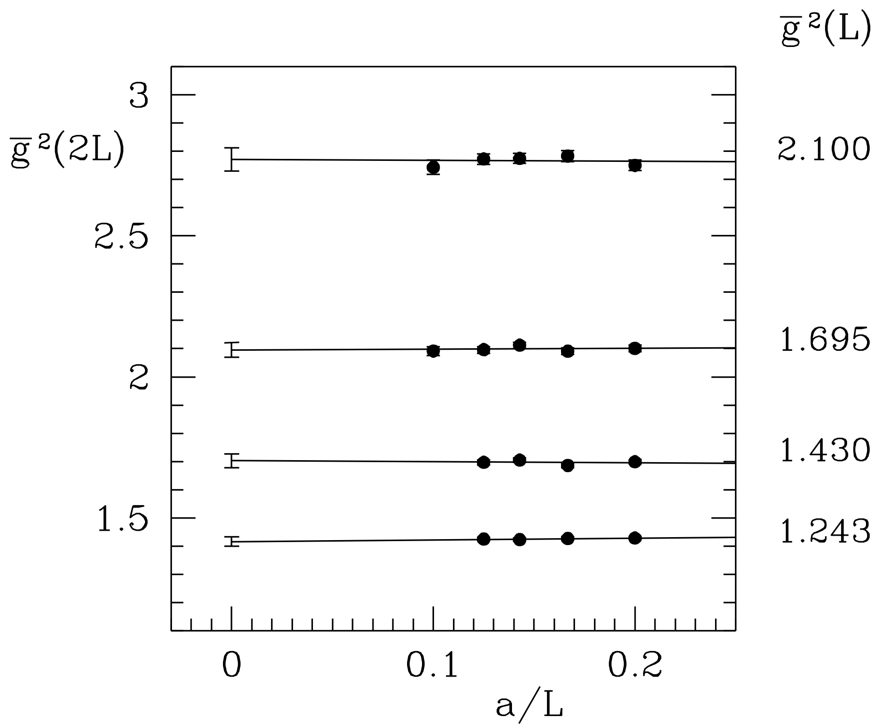
We want to compute $\sigma(u)$ on the lattice, extrapolated to the continuum, for a set of u values.

$$\Sigma(a/L, \bar{g}^2(L,a)) = \bar{g}^2(2L,a)|_{\bar{g}^2(L,a)=u, am_i=0}$$

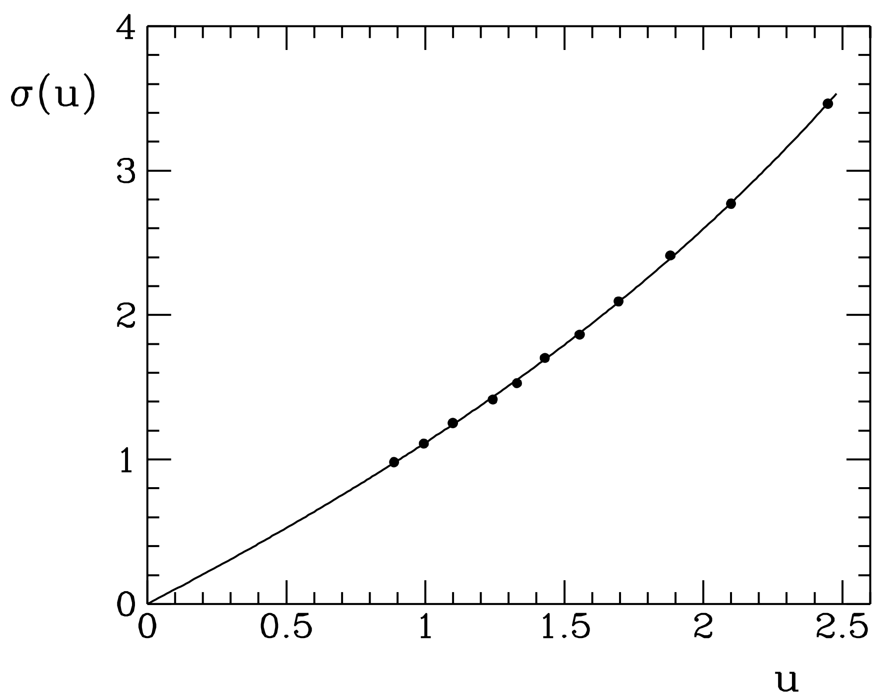
$$\sigma(\bar{g}^2(L)) = \lim_{a/L \to 0} \Sigma(a/L, \bar{g}^2(L,a))$$



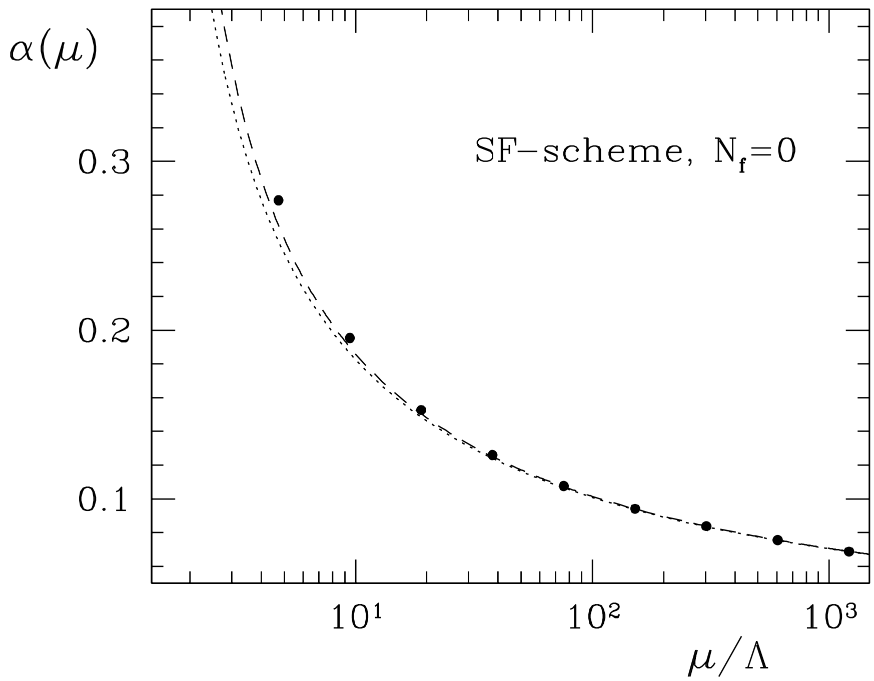
examples for the c.l. of Σ



$\sigma(u)$ can then be obtained by fit/interpolation



Starting from the most non–perturbative coupling $\bar{g}^2(L_{\rm max})$, we evolve downwards n times till the coupling $\bar{g}^2(2^{-n}L_{\rm max})$ is perturbative (we can check this !!).



from $\bar{g}^2(2^{-n}L_{\rm max})$ to "$\bar{g}^2(\mu=\infty)$" [Λ parameter] we can use the PT expression for the β-function

$$\Lambda L_{\rm max} = 2^n (b_0\overline{g}^2)^{-b_1/2b_0^2} \exp\left\{-\frac{1}{2b_0\overline{g}^2}\right\} \times \exp\left\{-\int_0^{\overline{g}} {\rm d}x \left[\frac{1}{\beta(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x}\right]\right\}$$

with $\bar{g}^2 = \bar{g}^2(2^{-n}L_{\rm max})$.

:-(Everything in units of a reference scale $L_{\rm max}$

To convert to MeV we measure low energy (hadronic) quantities on a volume (say) $2L_{\mathrm{max}}$

$$\text{e.g.} \quad r_0/L_{\mathrm{max}}, \quad F_\pi L_{\mathrm{max}} \ldots$$

at this point the Λ parameter is known in MeV.

What have we done ?

We have checked starting from first principles that QCD is the right theory to describe the low energy confining regime (hadronic input) and the high energy, asymptotically free regime (Λ parameter output)

The target is not to have the best estimate of the Λ parameter, this can be obtained from experiments, which however do not provide a first principle determination (in a sense QCD is assumed in the analysis of experimental data).

The lattice result “agrees” (it is quenched !!) with experiments. $N_{\mathrm{f}} = 2$ is on the way $\rightarrow$ talk.

basic reference: ALPHA, hep-lat/9810063

Scale independent renormalization

M. Testa, hep-lat/9803147:

- For global symmetries respected by the regulator the bare Noether currents are finite and normalized. For symmetries violated by the regularization even if the symmetry will be recovered in the continuum, currents require finite renormalization.
- Order by order in perturbation theory: power divergent mixings with lower dimensional operators never depend on the renormalization scale μ.

I'll discuss the example of $Z_{\rm A}$ in the SF scheme. Idea $\Rightarrow$ compute the renormalization constant by enforcing Chiral Ward Identities on the renormalized theory (up to $\mathrm{O}(a)$ or $\mathrm{O}(a^2)$).

WI in the continuum

Isospin vector and axial vector variations of the quark fields

$$\delta_V^a \psi(x) = \tfrac{1}{2}\tau^a \psi(x), \qquad \delta_V^a \overline{\psi}(x) = -\overline{\psi}(x)\frac{1}{2}\tau^a,$$
$$\delta_A^a \psi(x) = \tfrac{1}{2}\tau^a \gamma_5 \psi(x), \qquad \delta_A^a \overline{\psi}(x) = \overline{\psi}(x)\gamma_5\frac{1}{2}\tau^a,$$

are symmetries of the action for massless quarks. For the vector and axial currents:

$$V_\mu^a(x) = \overline{\psi}(x)\gamma_\mu \frac{1}{2}\tau^a \psi(x), \qquad A_\mu^a(x) = \overline{\psi}(x)\gamma_\mu\gamma_5 \frac{1}{2}\tau^a \psi(x),$$

$$\delta_V^a V_\mu^b(x) = -i\epsilon^{abc} V_\mu^c(x), \qquad \delta_A^a V_\mu^b(x) = -i\epsilon^{abc} A_\mu^c(x),$$

$$\delta_V^a A_\mu^b(x) = -i\epsilon^{abc} A_\mu^c(x), \qquad \delta_A^a A_\mu^b(x) = -i\epsilon^{abc} V_\mu^c(x).$$

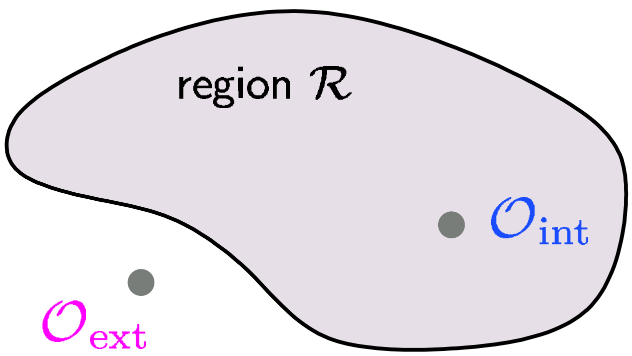
Applying a local axial transformation in a region R, the general (integrated) WI reads

$$\int_{\partial R} d\sigma_\mu(x) \langle A_\mu^a(x) \mathcal{O}_{int} \mathcal{O}_{ext} \rangle = -\langle (\delta_A^a \mathcal{O}_{int}) \mathcal{O}_{ext} \rangle + 2m \int_R d^4x \langle P^a(x) \mathcal{O}_{int} \mathcal{O}_{ext} \rangle$$

where the first and the third term come from the variation of the action.

Considering $\mathcal{O}_{int} = A^b_\nu(y)$ in the massless case

$$\int_{\partial R} d\sigma_\mu(x) \langle A^a_\mu(x) A^b_\nu(y) \mathcal{O}_{ext} \rangle = i\epsilon^{abc} \langle V^c_\nu(y) \mathcal{O}_{ext} \rangle$$



on the lattice . . .

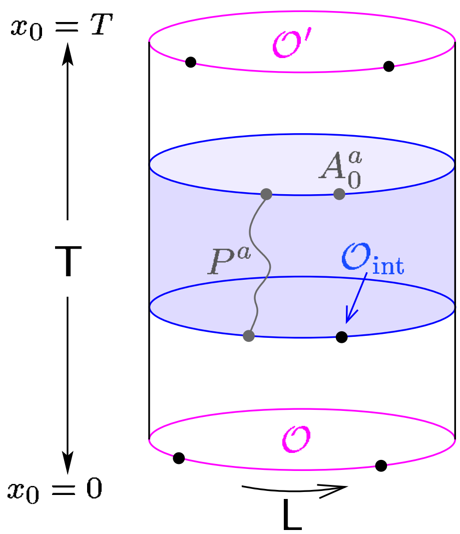
$$(A_R)^a_\mu = Z_A A^a_\mu \,, \qquad (V_R)^a_\mu = Z_V V^a_\mu$$

previous WI (when considering R = region among hyper-planes at $x_0 = y_0 \pm t$) reads (after summing over y and using twice the WI with no internal fields)

$$a^6 \sum_{\mathbf{x},\mathbf{y}} \epsilon^{abc} \langle (A_R)^a_0(x) (A_R)^b_0(y) \mathcal{O}_{ext} \rangle =$$

$$a^3 \sum_{\mathbf{y}} i \langle (V_R)^c_0(y) \mathcal{O}_{ext} \rangle + \mathrm{O}(a)$$

in the SF



Schrödinger Functional:

- QCD on a space-time cylinder $L^3 \times T$
- periodic b.c.'s in spatial directions
- fixed (Dirichlet) b.c.'s in time direction
- $\mathcal{O}^c_{\rm ext} = -\frac{1}{6L^6}\epsilon^{cde}\mathcal{O}'^d\mathcal{O}^e$
- $\mathcal{O}$ ($\mathcal{O}'$): zero-momentum pseudo–scalar states at $x_0 = 0$ ($x_0 = T$)

With this choice of $\mathcal{O}_{ext}$, the rhs of the integrated WI can be replaced by (hint: if properly normalized the charge generates infinitesimal isospin rotation)

$$f_1 = -\frac{1}{3L^6}\langle\mathcal{O}'^a\mathcal{O}^a\rangle$$

introducing the correlator

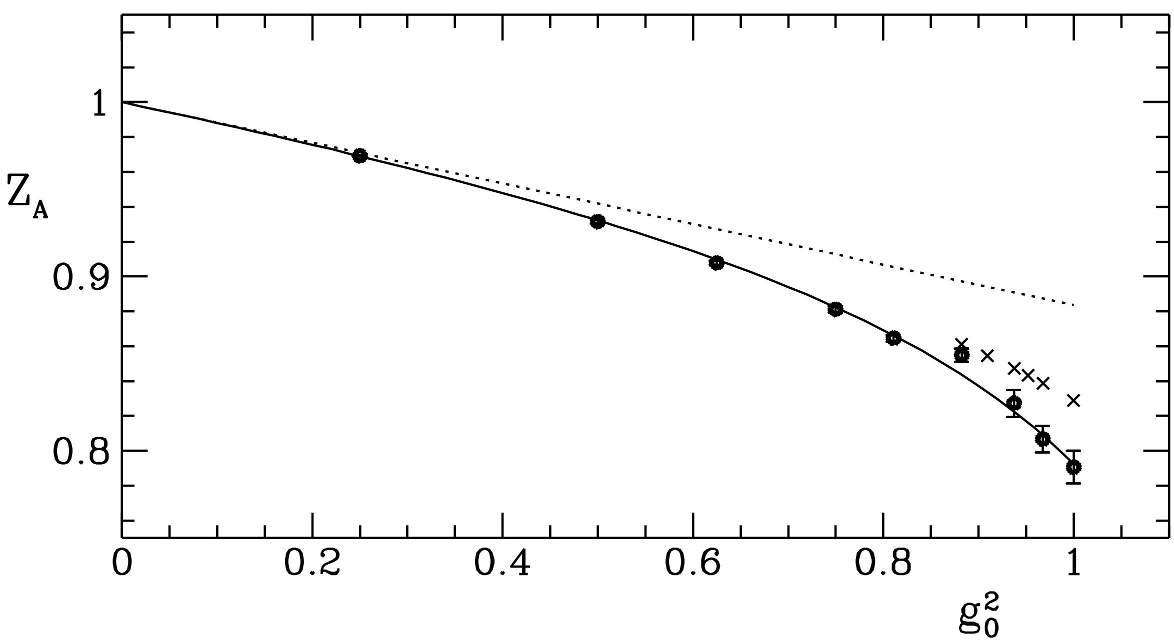
$$f_{AA}(x_0, y_0) = -\frac{a^6}{6L^6}\sum_{\mathbf{x},\mathbf{y}}\epsilon^{abc}\epsilon^{cde}\langle\mathcal{O}'^d A_0^a(x) A_0^b(y)\mathcal{O}^e\rangle$$

we rewrite the integrated WI

$$Z_A^2 f_{AA}(x_0, y_0) = f_1 + \mathrm{O}(a)$$

Results for O(a) improved Wilson fermions (quenched)

ALPHA: hep-lat/9611015



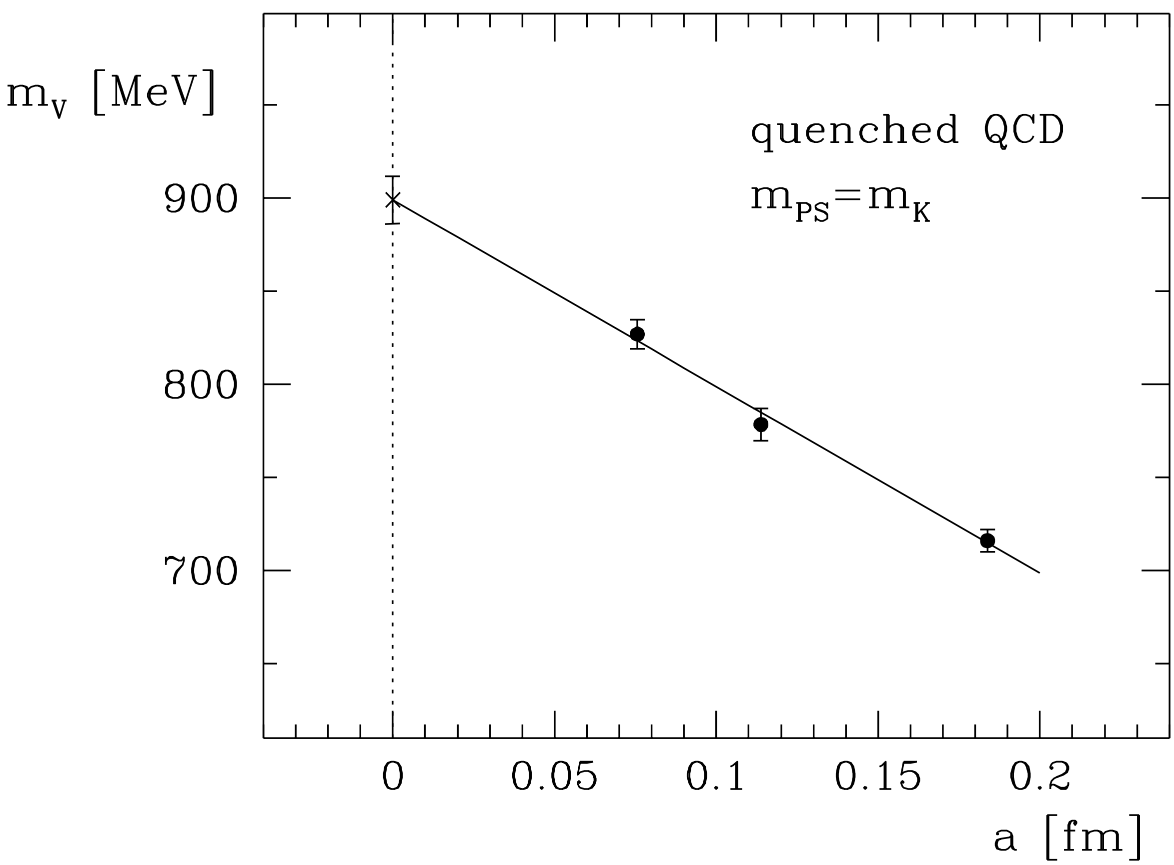
Even in small volumes, simulating at zero quark mass is very expensive. For the $N_{\mathrm{f}} = 2$ case we keep the volume integral in the WI and we extrapolate to the chiral point (at fixed a)

$\rightarrow$ talk

O(a) improvement programme à la Symanzik

Need for O(a) improvement

What is the systematic error when computing quantities at $a = 0.1\mathrm{fm}$? Using plain Wilson fermions the vector meson mass m_V for $m_{PS} = m_K$ [from GF11]



lattice artifacts appear of course already in PT

Symanzik's effective theory approach

The lattice action of QCD can be written as an effective continuum action.

$$S_{\text{eff}} = \int d^4x \left[\mathcal{L}_0 + a\mathcal{L}_1 + a^2\mathcal{L}_2 + \ldots\right]$$

The integration of the momenta from π/a to ∞ generates higher dimensional interaction terms $\mathcal{L}_k$.

For Wilson quarks , the symmetries of the action restrict $\mathcal{L}_1$ to be a combination of

$$\begin{aligned}
O_1 &= \overline{\psi} i\sigma_{\mu\nu}F_{\mu\nu}\psi \\
O_2 &= \overline{\psi}D_\mu D_\mu\psi + \overline{\psi}\overleftarrow{D}_\mu\overleftarrow{D}_\mu\psi \\
O_3 &= m\,\mathrm{tr}F_{\mu\nu}F_{\mu\nu} \\
O_4 &= m\left\{\overline{\psi}\gamma_\mu D_\mu\psi - \overline{\psi}\gamma_\mu\overleftarrow{D}_\mu\psi\right\} \\
O_5 &= m^2\overline{\psi}\psi
\end{aligned}$$

O_3 and O_5 amount to O(am) corrections to g_0^2 and m, in addition for on-shell quantities

$$\begin{aligned}
0 &= O_1 - O_2 + 2O_5 \\
0 &= O_4 + 2O_5
\end{aligned}$$

we may drop O_2 and O_4, $\Rightarrow$ only O_1 remains.

Lattice composite fields can also be represented as continuum effective fields renormalized at a scale $\mu = 1/a$

$$\Phi_{\rm eff} = \Phi_0 + a\Phi_1 + a^2\Phi_2 + \dots$$

example: isovector axial current

$$A^a_\mu(x) = \overline{\psi}(x)\gamma_\mu\gamma_5\frac{1}{2}\tau^a\psi(x)$$

the fields that appear at O(a) are then

$$\begin{aligned}(O_6)^a_\mu &= \overline{\psi}\gamma_5\frac{1}{2}\tau^a\sigma_{\mu\nu}D_\nu\psi - \overline{\psi}\overleftarrow{D}_\nu\sigma_{\mu\nu}\gamma_5\frac{1}{2}\tau^a\psi \\ (O_7)^a_\mu &= \partial_\mu\left\{\overline{\psi}\gamma_5\frac{1}{2}\tau^a\psi\right\} \\ (O_8)^a_\mu &= m\overline{\psi}\gamma_\mu\gamma_5\frac{1}{2}\tau^a\psi\end{aligned}$$

again O_8 is a mass dependent rescaling of the operator, O_6 is a combination of O_7 and O_8 $\Rightarrow$

$$(A_I)^a_\mu = A^a_\mu + c_A a\partial_\mu P^a$$

discretized on the lattice, has on-shell matrix elements approaching the c.l. with rate a^2, if the coeff c_A is properly tuned and the action is improved as well.

Indeed, a lattice n-point correlation function

$$G_n(x_1,\dots,x_n) = (Z_\Phi)^n \langle \Phi(x_1)\dots\Phi(x_n)\rangle_c$$

receives O(a) contrbutions from the composite fields as and from the action

$$\begin{aligned} G_n(x_1,\dots,x_n) &= \langle \Phi_0(x_1)\dots\Phi_0(x_n)\rangle_c \\ &-a\int d^4y \langle \Phi_0(x_1)\dots\Phi_0(x_n)\mathcal{L}_1(y)\rangle_c \\ &+a\sum_{k=1}^{n}\langle \Phi_0(x_1)\dots\Phi_1(x_k)\dots\Phi_0(x_n)\rangle_c \end{aligned}$$

$\langle . \rangle$ wrt $\mathcal{L}_0$.

Based on this, we want to modify the discretization of the operators and of the QCD action in order to cancel the leading O(a) effects. We have seen an example for the operator axial current.

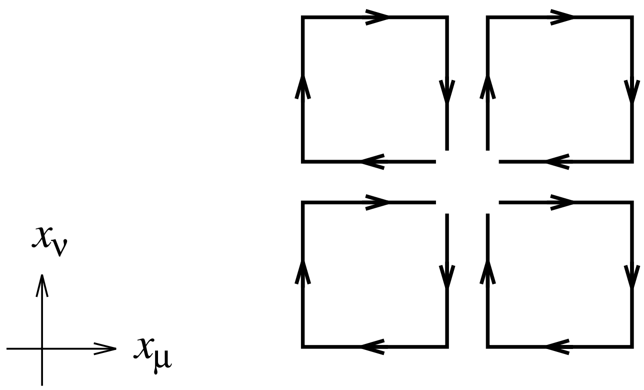
For the action

$$S = S_{\rm G} + S_{\rm F} + a^5 \sum_x c_{\rm SW} \overline{\psi}(x) \frac{i}{4} \hat{F}_{\mu\nu}(x) \psi(x)$$

$\hat{F}_{\mu\nu}(x)$ is the lattice expression for the gluon field strength tensor

$$\hat{F}_{\mu\nu}(x) = \frac{1}{8a^2} \{Q_{\mu\nu}(x) - Q_{\nu\mu}(x)\}$$

with $Q_{\mu\nu}(x)$ symmetrically defined around x



How do we tune $c_{\rm SW}$?

Chirality violating terms (like the Wilson term) are mostly responsible for O(a) effects. Indeed the SW term is chirality violating. $\Rightarrow$ Restoration of chiral symmetry can be used to fix the coefficient $c_{\rm SW}$.

A bare quark mass can be defined by the WI

$$m = \frac{\langle\alpha|\partial_\mu (A_I)^a_\mu O|\beta\rangle}{2\langle\alpha|P^a_I O|\beta\rangle}$$

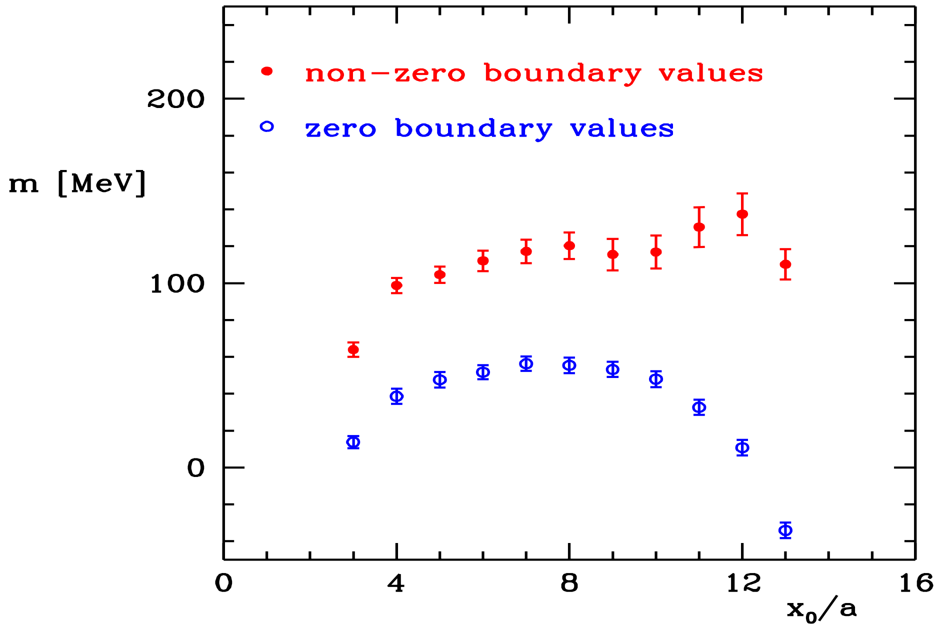
m is given by an operator identity. It should be independet from the states $|\alpha\rangle$ and $|\beta\rangle$.

In the SF this means that for a choice of g_0^2 and κ, we can tune $c_{\rm SW}$ by enforcing m to be independent from the boundary fields C or C'.

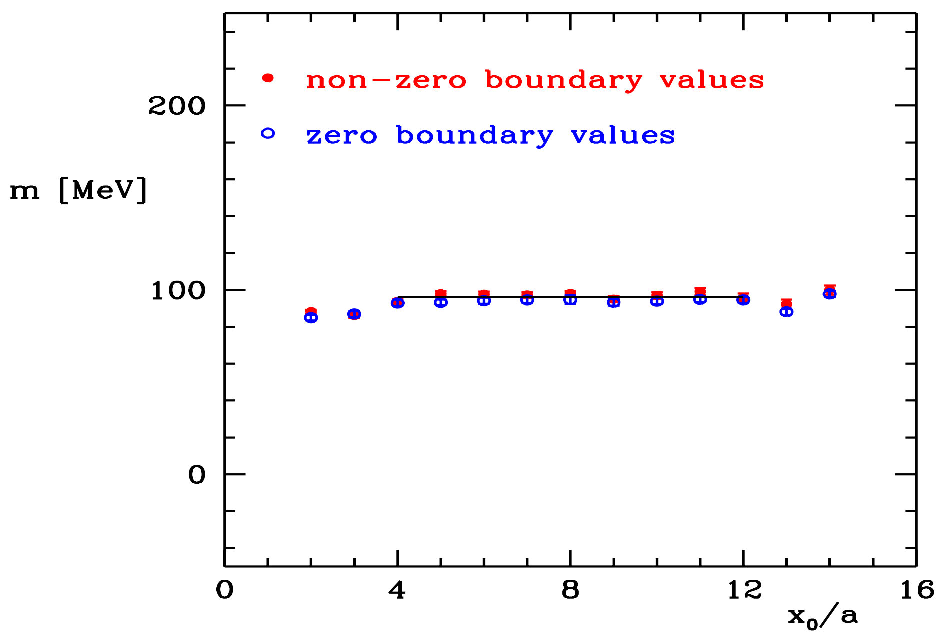
$$c_{\rm SW} \;\; \text{s.t.} \;\; \Delta m = m - m' = O(a^2)$$

Remark: I've been a bit sloppy here. m depends also on c_A . The condition is imposed on a quantity M closely related to the WI mass but independent from c_A.

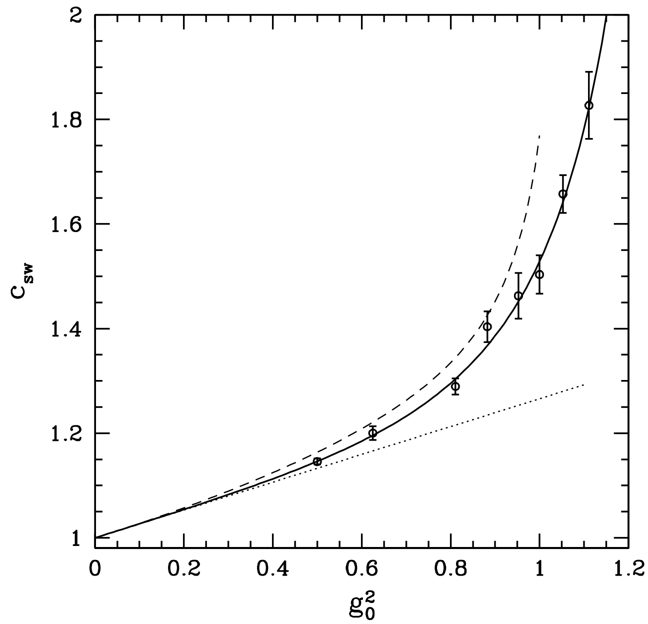
quark mass for $c_{\mathrm{SW}} = 0$



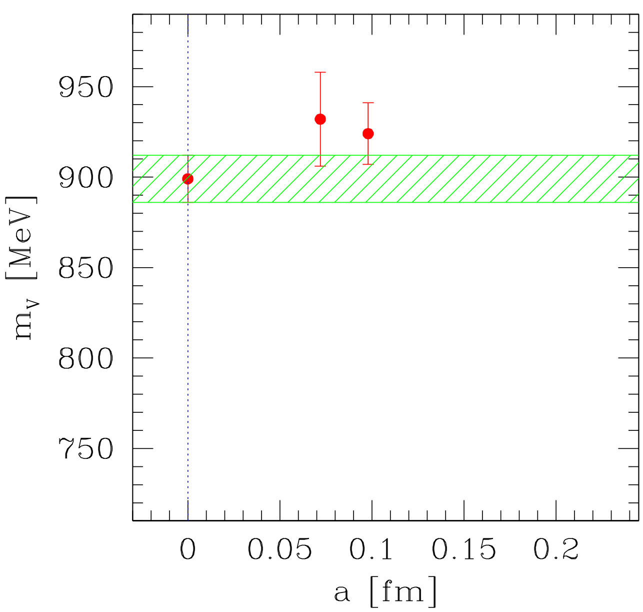
quark mass for c_{SW} properly tuned



Performing the tuning for various values of β



What about m_V ?



ALPHA: hep-lat/9605038, hep-lat/9609035

Remarks:

- Improvement helps in reducing lattice artefacts, but still extrapolating to the continuum is necessary
- We discussed Symanzik programme for Wilson fermions. The approach is general, but not the results. In other regularizations the coefficients have different values. In some they are even zero (Overlap, domain wall, tmQCD @ $\alpha = \pi/2, \ldots$).