Applications of Chiral Perturbation theory to lattice QCD (III)

How to extrapolate from what we can simulate to physical answers

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General strategy

Proceed in two steps: [Sharpe & Singleton]

Lattice Lagrangian:
Wilson, tm, staggered

Continuum effective Lagrangian:
continuum quark-level theory including explicit nonzero $a$ effects [Symanzik]

Chiral Lagrangian:
continuum $\chi$PT plus effects of additional operators induced by discretization
Power counting

- In chPT expand in \( \frac{p^2}{\Lambda^2} \sim \frac{m^2_{P\bar{G}B}}{\Lambda^2} \sim \frac{m_q}{\Lambda_{QCD}} \)
  - Second “equality” follows from \( \frac{m_K^2}{(4\pi f^2)^2} = 0.18 \approx \frac{m_{\text{phys}}}{\Lambda_{QCD}} \approx \frac{80}{300} = 0.27 \)

- How does \( \frac{m_q}{\Lambda_{QCD}} \) compare to \((a\Lambda_{QCD})^n\)?

- Equivalently, how does \( m_q \) compare to \( a\Lambda^2_{QCD}, a^2\Lambda^3_{QCD}, \ldots \)?
  - If \( a^{-1} = 2 \text{ GeV} \) and \( \Lambda_{QCD} = 300 \text{ MeV} \), then
    \[ a\Lambda^2_{QCD} = 45 \text{ MeV}, \quad a^2\Lambda^3_{QCD} = 7 \text{ MeV}, \quad a^3\Lambda^4_{QCD} = 1 \text{ MeV} \]

- Appropriate power counting is \( a^2\Lambda^2_{QCD} \lesssim m_q/\Lambda_{QCD} \lesssim a\Lambda_{QCD} \)

- **LESSON:** \( O(a) \) effects MUST BE REMOVED, and \( O(a^2) \) understood

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S. Sharpe, “XPT for LQCD (II)”, Nara, 11/7/2005 – p.25/70
Power counting terminology

- **Generic Small Mass (GSM) regime**: $\Lambda_{\text{QCD}} \gg m_q \gtrsim a\Lambda_{\text{QCD}}^2$
  - Includes $m_q \gg a\Lambda_{\text{QCD}}^2$ and $m_q \approx a\Lambda_{\text{QCD}}^2$ but not $m_q \ll a\Lambda_{\text{QCD}}^2$

- **Aoki regime**: $m_q \lesssim a^2\Lambda_{\text{QCD}}^3$
  - Includes $m_q \ll a^2\Lambda_{\text{QCD}}^3$

tmχPT at NLO

- Rewrite $\mathcal{L}_\chi$ in terms of $\chi'$ [Sharpe & Wu]

$$
\mathcal{L}_\chi = \frac{f^2}{4} \text{tr}(D_\mu \Sigma D_\mu \Sigma^\dagger) - \frac{f^2}{4} \text{tr}(\chi'^\dagger \Sigma + \Sigma^\dagger \chi')
- L_1 \text{tr}(D_\mu \Sigma D_\mu \Sigma^\dagger)^2 - L_2 \text{tr}(D_\mu \Sigma D_\nu \Sigma^\dagger)\text{tr}(D_\mu \Sigma D_\nu \Sigma^\dagger)
+ L_{45} \text{tr}(D_\mu \Sigma^\dagger D_\mu \Sigma)\text{tr}(\chi'^\dagger \Sigma + \Sigma^\dagger \chi') - L_{68} \left[ \text{tr}(\chi'^\dagger \Sigma + \Sigma^\dagger \chi') \right]^2
+ \tilde{W} \text{tr}(D_\mu \Sigma^\dagger D_\mu \Sigma)\text{tr}(\hat{A}^\dagger \Sigma + \Sigma^\dagger \hat{A}) - W \text{tr}(\chi'^\dagger \Sigma + \Sigma^\dagger \chi')\text{tr}(\hat{A}^\dagger \Sigma + \Sigma^\dagger \hat{A})
- W' \left[ \text{tr}(\hat{A}^\dagger \Sigma + \Sigma^\dagger \hat{A}) \right]^2 + W_{10} \text{tr}(\hat{D}_\mu \hat{A}^\dagger D_\mu \Sigma + D_\mu \Sigma^\dagger D_\mu \hat{A})
$$

- LECs are shifted:
  $$
  \tilde{W} = W_{45} - L_{45}, \quad W = W_{68} - 2L_{68}, \quad W' = W'_{68} - W_{68} + L_{68}
  $$

- $W_{10}$ is redundant: can be shifted into other LECs
  - If $\delta \Sigma = \frac{2}{f^2} W_{10} \left( \Sigma \hat{A}^\dagger \Sigma - \hat{A} \right)$
  - then $W \rightarrow W + W_{10}/4$, $\tilde{W} \rightarrow \tilde{W} + W_{10}/2$
  - Useful to keep $W_{10}$ as check on calculations, but will drop here
Comparing $tm\chi PT$ with data

$$m_{\pi \pm}^2 = |\chi'| + \text{cont. 1-loop chiral logs}$$

$$+ \frac{16}{f^2} \left[ |\chi'|^2 (2L_{68} - L_{45}) + |\chi'| \hat{a} \cos \omega_0 (2W - \bar{W}) + 2\hat{a}^2 (\cos \omega_0)^2 W' \right]$$

- Clear antisymmetry of \( \approx 30\% \sim a\Lambda^2 \) with \( \Lambda \approx 300 \text{ MeV} \)
- Non-vanishing minimum pion mass due to \( W' \)

[Farchioni et al., hep-lat/0410031]
More NLO results

- Dynamical Wilson fermion results for $a = 0.06 - 0.08$ with $m_\pi$ down to 280 MeV [Giusti]
- Earlier simulations for $a = 0.14 - 0.2$ fm (also with Wilson gauge action) found first-order phase transition, with $m_{\pi,\text{min}} \approx 600$ MeV at $a = 0.2$ fm
- According to tmχPT at NLO, $m_{\pi,\text{min}} \propto a + a^2 + \ldots$
- Thus expect $m_{\pi,\text{min}} \approx 240$ MeV at $a = 0.08$ fm

- $m_\pi^2$ vs. $m_{\text{PCAC}}$ might look like:
- $\delta W = \delta \tilde{W} = -0.3$, $\sqrt{|2w'|} = 250$ MeV, $L_i$ and chiral logs ignored

\[\text{mpi}^2 \text{ vs. PCAC mass, mu=0}\]
Relation between definitions of $\omega$

- At maximal twist:
  
  (i): $\omega_A = \pi/2 \implies \omega_0 = \omega_A + \delta W = \pi/2 + \delta W$
  
  (iii): $\omega_P = \pi/2 \implies \omega_0 = \omega_P = \pi/2$

Both methods lead to automatic $O(a)$ improvement

- In quenched simulations $\delta W \approx -0.35$ for $a^{-1} \approx 2$ GeV [Abdel-Rehim]

- Direct measure of LEC associated with discretization errors ($\Lambda \approx 0.7$ GeV)

- If fix $m' \sim O(a)$ ("method (iv)")}, then NOT AT MAXIMAL TWIST
Outline of Lecture 2

- Incorporating discretization errors into $\chi$PT
  - Why is this useful?
  - General two-step strategy

- Application to Wilson & twisted mass fermions
  - Symanzik effective action
  - Mapping Symanzik action into $\chi$PT
  - Results for $m_q \sim a\Lambda_{QCD}^2$
  - Defining the twist angle
  - Results for $m_q \sim a^2\Lambda_{QCD}^3$
Theoretical references for this section

Simulations relevant for this section

Power counting in the Aoki regime

- Power counting differs from GSM regime:
  - No $O(a)$ since absorbed into $m'$
  - LO: $m_q \sim a^2$
  - NLO: $m_q a \sim a^3$
  - NNLO: $m_q^2 \sim m_q a^2 \sim a^4$

- Reorders terms in $\mathcal{L}_\chi$:

  \[
  \mathcal{L}_\chi^{\text{LO}} = \frac{f^2}{4} \text{tr}(D_\mu \Sigma D_\mu \Sigma^\dagger) - \frac{f^2}{4} \text{tr}(\chi'^\dagger \Sigma + \Sigma^\dagger \chi') - W' [\text{tr}(\hat{A}^\dagger \Sigma + \Sigma^\dagger \hat{A})]^2
  \]

  \[
  \mathcal{L}_\chi^{\text{NLO}} = \bar{W} \text{tr}(D_\mu \Sigma^\dagger D_\mu \Sigma) \text{tr}(\hat{A}^\dagger \Sigma + \Sigma^\dagger \hat{A}) - W \text{tr}(\chi'^\dagger \Sigma + \Sigma^\dagger \chi') \text{tr}(\hat{A}^\dagger \Sigma + \Sigma^\dagger \hat{A})
  \]

  \[
  - \frac{W_{3,1}}{f^2} \text{tr}(\hat{A}^\dagger \hat{A}) \text{Tr}(\hat{A}^\dagger \Sigma + \text{p.c.}) - \frac{W_{3,3}}{f^2} \left[ \text{tr}(\hat{A}^\dagger \Sigma)^3 + \text{p.c.} \right]
  \]

- At LO have competition between continuum and "lattice" terms
- Two extra LECs at NLO, but $W_{3,1}$ can be absorbed by shift in $m'$
- Only parts of $W$, $\bar{W}$ terms containing sources are of NLO

S. Sharpe, “XPT for LQCD (II)”, Nara, 11/7/2005 – p.54/70
The Aoki regime at LO

\[ \mathcal{L}_{\chi}^{\text{LO}} = \frac{f^2}{4} \text{tr}(D_\mu \Sigma D_\mu \Sigma^\dagger) - \frac{f^2}{4} \text{tr}(\chi'^\dagger \Sigma + \Sigma^\dagger \chi') - W' \left[ \text{tr}(\hat{A}^\dagger \Sigma + \Sigma^\dagger \hat{A}) \right]^2 \]

- Along Wilson axis competition between terms odd and even under \( \Sigma \rightarrow -\Sigma \Rightarrow \) two possible phase structures [Creutz; Sharpe & Singleton]
- Extended to twisted mass-plane in [Münster; Scorzato; Sharpe & Wu]

\[ \alpha = 2B_0 m' / (16|W'|\hat{a}^2 / f^2), \quad \beta = 2B_0 \mu / (16|W'|\hat{a}^2 / f^2) \]

$W' < 0$: Aoki phase

Condensate: $\langle \Sigma \rangle = A_m + iB_m \tau_3$

Aoki phase washed out for $\mu \propto \beta \neq 0$

(a) Mass of $\pi_1$ and $\pi_2$

(b) Mass of $\pi_3$

S. Sharpe, “XPT for LQCD (II)”, Nara, 11/7/2005 – p.56/70
$W' > 0$: first-order transition

Along Wilson axis:

At top of phase transition: (dashed: charged pions; solid: neutral)

S. Sharpe, “XPT for LQCD (II)”, Nara, 11/7/2005 – p.57/70
More on $W' > 0$

Above phase transition: (dashed: charged pions; solid: neutral)

\[ \begin{align*}
\alpha & = 2B_0m'/(16|W'|\hat{a}^2/f^2), \\
\beta & = 2B_0\mu/(16|W'|\hat{a}^2/f^2)
\end{align*} \]

- $c_2 = -16W'\hat{a}^2 \Rightarrow m^2_\pi \sim a^2$ in all plots
- Pion mass-splitting depends on same LEC that determines phase structure and size of phase boundaries [Scorzato]

\[ m^2_\pi \pm - m^2_\pi = -\frac{32W'\hat{a}^2}{f^2}(\sin \omega_0)^2 + O(a^3) \]

- $W' < 0 \Rightarrow m_\pi \pm \leq m_\pi^0$ and Aoki phase
- $W' > 0 \Rightarrow m_\pi \pm \geq m_\pi^0$ and first-order
Stability of predictions

- Do higher order terms in tmχPT change the phase structure?
  - No! (As long as the higher order terms are smaller)
    - The presence of the first order transition is stable against small changes in the potential
    - The position of the line may move a small amount.
    - The properties of the second-order endpoints may be changed substantially (e.g. size of logarithmic corrections to scaling) [Aoki]
Lessons for lattice (I)

- Expect phase structure with potential isospin breaking $\sim a^2$
- Prediction of Aoki-phase made long ago (how can $m_\pi \to 0$ without chiral symmetry?) [Aoki]
- Old (80-90’s) quenched studies gave evidence for Aoki-phase scenario
- New results with dynamical tm quarks find first-order scenario, e.g. [Farchioni et al, hep-lat/0506025]

- Discontinuity decreases with $a$ qualitatively as expected
- Charged pion mass and $m_{PCAC}$ do not vanish
- Gives confidence in tm$\chi$PT
Lessons for lattice (II)

- $\text{tm}\chi\text{PT}$ gives reasonable description of data, e.g. [Farchioni et al, hep-lat/0410031]

- Detailed fits of $m_\pi$, $f_\pi$, $g_\pi$, $\omega_A$ ... appear to confirm this [Aoki & Bar; Farchioni et al]

- Fits should be done using full NLO forms (available for GSM and Aoki regimes [Sharpe & Wu; Aoki & Bar; Sharpe])
Examples of NLO results

- Contours of charged $m_\pi^2$ in $m'' - \mu$ (in GeV) plane:
  - $\delta W = \delta \overline{W} = -0.3$, $|W'| = 16\hat{a}^2|W'|/f^2 = (250 \text{ MeV})^2$, $W_{3,3} = 0$

- Aoki phase
- First-order

- Phase structure could severely impede chiral extrapolations!
Lessons for lattice (III)

- **Tune gauge action to reduce** $W' (\propto c_2)$ **and shrink phase boundaries and pion mass splitting**
  - Relies on tm$\chi$PT prediction that only one $O(a^2)$ LEC
  - $O(a)$ improving fermion action not enough, since $W'$ term is $O(a^2)$
  - If $W' \sim a$, then $O(a^3)$ terms in $V_\chi$ impact phase structure
    - Find one scenario with both Aoki-phase *and* first-order, but size $\sim a^3$ [Sharpe]
  - Does **not** remove $O(a)$ errors in physical quantities, e.g. along Wilson axis

- **Success with both DBW2 and tree-level improved Symanzik gauge actions**
  - Discontinuities much reduced [Farchioni *et al*]
  - Isospin splittings small, favor Aoki-phase scenario [McNeile]
  - Very encouraging for tmLQCD

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Lessons for lattice (IV): maximal twist

- How do definitions of maximal twist extrapolate into the Aoki regime?
- Does automatic $O(a)$ improvement still hold?

- Addressed by [Aoki & Bär; Sharpe & Wu; Sharpe]

- Methods (i) $[\omega_A = \pi/2]$, (iii) $[\omega_P = \pi/2]$, and (ii) still apply and lead to automatic $O(a)$ improvement

- $\langle \Sigma \rangle = i\tau_3 + O(a)$

- Must adjust $m$ with increasing accuracy as $\mu$ decreases

- Isospin breaking is $\sim 100\%$ as $\mu \to 0$

- $m_\pi \to 0$ definition fails
More on maximal twist in Aoki regime

- First-order scenario appears “worse”
- Automatic $O(a)$ improvement stops at end-points
- Infact just a reversal of roles of $\pi^\pm$ and $\pi^0$
  - Aoki-phase: $m_{\pi^\pm} \to 0$, $m_{\pi^0} \sim a$
  - First-order: $m_{\pi^\pm} \sim a$, $m_{\pi^0} \to 0$

S. Sharpe, “XPT for LQCD (II)”, Nara, 11/7/2005 – p.64/70
Isospin breaking at maximal twist

- Comparing the two scenarios:

Aoki phase

- First-order

NLO expressions for Aoki regime (chiral logs and $L_i$ dropped)
- Charged: solid lines; neutral: dashed.
- Method (i): thick lines; method (iii) thin.

$\delta_W = \delta_{\bar{W}} = -0.3$, $w' = 16a^2W'/f^2 = \mp(250\text{ MeV})^2$, $W_{3,3} = 0$

Lesson: What matters most is reducing size of $W'$—whether in Aoki or first order scenario is less important.
Bending near maximal twist

- “Bending” or apparent IR divergences occur if make poor choice of $m_c$
  - E.g. Aoki-phase scenario, using
    - method (i) [$O(a)$ improved]
    - $m_\pi = 0$ choice ($m'' = -5$ MeV)
    - Missing $m_c$ by 5 MeV ($m'' = -10$ MeV)

- $\omega_A \approx a^3$,

- $\omega_P \approx \pi/2$

- $\delta \omega \approx a^2$

- $\delta W = \delta \tilde{W} = -0.3$, $w' = -(250 \text{ MeV})^2$, $W_{3,3} = 0$ (Chiral logs, most $L_i \to 0$)

- Can correct for some of effect if determine $\omega$, but only method (ii) gives automatic improvement after correction
More on Bending

- Observed in simulations (and fit by [Aoki & Bär]), but removed using methods (i) or (ii) (figure from review [Shindler]).

- Bending caused by vacuum bending significantly away from $\langle \Sigma \rangle = i\tau_3$.

- If perturb about $\langle \Sigma \rangle = i\tau_3$, then find IR divergences due to $\pi^0$ poles [Frezzotti et al].

- These are summed up by tmχPT by expanding about correct vacuum.

- BOTTOM LINE: use a non-perturbative determination of maximal twist!

S. Sharpe, “XPT for LQCD (II)”, Nara, 11/7/2005 – p.67/70
Why does maximal twist work?

- Why are physical quantities automatically $O(a)$ improved?
- At quark level, maximal twist implies:

$$\mathcal{L}_{NLO}^{(4+5)} = \bar{\psi} \not{D} \psi + \mu \bar{\psi} i \gamma_5 \tau_3 \psi + ac \bar{\psi} i \sigma_{\mu \nu} F_{\mu \nu} \psi$$

$$= \bar{\psi}_{\text{phys}} \not{D} \psi_{\text{phys}} + \mu \bar{\psi}_{\text{phys}} \psi_{\text{phys}} + ac \bar{\psi}_{\text{phys}} \gamma_5 \tau_3 \sigma_{\mu \nu} F_{\mu \nu} \psi_{\text{phys}}$$

$$\Rightarrow O(a)$$ corrections necessarily violate parity and flavor
$$\Rightarrow$$ physical (parity-flavor conserving) quantities corrected only at $O(a^2)$

- At chiral Lagrangian level, potential $O(a)$ terms are

$$\mathcal{L}_{\chi,NLO} = \cdots + W \tr(D_\mu \Sigma^\dagger D_\mu \Sigma) \tr(\hat{A}^\dagger \Sigma + \Sigma^\dagger \hat{A})$$

$$- W \tr(\chi'^\dagger \Sigma + \Sigma^\dagger \chi') \tr(\hat{A}^\dagger \Sigma + \Sigma^\dagger \hat{A}) + \cdots$$

- $\tr(\hat{A}^\dagger \Sigma + \Sigma^\dagger \hat{A})$ vanishes when $\Sigma = i\tau_3$
- Expanding about $\langle \Sigma \rangle = i\tau_3$ get only odd powers of $\pi$

$$\Rightarrow$$ $O(a)$ terms only contribute to unphysical processes
Does it work in practice?

- Yes! Several successful scaling tests, e.g. quenched by [Farchioni et al]
Future issues

- **Challenges for simulations:**
  - Can $W'$ (a.k.a. $c_2$, $c_1$!) be tuned so phase-transition region shrinks significantly?
  - Can $\omega$ be tuned to $\pi/2$ accurately enough in practice?
  - Can isospin breaking be determined ($\Delta m_\pi^2$, $\Delta m_\Delta$, $\ldots$)?
  - Can fits test tm$\chi$PT (working not only at maximal twist)?
  - Inclusion of strange (and charm?) quarks (begun by [Farchioni lat05]

- **Challenges for tm$\chi$PT:**
  - Inclusion of $\Delta m_\pi^2$ into loops (analogue of including taste-breaking into staggered pion loops—which is very important in practice). Requires going to NNLO.
  - Extension to other hadrons (baryons done [Walker-Loud & Wu]
  - Extension to partially quenched theories (begun by [Münster et al]
  - Study discretization effects on calculations of weak matrix elements
Outline of 3 lectures

- **Lecture 1**
  - Overview and aims of 3 lectures
  - Chiral perturbation theory for (continuum) QCD

- **Lecture 2**
  - Incorporating lattice spacing errors
  - Application to Wilson & twisted mass fermions

- **Lecture 3**
  - Partial quenching and PQ\(\chi\)PT
  - Application to staggered fermions
Outline of Lecture 3

- Partial quenching and PQ\chi PT
  - What is partial quenching?
  - Developing PQ\chi PT
  - Results and outlook

- Application to staggered fermions
References for Partial Quenching

- S. R. Sharpe and N. Shoresh, “Partially quenched chiral perturbation theory without $\Phi_0$,” Phys. Rev. D 64, 114510 (2001)
What is Partially Quenched QCD?

- Explain with example of pion correlator:

\[ C_\pi(\tau) = -\left\langle \sum_{\vec{x}} \bar{u}\gamma_5 d(\vec{x}, \tau) \, d\gamma_5 u(0) \right\rangle \]

\[ \equiv -\frac{1}{Z} \int DU \prod_q DqD\bar{q} e^{-S_{\text{gauge}} - \int_x \sum_q \bar{q}(\not{D} + m_q)q} \sum_{\vec{x}} \bar{u}\gamma_5 d(\vec{x}, \tau) \, d\gamma_5 u(0) \]

\[ = \frac{1}{Z} \int DU \prod_q \det(\not{D} + m_q) e^{-S_{\text{gauge}}} \sum_{\vec{x}} \text{tr} \left[ \gamma_5 \left( \frac{1}{\not{D} + m_d} \right)_{x0} \gamma_5 \left( \frac{1}{\not{D} + m_u} \right)_{0x} \right] \]

\[ = \left\langle \sum_\gamma \gamma \right\rangle \]

\[ \propto f_\pi^2 e^{-m_\pi \tau} + \text{exp. suppressed} \]

- “sea” quarks in determinant; “valence” in propagators

- Partial Quenching: \( m_{\text{val}} \neq m_{\text{sea}} \)—many different \( m_{\text{val}} \) for each \( m_{\text{sea}} \)

- Numerically cheap—can we make use of this extra information?
PQQCD needs $\chi$PT

- Use PQQCD as a tool to learn about QCD, not as a model of QCD
  - PQQCD is unphysical, e.g. not unitary
  - Intermediate and external "states" differ, e.g. $\pi V \pi V \rightarrow \pi S \pi S \rightarrow \pi V \pi V$

- Need PQQ$\chi$PT in order to extrapolate to QCD
  - must be in the quark-mass regime where $\chi$PT is valid
  - Extends the range over which can match lattice and $\chi$PT

- Subspace with $m_{\text{val}} = m_{\text{sea}}$ are physical QCD-like theories
  - PQQ$\chi$PT must match $\chi$PT on subspace
  - LECs in PQQ$\chi$PT include those appearing in $\chi$PT, plus a few (sometimes none) additional unphysical ones
Historical comment on nomenclature

- Why called partially quenched? Why not partially unquenched?
- Bad old days: quenched approximation $m_{\text{sea}} \to \infty$
  \[ \Rightarrow \det(\mathbb{P} + m_q) \to \text{constant} \]
  \[ \Rightarrow \text{No quark loops} \]
  \[ \Rightarrow Z_{\text{QCD}} \to Z_{\text{QQCD}} = \int D U e^{-S_{\text{gauge}}} = Z_{\text{gauge}} \]
- Unphysical nature of quenched QCD shows up various ways, e.g.
  \[ \langle \bar{\psi} \psi \rangle \to \infty \text{ as } m_{\text{val}} \to 0 \]
- Partial quenching is in one sense a less extreme version of quenching, and thus the name
- If $m_{\text{sea}} \gg \Lambda_{\text{QCD}}$ then PQQCD, like quenched QCD, only qualitatively related to QCD
- Consider here only the case when $m_{\text{sea}} \ll \Lambda_{\text{QCD}}$ so one can use $\chi$PT and relate PQCD to QCD quantitatively

S. Sharpe, “$\chi$PT for LQCD (III)”, Nara, 11/7/2005 – p.7/46
Morel’s formulation of (P)QQCD

**IDEA:** commuting spin-$\frac{1}{2}$ fields (ghosts) $\tilde{q}$ give determinant which cancels that from valence quarks

$$\int D\bar{q} Dq \ e^{-\bar{q}(\not{D} + m_q)q} = \det(\not{D} + m_q)$$

$$\int D\bar{q}^\dagger D\bar{q} \ e^{-\bar{q}^\dagger(\not{D} + m_q)\bar{q}} = \frac{1}{\det(\not{D} + m_q)}$$

To formulate PQQCD need three types of “quark”

- valence quarks $q_{V1}, q_{V2}, \ldots q_{VN_V} \ (N_V = 2, 3, \ldots)$
- sea quarks $q_{S1}, q_{S2}, \ldots q_{SN} \ (N = 2, 3)$
- ghosts $\tilde{q}_{V1}, \tilde{q}_{V2}, \ldots \tilde{q}_{VN_V} \ (N_V = 2, 3, \ldots)$

- Ghosts are degenerate with corresponding valence quarks

Morel’s formulation (cont.)

Partition function reproduces that which is simulated:

\[
Z_{PQ} = \int DU e^{-S_{\text{gauge}}} \prod_{i=1}^{N_V} \left( D\bar{q}_V Dq_V D\bar{q}_V^+ D\bar{q}_V \right) \prod_{j=1}^{N} \left( D\bar{q}_S Dq_S \right) \times \\
\times \exp \left[ -\sum_{i=1}^{N_V} \bar{q}_V (\bar{q}_V + m_{Vi})q_V - \sum_{j=1}^{N} \bar{q}_S (\bar{q}_S + m_{Si})q_S - \sum_{k=1}^{N_V} \bar{q}_V^+ (\bar{q}_V + m_{V_k})q_V \right] \\
= \int DU e^{-S_{\text{gauge}}} \prod_{i=1}^{N_V} \left( \frac{\det(\bar{q}_V + m_{Vi})}{\det(\bar{q}_V + m_{Vi})} \right) \prod_{j=1}^{N} \det(\bar{q}_S + m_{Sj}) \\
= \int DU e^{-S_{\text{gauge}}} \prod_{j=1}^{N} \det(\bar{q}_S + m_{Sj}) \\
= Z_{\text{QCD-like}}
\]
Compact Notation

- Collect all fields into \((N + 2N_V)\)-dim vectors:

\[
Q = \begin{pmatrix} qV_1, qV_2, \ldots, qV_{N_V}, qS_1, qS_2, \ldots, qS_N, \tilde{q}V_1, \tilde{q}V_2, \ldots, \tilde{q}V_{N_V} \\ \text{valence} & \text{sea} & \text{ghost} \end{pmatrix}
\]

\[
Q^{tr} = \begin{pmatrix} \bar{q}V_1, \bar{q}V_2, \ldots, \bar{q}V_{N_V}, \bar{q}S_1, \bar{q}S_2, \ldots, \bar{q}S_N, \tilde{q}V_1, \tilde{q}V_2, \ldots, \tilde{q}V_{N_V} \\ \text{valence} & \text{sea} & \text{ghost} \end{pmatrix}
\]

\[
\mathcal{M} = \begin{pmatrix} mV_1, mV_2, \ldots, mV_{N_V}, mS_1, mS_2, \ldots, mS_N, mV_1, mV_2, \ldots, mV_{N_V} \\ \text{valence} & \text{sea} & \text{ghost valence} \end{pmatrix}
\]

- Then can write action and partition function as:

\[
S_{PQ} = S_{\text{gauge}} + \overline{Q}(\bar{\Psi} + \mathcal{M})Q
\]

\[
Z_{PQ} = \int D\bar{\Psi}D\Psi DQ \ e^{-S_{PQ}}
\]
Formal representation of PQ correlator

\[
C^{PQ}_\pi(\tau) = \left\langle \sum \gamma_5 \right. \begin{array}{c} x \\ \end{array} \begin{array}{c} 0 \\ \gamma_5 \end{array} \left( \begin{array}{c} d_V \\ u_V \end{array} \right) \right\rangle
\]

\[
= Z_{PQ}^{-1} \int DU \prod_{j=1}^N \det(\overline{\psi} + m_{Sj}) e^{-S_{\text{gauge}}}
\times \sum_{\vec{x}} \text{tr} \left[ \gamma_5 \left( \frac{1}{\overline{\psi} + m_{Vd}} \right) x_0 \gamma_5 \left( \frac{1}{\overline{\psi} + m_{Vu}} \right)_{0x} \right]
\]

\[
= Z_{PQ}^{-1} \int DU D\overline{Q} DQ \ e^{-S_{\text{PQ}}} \sum_{\vec{x}} \overline{u}_V \gamma_5 d_V (\vec{x}, \tau) \ \overline{d}_V \gamma_5 u_V (0)
\]

\[
Q = (q_{V1}, q_{V2}, \ldots, q_{VN_v}, q_{S1}, q_{S2}, \ldots, q_{SN}, \tilde{q}_{V1}, \tilde{q}_{V2}, \ldots, \tilde{q}_{VN_v})
\]

S. Sharpe, "\chi PT for LQCD (III)", Nara, 11/7/2005 – p.11/46
What have we learned about PQQCD?

- Well defined statistical system describing correlators in Euclidean space
  - Can use to represent individual contractions in complicated processes, e.g. \( \pi \pi \rightarrow \pi \pi \)

- Regained unitarity, but at the cost of introducing ghosts

- Shows in what way the PQ theory is unphysical
  - violate spin-statistics theorem
  - lose causality and positivity in Minkowski space
  - lose reflection positivity in Euclidean space

- Unphysical nature shows up in various ways:
  - Double poles in correlation functions
  - Correlators involving multi-particle states do not have exponential fall-off in time, and have contributions which diverge in infinite volume \( \Rightarrow \) cannot define scattering amplitudes \([\text{Lin et al}]\)

- Can we develop an EFT describing PQQCD including its unphysical nature?
Key property of PQQCD

- “Anchored” to physical QCD-like theories
- If $m_{V_u} = m_{S_j}$ and $m_{V_d} = m_{S_k}$ then valence correlator is physical:

$$C_{\pi}^{PQ}(\tau) = Z_{PQ}^{-1} \int DU D\bar{Q} DQ e^{-S_{PQ}} \sum_{\bar{x}} \bar{u}_V \gamma_5 d_V(\bar{x}, \tau) \bar{d}_V \gamma_5 u_V(0)$$

$$= Z_{PQ}^{-1} \int DU D\bar{Q} DQ e^{-S_{PQ}} \sum_{\bar{x}} \bar{q}_{S_j} \gamma_5 q_{S_k}(\bar{x}, \tau) \bar{q}_{S_k} \gamma_5 q_{S_j}(0)$$

$$= Z_{QCD-like}^{-1} \int DU \prod_{i=1}^{N} D\bar{q}_{Si} Dq_{Si} e^{-S_{QCD-like}}$$

$$\times \sum_{\bar{x}} \bar{q}_{S_j} \gamma_5 q_{S_k}(\bar{x}, \tau) \bar{q}_{S_k} \gamma_5 q_{S_j}(0)$$

$$= C_{\pi}^{QCD-like}(\tau)$$

- Example of enhanced ($V \leftrightarrow S$) symmetry in PQ theory
Outline of Lecture 3

- Partial quenching and PQχPT
  - What is partial quenching?
  - Developing PQχPT
  - Results and outlook
- Application to staggered fermions
Methods for developing PQ$\chi$PT

- “Supersymmetric” method based on Morel’s formulation [Bernard & Golterman]
- “Quark-line” method accounting by hand for quarks in loops [Sharpe]
- “Replica” method adjusting loop contributions by adjusting $N_{\text{sea}}$ [Damgaard & Splittorf]
- All give same results to date—likely equivalent
- Use supersymmetric method here (with addition of some quark-line method when considering staggered fermions)
Symmetries of PQQCD

\[
Q = \left( q_{V1}, q_{V2}, \ldots, q_{VN_V}, q_{S1}, q_{S2}, \ldots, q_{SN}, \tilde{q}_{V1}, \tilde{q}_{V2}, \ldots, \tilde{q}_{VN_V} \right)
\]

- Action of PQQCD looks like QCD
  \[
  S_{PQQCD} = S_{\text{gauge}} + \bar{Q}(\bar{\psi} + M)Q
  \]

- Naively, when \( M \to 0 \) have graded version of QCD chiral symmetry:
  \[
  Q_{L,R} \rightarrow U_{L,R}Q_{L,R}, \quad \bar{Q}_{L,R} \rightarrow \bar{Q}_{L,R}U_{L,R}^\dagger \quad U_{L,R} \in SU(N_V + N|N_V)
  \]

- Apparent symmetry is \( SU(N_V + N|N_V)_L \times SU(N_V + N|N_V)_R \times U(1)_V \)

- In fact, there are subtleties in the ghost sector, but can ignore in perturbative calculations [Sharpe & Shoresh]
Brief primer on graded Lie groups

- $U$ is graded: contains both commuting and anticommuting elements:
  \[
  U = \begin{pmatrix}
  A & B \\
  C & D \\
  \end{pmatrix}, \quad A, D \text{ commuting, } B, C \text{ anticommuting}
  \]

- If $U \in U(N_V + N|N_V)$ (fundamental representation) then
  \[
  UU^\dagger = U^\dagger U = 1, \quad [\text{with } (\eta_1 \eta_2)^* \equiv \eta_2^* \eta_1^*]
  \]

- Supertrace maintains cyclicity:
  \[
  \text{str} U \equiv \text{tr}A - \text{tr}D \quad \Rightarrow \quad \text{str}(U_1 U_2) = \text{str}(U_2 U_1)
  \]

- For $U \in SU(N_V + N|N_V)$, superdeterminant is unity:
  \[
  \text{sdet} U \equiv \exp[\text{str}(\ln U)] = \frac{\det(A - BD^{-1}C)}{\det(D)} \quad \Rightarrow \quad \text{sdet}(U_1 U_2) = \text{sdet}U_1 \text{sdet}U_2
  \]
Examples of $SU(N_V + N|N)$ matrices

$U = \begin{pmatrix} SU(N_V + N) & 0 \\ 0 & SU(N_V) \end{pmatrix} \Rightarrow \text{sdet}U = 1$

$U = \begin{pmatrix} e^{i\theta N_V} & 0 \\ 0 & e^{i\theta(N+N_V)} \end{pmatrix} \Rightarrow \text{sdet}U = \frac{(e^{i\theta N_V})^{N+N_V}}{(e^{i\theta(N+N_V)})^{N_V}} = 1$

- An overall phase rotation is not in $SU(N_V + N|N)$

$U = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \Rightarrow \text{sdet}U = \frac{e^{i\theta(N+N_V)}}{e^{i\theta N_V}} = e^{i\theta N}$

- Thus $U(N_V + N|N_V) = [SU(N_V + N|N_V) \otimes U(1)]/Z_N$

- Group structure different if $N = 0$ (quenched theory)
Follow same steps as for QCD

- Expand about $\mathcal{M} = 0$
  - *A posteriori* find that must take chiral limit with $m_V$ and $m_S$ in fixed ratio
  - Divergences if $m_V \to 0$ at fixed $m_S$ [Sharpe]

- Graded chiral symmetry is broken by condensate
  - Have Goldstone bosons and fermions (but both spin 0)

- Develop low-energy EFT based on symmetries and symmetry braking
  - Weaker theoretical basis than usual $\chi$PT since underlying theory is unphysical
  - PQ$\chi$PT matches unphysical features of PQQCD (e.g. double poles)

- Most LECs in PQ$\chi$PT are the same as those in $\chi$PT because QCD is a subset of PQQCD
  - Use PQQCD to determine physical parameters of QCD (and/or to improve chiral extrapolations)
Symmetry breaking in PQQCD

- Symmetry group \((M \to 0)\): \(G = SU(N_V + N|N_V)_L \times SU(N_V + N|N_V)_R\)
- For \(M\) diagonal, real and positive [Vafa & Witten] implies graded vector symmetry not spontaneously broken
  - Quark and ghost condensates equal if \(m_V = m_S \to 0\)
    \[\langle qV \bar{q}V \rangle = \langle \bar{q}V qV \rangle = \langle qS \bar{q}S \rangle = \omega\]
- Spontaneous chiral symmetry breaking in QCD \(\Rightarrow \omega \neq 0\)
  - We know pattern of symmetry breaking. Introducing order parameter
    \[\Omega_{ij} = \langle Q_{L,i,\alpha,c} \bar{Q}_{R,j,\alpha,c} \rangle_{PQ} \to_{G} U_L \Omega U_R^\dagger\]
    we know \(\Omega = \omega \times 1\) with standard masses \(\Rightarrow\) vacuum manifold is \(SU(N_V + N|N_V)\)
  - Symmetry breaking is \(G \to H = SU(N_V + N|N_V)_V\)
- Can derive Goldstone’s theorem using Ward identities for two-point Euclidean correlators
  - \((N + 2N_V)^2 - 1\) Goldstone “particles” created by operators \(\bar{Q}\gamma_\mu\gamma_5 T^a Q\)
    with \(T^a\) a traceless generator of \(SU(N_V + N|N_V)\)
Moving to EFT

- In QCD, proceed as follows:
  - having established GB poles in two-point functions, we know that they will also be present in higher-order correlation functions, and in cuts
  - $\chi$PT reproduces this behavior, while incorporating the chiral Ward identities, and yielding physical S-matrix

- In PQQCD, situation is worse:
  - Have GB poles in two-point functions
  - Have Ward identities between correlation functions
  - No Hamiltonian so cannot show that same poles appear in higher-order correlation functions, or in cuts (no complete sets of states)
  - In fact, can show that there are double poles (but no higher) in neutral correlators [Sharpe & Shoresh]
  - Cannot rely on Weinberg’s argument to determine EFT since no S-matrix
  - Only “anchor” is fact that know EFT for QCD-like subspace

- For PQQCD must simply assume minimal change from QCD: assume that have local $\mathcal{L}_{\text{eff}}$, constrained by symmetries
  - Saturates Ward identities and reproduces double poles

S. Sharpe, “$\chi$PT for LQCD (III)”, Nara, 11/7/2005 – p.21/46
Constructing $\mathcal{L}_{PQ}$: choice of $\Sigma$

- Follow method used for QCD:

$$\Omega/\omega \to \Sigma(x) \in SU(N_V + N|N) , \quad \Sigma \rightarrow U_L \Sigma U_R^\dagger$$

- For standard masses, $\langle \Sigma \rangle = 1$, so define Goldstones by

$$\Sigma = \exp \left[ \frac{2i}{f} \Phi(x) \right] , \quad \Phi(x) = \begin{pmatrix} \phi(x) & \eta_1(x) \\ \eta_2(x) & \tilde{\phi}(x) \end{pmatrix}$$

\[ \text{sdet} \Sigma = 1 \Rightarrow \text{str} \Phi = \text{tr} \phi - \text{tr} \tilde{\phi} = 0 \]

- QCD GBs contained in $\Phi$

$$\Phi(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \pi(x) & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \Sigma_{QCD} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Building blocks for PQ$\chi$PT as for $\chi$PT, e.g.

$$L_\mu = \Sigma D_\mu \Sigma^\dagger \rightarrow U_L L_\mu U_L^\dagger , \quad \text{str}(L_\mu) = 0$$

- Power counting as in $\chi$PT

S. Sharpe, “$\chi$PT for LQCD (III)”, Nara, 11/7/2005 – p.22/46
PQ chiral Lagrangian  [Bernard & Golterman]

\[ L^{(2)} = \frac{f^2}{4} \text{str} \left( D_\mu \Sigma D_\mu \Sigma^\dagger \right) - \frac{f^2}{4} \text{str}(\chi \Sigma^\dagger + \Sigma \chi^\dagger) \]

\[ L^{(4)} = -L_1 \text{str}(D_\mu \Sigma D_\mu \Sigma^\dagger)^2 - L_2 \text{str}(D_\mu \Sigma D_\nu \Sigma^\dagger)\text{tr}(D_\mu \Sigma D_\nu \Sigma^\dagger) \]

\[ + L_3 \text{str}(D_\mu \Sigma D_\mu \Sigma^\dagger D_\nu \Sigma D_\nu \Sigma^\dagger) \]

\[ + L_4 \text{str}(D_\mu \Sigma^\dagger D_\mu \Sigma)\text{str}(\chi^\dagger \Sigma + \Sigma^\dagger \chi) + L_5 \text{str}(D_\mu \Sigma^\dagger D_\mu \Sigma)[\chi^\dagger \Sigma + \Sigma^\dagger \chi] \]

\[ - L_6 [\text{str}(\chi^\dagger \Sigma + \Sigma^\dagger \chi)]^2 - L_7 [\text{str}(\chi^\dagger \Sigma - \Sigma^\dagger \chi)]^2 - L_8 \text{str}(\chi^\dagger \Sigma \chi^\dagger \Sigma + p.c.) \]

\[ + L_9 i\text{str}(L_{\mu\nu} D_\mu \Sigma D_\nu \Sigma^\dagger + p.c.) + L_{10} \text{str}(L_{\mu\nu} \Sigma R_{\mu\nu} \Sigma^\dagger) \]

\[ + H_1 \text{str}(L_{\mu\nu} L_{\mu\nu} + p.c.) + H_2 \text{str}(\chi^\dagger \chi) + WZW_{PQ} \]

\[ + L_{PQ} \mathcal{O}_{PQ} \]

- \( \chi = 2B_0 \mathcal{M} \)
- Same form as for QCD with \( \text{tr} \rightarrow \text{str} \) plus one extra term (\( \mathcal{O}_{PQ} \))
- How do the LECs related to those of QCD?
Relating PQ\chiPT to \chiPT

- If choose \Sigma to lie in QCD subspace

\[ \Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \Sigma_{\text{QCD}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

and sources do not connect subspaces, then

\[ \mathcal{L}_{\text{PQ}\chiPT}^{(2,4,...)}(\Sigma) \rightarrow \mathcal{L}_{\chiPT}^{(2,4,...)}(\Sigma_{\text{QCD}}) \]

- If external fields in correlation function are from sea sector, then can show that all valence and ghost contributions cancel in intermediate states
  \[ \Rightarrow \Sigma \text{ takes the form given above} \]
  \[ \Rightarrow \text{PQ\chiPT calculation collapses to one in } \chiPT \]

- Thus LECs in PQ\chiPT are equal to those in \chiPT
  \[ \Rightarrow \text{Results in the chiral regime from PQQCD give information about physical LECs} \]
What about $\mathcal{O}_{\text{PQ}}$?

- Starting at NLO, at each order there are an increasing number of PQ operators that vanish on QCD subspace
- At NLO, only one such operator [Sharpe & Van de Water]

$$
\mathcal{O}_{\text{PQ}} = \text{str} \left( D_\mu \Sigma D_\nu \Sigma^\dagger D_\mu \Sigma D_\nu \Sigma^\dagger \right) \\
- \frac{1}{2} \text{str} \left( D_\mu \Sigma D_\mu \Sigma^\dagger \right)^2 - \text{str} \left( D_\mu \Sigma D_\nu \Sigma^\dagger \right) \text{str} \left( D_\mu \Sigma D_\nu \Sigma^\dagger \right) \\
+ 2 \text{str} \left( D_\mu \Sigma D_\nu \Sigma^\dagger D_\mu \Sigma D_\nu \Sigma^\dagger \right)
$$

- Vanishes if $\Sigma \to \Sigma_{\text{QCD}}$ due to Cayley-Hamilton relations for $3 \times 3$ matrices
- Does not vanish for general $\Sigma_{\text{PQ}}$
- Appears in $\mathcal{L}_{\text{PQX}}^{(4)}$ with additional LEC
- Same is true for standard $\chi$PT if $N \geq 4$
- $\mathcal{O}_{\text{PQ}}$ contributes to $\pi \pi$ scattering at NLO, but to $m_\pi$ and $f_\pi$ only at NNLO
Why is $\mathcal{O}_{PQ}$ present?

- Because PQQCD allows isolation of individual Wick contractions, unlike QCD.
- For example, $\pi^+ K^0$ scattering in QCD has two contractions:

\[
\begin{align*}
\text{Contractions in QCD:} & \quad + \\
\end{align*}
\]

- Can separate these contractions in PQQCD, e.g.

\[
\begin{align*}
\text{Contractions in PQQCD:} & \quad + \\
\end{align*}
\]

- $\mathcal{O}_{PQ}$ contributes to the PQQCD process, but not that in QCD.
- Shows how PQQCD differs from QCD even if $m_V = m_S$. 
Calculating in PQ\(\chi PT\)

- PQ Lagrangian at LO:

\[
\mathcal{L}^{(2)} = \frac{f^2}{4} \text{str} \left(D_\mu \Sigma D_\mu \Sigma^\dagger\right) - \frac{f^2}{4} \text{str}(\chi \Sigma^\dagger + \Sigma \chi^\dagger)
\]

- Insert expansion in Goldstone fields:

\[
\Sigma = \exp \left[\frac{2i}{f} \Phi(x)\right], \quad \Phi(x) = \begin{pmatrix} \phi(x) & \eta_1(x) \\ \eta_2(x) & \tilde{\phi}(x) \end{pmatrix}, \quad \text{str}\Phi = 0
\]

\[
\mathcal{L}^{(2)} = \text{str}(\partial_\mu \Phi \partial_\mu \Phi) + \text{str}(\chi \Phi^2) + \ldots
\]

\[
= \text{tr}(\partial_\mu \phi \partial_\mu \phi + \partial_\mu \eta_1 \partial_\mu \eta_2 - \partial_\mu \eta_2 \partial_\mu \eta_1 - \partial_\mu \tilde{\phi} \partial_\mu \tilde{\phi})
\]

\[
+ \text{tr} \left[ (\phi^2 + \eta_1 \eta_2) \begin{pmatrix} m_V & 0 \\ 0 & m_S \end{pmatrix} \right] - \text{tr}(\tilde{\phi}^2 m_V) - \text{tr}(\eta_2 \eta_1 m_V)
\]

- \(\phi\) part is like in QCD, except includes both valence and sea quarks

- Propagator for “charged” meson \(\bar{q}_1 q_2\) (either valence or sea) is

\[
\frac{1}{(p^2 + m_{12}^2)} , \quad m_{12}^2 = (\chi_1 + \chi_2)/2
\]
LO calculation (cont.)

\[ \mathcal{L}^{(2)} = \text{tr}(\partial_\mu \phi \partial_\mu \phi + \partial_\mu \eta_1 \partial_\mu \eta_2 - \partial_\mu \eta_2 \partial_\mu \eta_1 - \partial_\mu \tilde{\phi} \partial_\mu \tilde{\phi}) + \text{tr} \left[ (\phi^2 + \eta_1 \eta_2) \begin{pmatrix} m_V & 0 \\ 0 & m_S \end{pmatrix} \right] - \text{tr}(\tilde{\phi}^2 m_V) - \text{tr}(\eta_2 \eta_1 m_V) \]

- \tilde{\phi} terms have wrong signs
  - Naively, propagator for “charged” ghost mesons \( \bar{q}_1 q_2 \) is \(-1/(p^2 + m_{12}^2)\), \( m_{12}^2 = (\chi_1 + \chi_2)/2 \)
  - But potential not minimized and functional integral not convergent!
  - More careful treatment of symmetries of PQQCD, maintaining convergence of ghost functional integral, concludes that naive result is OK in perturbation theory (but not non-perturbatively, e.g. in \( \epsilon \)-regime, where should change \( \tilde{\phi} \rightarrow i\tilde{\phi}, \Sigma^\dagger \rightarrow \Sigma^{-1} \) [Sharpe & Shoresh]

- Goldstone fermion propagators can have either sign (no convergence problems); actual signs important for cancellations
What about $\Phi_0$?

- How implement $\text{str}(\Phi) = \text{tr}(\phi) - \text{tr}(\tilde{\phi}) = 0$?
  
  1. Use a basis of generators which is straceless:
     
     $\Phi = \sum_a \Phi_a T^a$ with $\text{str}(T^a) = 0$
     
     ▶ Analagous to not including the $\eta'$ in QCD $\chi$PT
     
     ▶ Clumsy in practice and not used
  
  2. Include identity component but then “integrate out”
     
     $\Phi \rightarrow \Phi + \Phi_0/\sqrt{N}$ so that $\text{str} \Phi = \sqrt{N} \Phi_0$
     
     $\mathcal{L}_{PQX} \rightarrow \mathcal{L}_{PQX} + m_0^2 \text{str}(\Phi)^2/N$
     
     ▶ Calculate propagators, then send $m_0^2 \rightarrow \infty$ within them
     
     ▶ To make formally correct, must regularize with a cut-off (e.g. lattice)
       so that $(\partial_\mu \Phi_0)^2 < m_0^2 \Phi_0^2$ (trivial decoupling)
     
     ▶ Really just a trick to implement stracelessness
     
     ▶ Method used in practice

- Introducing $\Phi_0$ has advantage of allowing use of “quark line” basis:
  
  $\Phi_{ij} \sim Q_i \overline{Q}_j$ for all $i, j$
Quark lines and double poles

- "Charged" particle propagators are simple:

\[ \langle \Phi_{ij} \Phi_{ji} \rangle = \pm \frac{1}{p^2 + (\chi_i + \chi_j)/2} = \]

- Neutral propagators have double poles:

\[ \mathcal{L}^{(2)} = \sum_{j=1}^{N+2N_V} \epsilon_j (\partial_\mu \Phi_{jj} \partial_\mu \Phi_{jj} + m_j \Phi_{jj}^2) + (m_0^2/N)(\sum_j \epsilon_j \Phi_{jj})^2 \]

\[ \epsilon_j = \begin{cases} +1 & \text{valence or sea quarks} \\ -1 & \text{ghosts} \end{cases} \]

- Can simply invert with linear algebra tricks. Schematically, for external valence quarks have "hairpin" sum:

\[ \text{Valence quarks: } \begin{array}{c} V \\ V \end{array} + \begin{array}{c} V \\ V \end{array} + \begin{array}{c} V \\ S \end{array} + \begin{array}{c} V \end{array} + \ldots \]
Neutral propagator

- Result after $m_0^2 \to \infty$ for $N = 3$ [Bernard & Golterman; Sharpe & Shoresh]

\[
\langle \Phi_{ii} \Phi_{jj} \rangle = \frac{\epsilon_i \delta_{ij}}{p^2 + \chi_i} - \frac{1}{N} \frac{1}{(p^2 + \chi_i)(p^2 + \chi_j)} \frac{(p^2 + \chi_{S1})(p^2 + \chi_{S2})(p^2 + \chi_{S3})}{(p^2 + M_{\eta_0}^2)(p^2 + M_\eta^2)}
\]

- Simplifies for degenerate sea quarks:

\[
\langle \Phi_{ii} \Phi_{jj} \rangle = \frac{\epsilon_i \delta_{ij}}{p^2 + \chi_i} - \frac{1}{N} \frac{1}{(p^2 + \chi_i)(p^2 + \chi_j)} (p^2 + \chi_S)
\]

  - Manifestly unphysical double pole for $\chi_i = \chi_j$
  - Residue is then $(\chi_i - \chi_S)/N$, so vanishes for physical subspace
  - Can show from symmetries of PQQCD that if charged propagators have single poles, then neutral have double (and no higher) poles [Sharpe & Shoresh]

- Propagator becomes physical if $i, j$ are sea quarks, e.g. for degenerate sea

\[
\langle \Phi_{SS} \Phi_{SS} \rangle = \frac{1}{p^2 + \chi_S} \left( 1 - \frac{1}{N} \right)
\]

  - Recover projection against $\eta'$

S. Sharpe, "\textit{χPT for LQCD (III)}", Nara, 11/7/2005 – p.31/46
Outline of Lecture 3

- Partial quenching and PQ\chi PT
  - What is partial quenching?
  - Developing PQ\chi PT
  - Results and outlook
- Application to staggered fermions
Sample calculation: $m^2_\pi$

- Calculations are straightforward extension of standard $\chi$PT
- Mass-squared of “pion” composed of valence quarks $V_1, V_2$
- Quark-line diagrams for 1-loop contributions

$\chi$PT

- LO four-pion vertices have single strace, so are "connected"
- Manifest cancellation between contributions from commuting and anticommuting particles
NLO result for $m_{\pi}^2$

To simplify expression for loop contributions, assume $N$ degenerate sea quarks and $m_{V1} = m_{V2} \neq m_S$

$$m_{VV}^2 = \chi_V \left( 1 + \frac{1}{N} \frac{2\chi_V - \chi_S}{\Lambda^2_x} \ln(\chi_V/\mu^2) + \frac{\chi_V - \chi_S}{N\Lambda^2_x} \right)$$

- Reduces to QCD-like result when $\chi_V \rightarrow \chi_S$
- $\chi_V$ and $\chi_S$ provide separate dials for determining $2L_8 - L_5$ and $2L_6 - L_4$
- Result in PQ mass-plane depends on physical LECs
- Unphysical nature of result clear from divergence in $\chi_S \ln \chi_V$ as $\chi_V \rightarrow 0$
- In practice, expansion breaks down only for very small $\chi_V$

Has been used to determine $2L_8 - L_5$ which, using continuum $\chi$PT, constrains physical $m_u$

- Cannot determine using continuum $\chi$PT alone [Kaplan & Manohar]
- Need to use $N = 3$ or $N = 2 + 1$ sea quarks (not $N = 2$)
- Lattice results $\Rightarrow m_u \neq 0$
Status of PQ\chiPT calculations

- It is now standard to extend any \chiPT calculation to PQ\chiPT
  - Many quantities considered at NLO: pions, baryons, vector mesons, scalar mesons, heavy-light hadrons, weak matrix elements ($B_K$, $K \rightarrow \pi\pi$), NEDM, pion scattering, ...
  - First calculations at NNLO for pion properties
  - PQ effects also included in tm\chiPT, staggered \chiPT and mixed action \chiPT
  - Most non-trivial example is baryons, where need to use a set-up in which all three quark lines are explicit
  - Most striking result is for scalar meson correlators, where hairpin propagators lead to unphysical negative contributions at long distances

- In general, can use PQ\chiPT to determine form of expected results for individual contractions (e.g. connected and disconnected contributions to $\pi_0$ propagators in tmLQCD)
- Most extensive practical use is in MILC improved staggered simulations
- Potentially a powerful practical tool, but important to test given incomplete theoretical justification
A final fun example: $L_7$

$$\mathcal{L}^{(4)}_\chi = \cdots - L_7 \text{str} (\chi \Sigma^\dagger - \Sigma \chi^\dagger)^2 + \cdots$$

- Contributes to PGB masses only for non-degenerate quarks
- In QCD, only significant contribution is to $m_\eta$

$$4m_K^2 - 2m_\pi^2 - 3m_\eta^2 = \frac{32(m_K^2 - m_\pi^2)^2}{3f^2}(L_5 - 6L_8 - 12L_7) + \text{chiral logs}$$

- Direct lattice calculation of $m_\eta$ possible but challenging
- Can we determine $L_7$ and thus $m_\eta$ indirectly using PQQCD?

- **Yes, from residue of PQ double pole** [Sharpe & Shoresh]

$$\frac{\int d^3x \langle \Phi_{V_1,V_1}(t, x) \Phi_{V_2,V_2}(0) \rangle}{\int d^3x \langle \Phi_{V_1,V_2}(t, x) \Phi_{V_2,V_1}(0) \rangle_{m_{V_1}=m_{V_2}}} \rightarrow \frac{Dt}{2M_{VV}}$$

- With $N = 3$ degenerate sea quarks find:

$$D = \frac{\chi_V - \chi_S}{N} - \frac{16}{f^2} \left(L_7 + \frac{L_5}{2N}\right) (\chi_V - \chi_S)^2 + \text{known chiral logs}$$

- PQ simulations allow use of multiple $\chi_V \Rightarrow \text{better signal}$?
Outline of Lecture 3

- Partial quenching and PQ\chi PT
  - What is partial quenching?
  - Developing PQ\chi PT
  - Results and outlook
- Application to staggered fermions
References for staggered fermions

References for staggered $\chi$PT

References for staggered $\chi$PT (cont.)

Summary on SχPT

- Need to use PQχPT + add in $a^2$ terms + account for 4-th root by factors of 1/4 determined using quark line diagrams

- Dominant effect is to chiral logarithms:
  - pions in loops have $a^2$ corrections to their masses,
  - most do not become massless in the chiral limit
  - reduces the curvature due to the chiral logs
  - essential to fit the MILC pion, kaon and heavy-light data

- For operators, also have extra mixing at $O(a^2)$

- Staggered complications ⇒ mixed action (overlap/DW valence and staggered sea)?
Taste Symmetry Breaking

- Staggered quarks come in 4 tastes ⇒ staggered mesons come in 16 tastes
- Labeled by the taste matrix in the lattice operator: \( \pi_T \equiv \bar{Q}_i (\gamma_5 \otimes \xi_T) Q_j \)

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<tbody>
<tr>
<td>1 Singlet</td>
<td>( \xi_1 )</td>
</tr>
<tr>
<td>1 Goldstone</td>
<td>( \xi_5 )</td>
</tr>
<tr>
<td>4 Vector</td>
<td>( \xi_\mu )</td>
</tr>
<tr>
<td>4 Axial</td>
<td>( \xi_{\mu 5} )</td>
</tr>
<tr>
<td>6 Tensor</td>
<td>( \xi_{\mu \nu} )</td>
</tr>
</tbody>
</table>

- On the lattice, quarks of one taste can turn into another by exchanging high-momentum gluons
- Breaks the continuum \( SU(4) \) taste symmetry at \( O(a^2) \)
- Discretization errors numerically significant at present lattice spacings (MILC)
- Fits of staggered lattice data show that must account for taste violations in continuum and chiral extrapolations ⇒ staggered chiral perturbation theory (S\(\chi\)PT)

S. Sharpe, “\(\chi\)PT for LQCD (III)”, Nara, 11/7/2005 – p.42/46
SχPT Power-Counting

- Because the Symanzik action contains explicit powers of the lattice spacing, must incorporate $a$ into the SχPT power-counting scheme.

- Taste breaking discretization errors are $\propto a^2 \alpha_V^2 (q^* = \pi/a)$
  - $a^2$ because come from dimension 6 operators in the effective action
  - $\alpha^2$ because require two gluon exchange

- We observe in lattice simulations that the lattice Goldstone PGB mass is of the same size as the splittings among the other tastes.
  - $\Rightarrow a^2 \alpha_V^2 \sim m_{PGB}^2$
  - Define $a_\alpha^2 \equiv a^2 \alpha_V^2 (q^* = \pi/a)$ for later use

- Therefore use the following (standard) SχPT power-counting scheme:
  - $p_{PGB}^2 \sim m_q \sim a_\alpha^2$

★ Note that this scheme is phenomenologically based on the parameters of current staggered simulations ★
Step 3: The Staggered Chiral Lagrangian

- The lowest-order, $\mathcal{O}(p_{\text{PGB}}^2, m_q, a_\alpha^2)$, staggered chiral Lagrangian is:

$$\mathcal{L}_{S\chi\text{PT}} = \frac{f^2}{8} \text{Tr} (\partial_\mu \Sigma \partial_\mu \Sigma^\dagger) - \frac{1}{4} \mu f^2 \text{Tr} (\mathcal{M} \Sigma + \mathcal{M} \Sigma^\dagger) + a_\alpha^2 \mathcal{V}$$

- Kinetic energy and mass terms just like the LO continuum chiral Lagrangian – but with more tastes

- **Staggered potential**, $\mathcal{V}$, comes from dimension 6 operators in the Symanzik action
  
  ▶ Contains lattice effects
  ▶ Determine $\mathcal{V}$ by promoting taste matrices to spurion fields – just like quark mass matrix

- Staggered potential leads to important PGB properties, one of which is that it *splits the tree-level PGB masses into degenerate groups*:

$$\left( m_\pi^2 \right)_{\text{LO}} = 2\mu \frac{m_i + m_j}{2} + a_\alpha^2 \Delta_F$$

▶ $\Delta_F$ different for each $SO(4)$-taste irrep: $\xi_5, \xi_{\mu 5}, \xi_{\mu \nu}, \xi_\mu, \xi_I$

▶ *Recall that we observe this splitting in the lattice data!*

- Also produces interaction vertices at higher-orders in $S\chi\text{PT}
$B_K$ at NLO – Explicit Example

- Show for *degenerate valence quarks*
  - Simplest example – only 5 linear combinations of operators contribute
  - Sum over PGB tastes ($B'$) in logs generic trait of $S\chi$PT expressions

$$
B_K = B_0 \left\{ 1 + \frac{1}{1024\pi^2 f_{K_P}^2} \sum_{B'} f^{B'} m_{K_{B'}}^2 \left[ 3 \log \left( \frac{m_{K_{B'}}^2}{\Lambda_{\chi}^2} \right) + 1 \right] \right\} + A \\
+ B \left[ \frac{3 m_{K_P}^2}{16 f_{K_P}^2} + D \frac{3(m_u + m_d + m_s)}{16 f_{K_P}^2} \right] \\
+ \frac{3}{512\pi^2 f_{K_P}^2} \sum_{B'} \left[ \log \left( \frac{m_{K_{B'}}^2}{\Lambda_{\chi}^2} \right) + 1 \right] \sum_B \left( \frac{C_{\chi}^{1B}}{f^4} g^{BB'} - \frac{C_{\chi}^{2B}}{f^4} h^{BB'} \right)
$$

- **Additive** “corrections” from perturbative and discretization errors can in principle be determined and removed separately at each lattice spacing
- Removal of **Multiplicative** “corrections” from perturbative and discretization errors requires fit to multiple lattice spacings

S. Sharpe, “$\chi$PT for LQCD (III)”, Nara, 11/7/2005 – p.45/46
References for mixed action $\chi$PT