Chiral Lattice Gauge Theories
Constructions and Applications

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Outline

Introduction
- Chiral fermions and Chiral gauge theories
- Ginsparg-Wilson relation and Weyl fermions on the lattice
- Chiral lattice gauge theories with exact gauge invariance

Issues in the construction of chiral lattice gauge theories
- GW relation and Exact chiral symmetry on the lattice
- Path integral measure of Weyl fermions
- Requirements for the measure

U(1) chiral lattice gauge theories
- "Constructive" proof of the existence of the lattice theories
- Explicit construction of U(1) chiral lattice gauge theories

SU(2)×U(1) lattice Electroweak theory
- SU(2)×U(1) Electroweak theory on the lattice
- Construction of $\mathcal{L}_\eta$ in SU(2)×U(1) EW theory

Practical Implementation of U(1) chiral gauge theories
- Cohomological analysis within $V = L^d$
- Numerical computations in 2 dim.
Chiral fermion

- Dirac fermion

\[ \mathcal{L}(x) = \bar{\psi}(x) \left\{ i \gamma^\mu \partial_\mu - m \right\} \psi(x) \]

Massless limit ⇒ Chiral symmetry

\[ \delta \mathcal{L}(x) = 0; \quad \delta \psi(x) = i \alpha \gamma_5 \psi(x), \quad \delta \bar{\psi}(x) = i \alpha \bar{\psi}(x) \gamma_5 \]

\[ D \equiv (i \gamma^\mu \partial_\mu); \quad \gamma_5 D + D \gamma_5 = 0 \]

- Weyl fermion (Chiral fermion)

\[ \gamma_5 \psi_L(x) = -\psi_L(x), \quad \gamma_5 \psi_R(x) = +\psi_R(x), \]

- minimal (two-dim.) Spinor rep. of Lorentz symmetry

- Unit of fermions in the standard model

\[ G = SU(3)_C \times SU(2)_L \times U(1)_Y \]

\[
\left( \begin{array}{c}
 u_{Li}(x) \\
 d_{Li}(x)
\end{array} \right)_{Y = -\frac{1}{6}} \quad u_{Ri}(x)_{Y = \frac{2}{3}} \quad \left( \begin{array}{c}
 \nu_L(x) \\
 e_L(x)
\end{array} \right)_{Y = -\frac{1}{2}} \quad [\nu_R(x)]_{Y = -1} \quad e_R(x)_{Y = -1}
\]
Chiral gauge theories

\[ \mathcal{L}(x) = -\frac{1}{2} \text{Tr} F_{\mu\nu}(x)^2 + \sum_r \bar{\psi}_L(x) [\gamma_\mu (\partial_\mu + igA_\mu)] \psi_L(x) \]

▷ Examples
  ▷ \( G = SU(3)_C \times SU(2)_L \times U(1)_Y \) (SM)
  ▷ \( G = SU(5), \ G = SO(10) \) (GUT models)
  ▷ \( G = SU(N)_{TC} \times SU(2)_L \) (Technicolor model)

▷ Gauge anomaly cancellation \( \sum_r \text{Tr}(T^a \{ T^b, T^c \}) = 0 \)

\[ i \Gamma_{\text{eff}}[A] = \text{Indet} \left( [\gamma_\mu (\partial_\mu + igA_\mu P_L)] \right) \]

\[ \delta_\omega \Gamma_{\text{eff}}[A] = \int_x \frac{i}{24\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{Tr} \omega \left\{ \partial_\mu A_\nu \partial_\rho A_\sigma + \frac{1}{2} \partial_\mu (A_\nu A_\rho A_\sigma) \right\} \]

▷ can not be removed by local counter terms
▷ cancellation in the standard model

\[ \sum_L Y^3 - \sum_R Y^3 = 0, \quad \sum_{\text{doublet}(L)} Y = 0, \quad \sum_{\text{singlet}(R)} Y = 0 \]
Dynamics in chiral gauge theories

- Fermion number non-conservation (Chiral anomaly)
  - Baryon number is not conserved due to anomaly in the SM
  - Sphaleron effect at $T \neq 0$, EW phase transition at $T = T_c$
  - $\Rightarrow$ Baryon Asymmetry of Universe

- Dynamical chiral / gauge symmetry breaking
  - Spontaneous breaking of chiral symmetry
    $SU(3) \times SU(2)_L \times U(1)_Y \rightarrow \text{SU}(5)$
    $\{ (\underline{3} \oplus \overline{3}^*) \times N_f \}$ (Vector-like, Confinement)
  - Spontaneous breaking of gauge symmetry
    $SU(N) \times SU(2)_L \rightarrow \text{SU}(5)$
    $\{ (\underline{N} \oplus \overline{N}^* \oplus \underline{1} \oplus \overline{1}) \times N \}$
    $\Lambda_N > \Lambda_2$ (QCD like, Weakly coupled $SU(2)_L$)
  - $SU(5) \times SU(2)_L$ (Confinement)

- Composite massless fermions
  - ’t Hooft anomaly matching condition
    $SU(5) \rightarrow \text{Confinement}$
Dynamical chiral / gauge symmetry breaking

- Spontaneous breaking of chiral symmetry
  \( SU(3) \); \( \{(3 \oplus 3^*) \times N_f \} \) Vector-like (Confinement)
  - Chiral symmetry: \( U(1)_A \) anomaly
    \( U(1)_V \) \( Q_3 = +1, Q_3^* = -1 \)
    \( SU(N_f)_L \times SU(N_f)_R \)
  - Asymptotic freedom \( \Rightarrow \) strong coupling at IR
  - Fermion condensate

\[ \langle \psi^i(x)\psi_i(x) \rangle \neq 0 \]

\( SU(N_f)_L \times SU(N_f)_R \rightarrow SU(N_f)_V \)
Nambu-Goldstone bosons (cf. \( \pi, K \))

- Spontaneous breaking of gauge symmetry
  \( SU(N) \times SU(2)_L \); \( \{(N, 2) \oplus (N^*, 1) \times 2, \ldots \} \)
  \( \Lambda_N > \Lambda_2 \) (QCD like, Weakly coupled \( SU(2)_L \))

\[ \langle \psi^{ia}(x)\psi_i(x) \rangle = \langle \phi^a(x) \rangle \neq 0 \Rightarrow SU(2)_L \text{ break down} \]
Dynamical chiral / gauge symmetry breaking [cont’d]

$SU(5) \{ 10 + 5^* \}$

Most Attractive Channel (MAC)

$$V(r) \approx \frac{g^2(\mu)}{r} T^a_{r_1} T^a_{r_2}$$

$$T^a_{r_1} T^a_{r_2} = \frac{1}{2} \left\{ (T^a_{r_1} \oplus T^a_{r_2})^2 - T^a_{r_1} T^a_{r_1} - T^a_{r_2} T^a_{r_2} \right\}$$

$$10 \otimes 10 \rightarrow 5^* \oplus 45 \oplus 50$$

$$\langle \epsilon_{ijklm} \psi^{[jk]}(x) \psi^{[lm]}(x) \rangle \neq 0 \Rightarrow SU(5) \text{ break down}$$
Massless composite fermions (Confinement)

\[ SU(5) ; \{ 10 \oplus 5^* \} \]

- conserved Fermion(“Baryon”) number:
  \[ Q_{10} = +1, \quad Q_{5^*} = -3 \]

\[ \langle J_\mu(q) J_\nu(k) J_\lambda(p) \rangle |_{k^2=p^2=0} \]

\[ \cong \frac{\sum_r Q_r^3 D_r}{4\pi} \delta(q^2)(\epsilon_{\mu\nu\alpha\beta} k_\lambda - \epsilon_{\mu\lambda\alpha\beta} p_\nu) k^\alpha p^\beta \]

(Zero Mass Physical Threshold)

\[ 10Q_{10}^3 + 5Q_{5^*}^3 = 125 = 5^3 \]

- “Baryon” candidates with \( Q = 5 \)
  \[ \epsilon^{ijklm} \psi_{[ij]} \psi_{[kl]} \chi_m, \quad \psi_{[ij]}^\dagger \chi_i^\dagger \chi_j^\dagger, \ldots \]

- NG bosons (Spontaneous breaking)
Fermion number non-conservation (Chiral anomaly)

- Baryon number is not conserved due to anomaly in the SM
- Sphaleron effect at $T \neq 0$, EW phase transition at $T = T_c$
- $\Rightarrow$ Baryon Asymmetry of Universe

Rate of Baryon number non-conservation at finite $T$

- $T < T_c$ (Broken phase)
  Sphaleron contribution in SU(2) × U(1) Electroweak theory
  \[ \Gamma \approx \frac{\omega_-}{\pi} \beta \text{Im} F \approx \frac{\omega_-}{\pi} \frac{\text{Im} Z_{\text{barrier}}}{Z_0} \]
  Fermion contribution? (Large) Yukawa coupling to Higgs

- $T > T_c$ (Symmetric phase)
  "Lattice" simulation by Bodeker et al., (PRD61, 056003)
GW relation and Weyl fermions on the lattice

- Ginsparg-Wilson relation: \( \gamma_5 D + D \gamma_5 = 2aD \gamma_5 D \)

\[
S = a^4 \sum_x \bar{\psi}(x) D \psi(x)
\]

\[
\delta_{\alpha} \psi(x) = i\alpha \gamma_5 (1 - 2aD) \psi(x), \quad \delta_{\alpha} \bar{\psi}(x) = i\alpha \bar{\psi}(x) \gamma_5
\]

- Chiral projections: \( \hat{\gamma}_5 \equiv \gamma_5 (1 - 2aD), \quad \{\hat{\gamma}_5\}^2 = 1 \)

\[
\left(\frac{1 - \hat{\gamma}_5}{2}\right) \psi_L(x) = \psi_L(x), \quad \bar{\psi}_L(x) \left(\frac{1 + \gamma_5}{2}\right) = \bar{\psi}_L(x)
\]

\[
a^4 \sum_x \bar{\psi}(x) D \psi(x) = a^4 \sum_x \{ \bar{\psi}_L(x) D \psi_L(x) + \bar{\psi}_R(x) D \psi_R(x) \}
\]

- Weyl fermion action

\[
S_w = a^4 \sum_x \bar{\psi}_L(x) D \psi_L(x)
\]
Chiral lattice gauge theories with exact gauge invariance

- Anomaly-free U(1) chiral gauge theories
  - A complete construction on the finite-volume lattice

- SU(2) × U(1) gauge theory of EW interaction
  - Local cohomology problem in 4+2 dim. is solved for $L = \infty$
    ⇒ Exact cancellations of the gauge anomalies
      U(1)$^3$, SU(2)$^2 \times U(1)$-mixed type
  - Measure term is constructed for $L < \infty$, $m_{\mu\nu} = 0$
  - Global integrability ?

- Non-abelian chiral gauge theories (SU(N), SO(10) etc.)
  - Non-perturbative construction is not obtained yet
  - In all orders in the weak coupling expansion
GW relation and Exact chiral symmetry on the lattice

- Ginsparg-Wilson relation: $\gamma_5 D + D \gamma_5 = 2aD \gamma_5 D$

  $$S = a^4 \sum_x \bar{\psi}(x) D \psi(x)$$

  $$\delta_\alpha \psi(x) = i\alpha \gamma_5 (1 - 2aD) \psi(x), \quad \delta_\alpha \bar{\psi}(x) = i\alpha \bar{\psi}(x) \gamma_5$$

- Gauge-covariant solution: overlap Dirac operator

  $$D = \frac{1}{2a} \left( 1 + \gamma_5 \frac{H_w}{\sqrt{H_w^2}} \right)$$

  $$H_w = \gamma_5 (D_w - m_0/a), \quad 0 < m_0 < 2$$

- Locality

  $$\| D(x, y) \| \leq \kappa (1 + \| x - y \|^{\nu}) e^{-\| x - y \|/\varrho} \quad \varrho/a \simeq O(1)$$

  $$\| 1 - U_{\Box} \| \leq \epsilon \quad \epsilon < \frac{1}{30}$$

  $$\epsilon < \frac{1}{30} \quad \therefore \quad (aD_w - 1)(aD_w - 1)^\dagger \geq 1 - 30\epsilon$$
Chiral anomaly

\[ \delta_\alpha \left[ \prod_x d\psi(x)d\bar{\psi}(x) \right] = \left[ \prod_x d\psi(x)d\bar{\psi}(x) \right] \times \sum_x \alpha(x)(-2) \text{tr} \{ \gamma_5 (1 - aD)(x, x) \} \]

Atiyah-Singer Index theorem on the lattice

\[ 2 \text{Tr} \{ \gamma_5 (1 - aD) \} = 2 \text{Index}(D) = 2(n_+ - n_-) \]

Eigenvalues of Dirac op.

\[ D + D^\dagger = 2aD^\dagger D = 2aDD^\dagger \text{ (normal)}, \quad D^\dagger = \gamma_5 D\gamma_5 \text{ (} \gamma_5 \text{-conjugate)} \]

\[ \lambda + \lambda^* - 2a\lambda^* \lambda = (-2a) \left[ (\lambda - 1/2a)(\lambda - 1/2a)^* - (1/2a)^2 \right] = 0 \]

∴ the circle with the center \((1/2a, 0)\) and the radius \(1/2a\)

\[ \lambda = 0 : \quad \gamma_5 \psi_\lambda(x) = \pm \psi_\lambda(x) \quad n_{\pm} \]

\[ \lambda = 1/a : \quad \gamma_5 \psi_\lambda(x) = \pm \psi_\lambda(x) \quad N_{\pm} \]

\[ \lambda \neq 0, 1/a : \quad \text{pair-wise} \quad \left\{ \begin{array}{l} \lambda \rightarrow \psi_\lambda \\ \lambda^* \rightarrow \gamma_5 \psi_\lambda \end{array} \right. \]
cf. \( D_E = \gamma_\mu D_\mu \)

\[ \lambda \]

\[ \frac{1}{a} \]

Figure: Eigenvalues of \( D \)
Weyl fermions on the lattice

- **Chiral projections**

\[ \hat{\gamma}_5 \equiv \gamma_5 (1 - 2aD_L), \quad \{\hat{\gamma}_5\}^2 = 1 \]

\[
\left( \frac{1 - \hat{\gamma}_5}{2} \right) \psi_-(x) = \psi_-(x), \quad \bar{\psi}_-(x) \left( \frac{1 + \gamma_5}{2} \right) = \bar{\psi}_-(x)
\]

\[
a^4 \sum_x \bar{\psi}(x) D\psi(x) = a^4 \sum_x \{ \bar{\psi}_-(x) D\psi_-(x) + \bar{\psi}_+(x) D\psi_+(x) \}
\]

- **Weyl fermion action**

\[
S_w = a^4 \sum_x \bar{\psi}_-(x) D\psi_-(x)
\]
Path-Integral Measure

- Chiral basis: \( \{ v_i(x) \} \) and \( \{ \bar{v}_k(x) \} \)

\[
\hat{P}_- v_i(x) = v_i(x), \quad \hat{P}_- = \left( \frac{1 - \hat{\gamma}_5}{2} \right)
\]

\[
\bar{v}_k(x) P_+ = \bar{v}_k(x), \quad P_+ = \left( \frac{1 + \gamma_5}{2} \right)
\]

- Mode expansion with grassman number coefficients

\[
\psi_-(x) = \sum_i v_i(x) c_i, \quad \bar{\psi}_-(x) = \sum_k \bar{c}_k \bar{v}_k(x)
\]

- The definition of the measure

\[
\mathcal{D}[\psi_-]\mathcal{D}[\bar{\psi}_-] \equiv \prod_i dc_i \prod_k d\bar{c}_k [U_\mu(x)]
\]
Chiral determinant (Partition function)

\[ Z_{\text{Weyl}} = \int \prod_i dc_i \prod_k d\bar{c}_k \ e^{-\bar{c}_k (a^4 \sum_x \bar{\nu}_k(x)Dv_i(x))c_i} = \det \{M_{ki}\} \]

\[ M_{ki} = a^4 \sum_x \bar{\nu}_k(x)Dv_i(x) \]

Fermion zero modes / Index theorem on the lattice

- \( i = 1, \cdots, N_- \), \( k = 1, \cdots, N_0 \) \( (N_0 = 2N_cL^4) \)
- \( \text{Index}D = 2(N_0 - N_-) \)
- \( N_- \) (the shape of the matrix \( M_{ki} \)) is a dynamical variable
- Fermion zero modes, Fermion number nonconservation
change of the chiral basis by a unitary transformation

\[ \tilde{v}_i(x) = v_j(x) \left( \tilde{Q}^{-1} \right)_{ji} \quad \tilde{c}_i = \tilde{Q}_{ij} c_j \]

\[ \mathcal{D}[\psi_-] \mathcal{D}[\bar{\psi}_-] \implies \mathcal{D}[\psi_-] \mathcal{D}[\bar{\psi}_-] \det \tilde{Q} \left[ U_\mu(x) \right] \]

cf. in Lattice QCD

\[ \mathcal{D}[\psi] \mathcal{D}[\bar{\psi}] = \prod_x d\psi(x) d\bar{\psi}(x) \]

Requirements for the measure

- Locality
- Gauge-invariance
- Integrability
- Lattice symmetries
Gauge-field dependence of Path-Integral Measure

- Effective action
  \[ \exp(\Gamma_{\text{eff}}) = \int \prod_i dc_i \prod_k d\bar{c}_k \ e^{-\bar{c}_k (a^4 \sum_x \bar{v}_k(x)Dv_i(x))c_i} = \det \{M_{ki}\} \]
  \[ M_{ki} = a^4 \sum_x \bar{v}_k(x)Dv_i(x) \]

- Variation of the effective action w.r.t. gauge field
  \[ \delta U(x, \mu) = i\eta_\mu(x)U(x, \mu) \quad \eta_\mu(x) = \eta_{\mu}^a(x)T^a \]
  \[ \delta_\eta \Gamma_{\text{eff}} = \delta_\eta \text{Tr} \ln(\bar{v}_k, Dv_i) \]
  \[ = \text{Tr} \left\{ (\delta_\eta D)\hat{P}D^{-1}P_+ \right\} + \sum_i (v_i, \delta_\eta v_i) \]

- the measure term
  \[ \mathcal{L}_\eta = i \sum_i (v_i, \delta v_i) = a^4 \sum_x \eta_\mu(x) \cdot j_\mu(x) \]
Requirements for $\mathcal{L}_\eta$ : [Locality]

- Field equations (Schwinger-Dyson equation)

$$ \delta S_G[U(x, \mu)] + \left\langle a^4 \sum_x \bar{\psi}_-(x) \delta D\psi_-(x) \right\rangle_F + ia^4 \sum_x \eta_\mu(x) \cdot j_\mu(x) = 0 $$

- $j_\mu(x)$ should be a local function of $U(x, \mu)$!
Requirements for $\mathcal{L}_\eta$ : [Gauge invariance]

- **Gauge anomaly cancellation**: $\eta_\mu = -i \nabla_\mu \omega$, $\delta_\eta D = i[\omega, D]$

  $$\delta_\eta \Gamma_{\text{eff}} = i \text{Tr} \omega \gamma_5 (1 - aD) - i \sum_x \omega \cdot \nabla^* j_\mu = 0$$

- $j_\mu^a(x)$ should satisfies the anomalous conservation law:

  $$\{\nabla^* j_\mu\}^a(x) = \text{tr}\{ T^a \gamma_5 (1 - aD)(x, x) \}$$

- This requires the perturbative anomaly cancellation condition:

  $$\sum_R d^{abc} = \sum_R 2i\text{tr}\{ T^a[T^b T^c + T^c T^b] \} = 0$$

  $$\text{tr}\{ T^a \gamma_5 (1 - aD)(x, x) \} \xrightarrow{a \to 0} \frac{-1}{128\pi^2} d^{abc} \epsilon_{\mu\nu\rho\sigma} F^{b}_{\mu\nu}(x) F^{c}_{\rho\sigma}(x) + O(a)$$

  \[\therefore\] $j_\mu^a(x)$ should be local \[\therefore\] $j_\mu(x) = O(a)$
Requirements for $\mathcal{L}_\eta$ : [Integrability]

- Local integrability condition (for contractible loops)

$$
\delta_\eta \mathcal{L}_\zeta - \delta_\zeta \mathcal{L}_\eta + a \mathcal{L}_{[\eta,\zeta]} = i \text{Tr} \left\{ \hat{P}_- [\delta_\eta \hat{P}_-, \delta_\zeta \hat{P}_-] \right\}
$$

- For a measure smooth on the space of gauge fields

$$
\nu_i |_{t} = Q_t \sum_{l} \nu_l |_{t=0} (S^{-1})_{li}
$$

- $Q_t$ : the evolution op. of the projection op. $\hat{P}_-$

$$
P_t = Q_t P_0 Q_t^{-1}, \quad P_t = \hat{P}_- |_{U=U_t} \equiv \left( \frac{1 - \gamma_5}{2} \right) |_{U=U_t}
$$

$$
\partial_t Q_t = [\partial_t P_t, P_t] Q_t, \quad Q_0 = 1
$$

- $S_{ij}$ : Unitary transformation of the basis vectors $\{ \nu_i \}$

$$
S |_{t=0} = 1
$$
Requirements for $\mathcal{L}_\eta$ : [Integrability] (cont’ d)

- the measure term

\[
\mathcal{L}_\eta = i \sum_{l} (v_{l}|_{t=0}, Q^{-1}_{t} \partial_{t} Q_{t} v_{l}|_{t=0}) + i \sum_{i,l} S_{il} \partial_{t} (S^{-1})_{li}
\]

\[= -i \partial_{t} \ln \det S\]

- For any closed loop

\[
W \equiv \exp \left\{ i \int_{0}^{1} dt \mathcal{L}_\eta \right\} = \det S|_{t=1}
\]

\[
Q^{-1}_{1} v_{i}|_{t=1} = \sum_{l} v_{l}|_{t=0} (S^{-1})_{li}
\]

\[\therefore \quad W = \det \{ 1 - P_{0} + P_{0} Q_{1} \} \]
Geometrical interpretation of the integrability
under change of the chiral basis by a unitary transformation,

\[ \tilde{v}_i(x) = v_j(x) \left( \tilde{Q}^{-1} \right)_{ji} \quad \tilde{c}_i = \tilde{Q}_{ij} c_j, \]

\[ \mathcal{D}[\psi_-] \mathcal{D}[\bar{\psi}_-] \implies \mathcal{D}[\psi_-] \mathcal{D}[\bar{\psi}_-] \det \tilde{Q} [U_\mu(x)] \]

This defines a U(1) bundle over the space of gauge fields

- **connection (vector field)**
  \[ \mathcal{L}_\eta \implies \mathcal{L}_\eta - ig^{-1} \delta_\eta g, \quad g = \det \tilde{Q} \]

- **curvature (field tensor)**
  \[ i \text{Tr} \left\{ \hat{P}_- [\delta_\eta \hat{P}_-, \delta_\zeta \hat{P}_-] \right\} = \delta_\eta \mathcal{L}_\zeta - \delta_\zeta \mathcal{L}_\eta + a \mathcal{L}_{[\eta, \zeta]} \]

- **Wilson lines**
  \[ \det \{ 1 - P_0 + P_0 Q_1 \} = \exp \left\{ i \int_0^1 dt \mathcal{L}_\eta \right\} \]
Summary of the requirements for $\mathcal{L}_\eta$

$$\mathcal{L}_\eta = i \sum_i (v_i, \delta v_i) = a^4 \sum_x \eta^a_\mu(x) j^a_\mu(x)$$

- Local and smooth function of link variables $U_\mu(x)$, which transforms as an axial vector current
- Anomalous conservation law
  $$\{ \nabla^*_\mu j_\mu \}^a(x) = \text{tr} \{ T^a \gamma_5 (1 - aD)(x, x) \}$$
- Integrability condition
  For any local variations of gauge fields, $\eta$ and $\zeta$ :
  $$\delta_\eta \mathcal{L}_\zeta - \delta_\zeta \mathcal{L}_\eta + a\mathcal{L}_{[\eta, \zeta]} = i \text{Tr} \left\{ \hat{P}_- [\delta_\eta \hat{P}_-, \delta_\zeta \hat{P}_-] \right\}$$
  For any closed loop in the space of gauge fields :
  $$W = \det \{ 1 - P_0 + P_0 Q_1 \} , \quad W \equiv \exp \left\{ i \int_0^1 dt \mathcal{L}_\eta \right\}$$
Reconstruction of the measure from $\mathcal{L}_\eta$

- One-parameter interpolation (within a given topological sector)
  
  \[ U_t(x, \mu) \quad t \in [0, 1] \]

- Choice of basis vectors
  
  \[ v_i(x) = \begin{cases} 
  Q_1 v_i^0(x) W^{-1} & \text{if } i = 1 \\
  Q_1 v_i^0(x) & \text{otherwise} 
  \end{cases} \]

- $Q_t$: Evolution op.
  
  \[ P_t = Q_t P_0 Q_t^{-1}, \quad P_t = \hat{P}_L|_{U=U_t} \]

- $W$: "Local counter term"
  
  \[ W = \exp \left\{ i \int_0^1 dt \, \mathcal{L}_\eta \right\} \]

  \[ \mathcal{L}_\eta = a^4 \sum_x \eta^a_\mu(x) j^a_\mu(x), \quad i\eta_\mu(x) = \partial_t U_t(x, \mu) U_t(x, \mu)^{-1} \]
U(1) chiral lattice gauge theories with exact gauge invariance

- Anomaly-free U(1) chiral gauge theories

\[ \psi_{L,\alpha}(x) \quad (\alpha = 1, \ldots, N) \quad \sum_{\alpha=1}^{N} e_{\alpha}^{3} = 0 \]

- On a finite lattice of the size \( L \) with P.B.C.

\[ \Gamma_{4} = \{ x = (x_{0}, \ldots, x_{3}) \in \mathbb{Z}^{4} \mid 0 \leq x_{\mu} < L \} \]

- A complete construction on the finite-volume lattice

M. Lüscher,

- cohomological classification of chiral anomalies in 4D (local cohomology problem in the infinite lattice)
- proof of the exact cancellation of gauge anomaly
- proof of the global integrability
- cover all topologically non-trivial sectors
Statement of the result

In an anomaly-free U(1) chiral gauge theories, the *local* current $j_\mu(x)$ which satisfies the following properties exists in all topological sectors:

- $j_\mu(x)$ is defined for all admissible gauge fields and depends smoothly on the link variables
- $j_\mu(x)$ is gauge-invariant and transforms as an axial vector current under the lattice symmetries
- $j_\mu(x)$ satisfies the anomalous conservation law
- The linear functional $\mathfrak{L}_\eta = \sum_{x \in \Gamma_4} \eta_\mu(x)j_\mu(x)$ satisfies the local integrability condition

It follows from these properties that $j_\mu(x)$ also satisfies the global integrability condition provided

$$N_e \text{ is even for all odd } e$$

($N_e$ denotes the number of fermion flavors with $|e_\alpha| = e$)
▶ An explicit (simplified) construction
D. Kadoh, Y. Kikukawa and Y. Nakayama,
“Solving the local cohomology problem in U(1) chiral gauge
theories within a finite lattice”
JHEP 0412 (2004) 006, hep-lat/0309067
Explicit construction of $\mathcal{L}_\eta$

- One-parameter family of the admissible U(1) gauge fields

$$ U_t(x, \mu) = e^{it\tilde{A}_\mu(x)} U_{[w]}(x, \mu) V_{[m]}(x, \mu) \quad t \in [0, 1] $$

- Cohomologically trivial part of the anomaly

  For an anomaly-free multiples with $\sum_\alpha e^3_\alpha = 0$

$$ \sum_\alpha e_\alpha q^\alpha(x) = \partial^*_\mu k_\mu(x), \quad k_\mu(x) = \sum_\alpha e_\alpha \{ \tilde{k}_\mu(x) \} |_{U \to U^{e_\alpha}} $$

- Explicit formula of the measure term

$$ \mathcal{L}_\eta = i \int_0^1 dt \ Tr_L \left\{ \hat{P}_- [\partial_t \hat{P}_-, \delta_\eta \hat{P}_-] \right\} $$

$$ + \int_0^1 dt \sum_{x \in \Gamma_4} \left\{ \eta_{\mu}(x) \bar{k}_\mu(x) + \tilde{A}_\mu(x) \delta_\eta \bar{k}_\mu(x) \right\} + \bar{\mathcal{L}}_\eta |_{U=U_{[w]} V_{[m]}} $$
Admissible $U(1)$ gauge fields on the finite lattice

- Admissibility condition

\[ \| 1 - U_\Box \| \leq \epsilon \quad \epsilon < \frac{1}{30} \]

- Topological invariant of admissible $U(1)$ gauge fields

\[ m_{\mu \nu} = \frac{1}{2\pi} \sum_{s,t} F(x + s\hat{u} + t\hat{v}) \]

\[ V[m](x, \mu) = e^{-\frac{2\pi i}{L^2} [L\delta_{x\mu}, L^{-1} \sum_{\nu > \mu} m_{\mu \nu} x_\nu + \sum_{\nu < \mu} m_{\mu \nu} x_\nu]} \]

\[ \frac{1}{2\pi i} \ln \left\{ V[m]_\Box \right\} = m_{\mu \nu} \]
a unique parametrization of \( U(x, \mu) \)

\[
U(x, \mu) = e^{i A_T^\mu(x)} \land(x) U_{[w]}(x, \mu) \land(x + \hat{\mu})^{-1} V_{[m]}(x, \mu)
\]

Transverse vector potentials

\[
\partial^* \mu A_T^\mu(x) = 0, \quad \sum_{x \in \Gamma} A_T^\mu(x) = 0,
\]

\[
\partial_\mu A_T^\nu(x) - \partial_\nu A_T^\mu(x) + 2\pi m_{\mu\nu} / L^2 = F_{\mu\nu}(x)
\]

Wilson lines

\[
U_{[w]}(x, \mu) = \begin{cases} w_\mu & \text{if } x_\mu = 0 \text{ mod } L \\ 1 & \text{otherwise} \end{cases}
\]

Topology of the space of the admissible gauge fields

\[
\mathcal{M}[m] \cong U(1)^4 \times U(1)^{L^4} \times \mathcal{A}[m]
\]
Vector potential representation of admissible $U(1)$ gauge fields

- Vector potential representation of the link variables
  - the magnetic fluxes and the Wilson lines are subtracted
  - $\tilde{A}_\mu(x)$ is periodic and bounded
  - $\tilde{A}_\mu(x)$ is unique up to gauge transformation functions with the value of integer multiple of $2\pi$

\[
U_\mu(x) = e^{i\tilde{A}_\mu(x)} U_w(x, \mu) V_m(x, \mu)
\]

\[
F_{\mu\nu}(x) = \partial_\mu \tilde{A}_\nu(x) - \partial_\nu \tilde{A}_\mu(x) + \frac{2\pi m_{\mu\nu}}{L^2}
\]

\[
\begin{cases} 
|\tilde{A}_\mu(x)| \leq \pi (1 + 4 \| x \|) & (\| x \| \leq L/2) \\
|\tilde{A}_\mu(x)| \leq \pi (1 + 2L + 2(n - 1)L^2) & \text{(otherwise)}
\end{cases}
\]
Construction of $k_\mu(x)$: Local cohomology problem

- Topological properties of U(1) gauge anomaly

$$q(x) = \text{tr} \{ \gamma_5 (1 - aD)(x, x) \}$$

$$\sum_x \delta q(x) = 0 \iff \delta \gamma_5 \gamma_5 + \gamma_5 \delta \gamma_5 = 0$$

- For an anomaly-free multiples with $\sum_\alpha e_\alpha^3 = 0$,

$$\sum_\alpha e_\alpha q^\alpha(x) = \partial^*_\mu k_\mu(x), \quad k_\mu(x) = \sum_\alpha e_\alpha \{ \tilde{k}_\mu(x) \} |_{U \to U^{e_\alpha}}$$
Cohomological analysis (infinite lattice)

- Separation of the finite volume contribution

\[ D(x, y) = \sum_{n \in \mathbb{Z}^4} D_{\infty}(x, y + nL), \quad D(x, y) = D_{\infty}(x, y) + O(e^{-L/\rho}) \]

\[ q(x) = q_{\infty}(x) + \Delta q(x), \quad |\Delta q(x)| < cL^\sigma e^{-L/\rho} \]

- Vector potential representation of the link variables

\[ U_\mu(x) = e^{iA_\mu(x)}, \quad |A_\mu(x)| \leq \pi(1 + 4 \| x \|) \]

\[ F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \]
One-parameter family of admissible gauge fields

\[ U^t_\mu(x) = e^{itA_\mu(x)}, \quad t \in [0, 1] \]

Topological property of \( q_\infty(x) \)

\[ q_\infty(x) = \sum \limits_y j_\mu(x, y)A_\mu(y), \quad j_\mu(x, y) = \int_0^1 dt \left. \frac{\partial q(x)}{\partial A_\mu(y)} \right|_{A \to tA} \]

\[ \sum \limits_x j_\mu(x, y) = 0, \quad j_\mu(x, y) \partial_\mu^* = 0 \]
Poincaré lemma on the lattice (the infinite lattice)

**Lemma** Let $f$ be a $k$-form which satisfies

\[ d^* f = 0 \quad \text{and} \quad \sum_{x \in \Gamma_n} f(x) = 0 \quad \text{if} \quad k = 0 \]

Then there exist a form $g \in \Omega_{k+1}$ such that

\[ f = d^* g \]

Result in the infinite lattice

\[ q_\infty(x) = \frac{1}{32\pi^2} \varepsilon_{\mu\nu\lambda\rho} F_{\mu\nu}(x) F_{\lambda\rho}(x + \hat{\mu} + \hat{\nu}) + \partial^* k_{\infty\mu}(x) \]

Finite volume correction (Suzuki et al.)

\[ \Delta q(x) = q(x) - q_\infty(x) = \partial^* \Delta k_{\infty\mu}(x) \]

\[ |\Delta q(x)| < c L^\sigma e^{-L/\rho}, \quad \sum_{x \in \Gamma_4} \Delta q(x) = 0 \]
Measure term for the Wilson lines

- the Wilson lines

\[ U_{[w]}(x, \mu) = \begin{cases} \ w_{\mu} & \text{if } x_{\mu} = 0 \\ 1 & \text{otherwise} \end{cases} \]

\[ w_{\mu} = \exp(it_{\mu}), \quad t_{\mu} \in [0, 2\pi) \]

- the variational parameters in these directions

\[ \eta_{\mu}(x)^{(\nu)} = \frac{1}{i} \partial_{t_{\nu}} U_{[w]}(x, \mu) \cdot U_{[w]}(x, \mu)^{-1} \quad (\nu = 1, 2, 3, 4). \]

- properties of the curvature

\[ \mathcal{C}_{\mu\nu} \equiv i \text{Tr} \hat{P}_{-} [ \partial_{t_{\mu}} \hat{P}_{-}, \partial_{t_{\nu}} \hat{P}_{-} ] \bigg|_{U=U_{[w]} V_{[m]}} \]

In anomaly-free theories, \( \mathcal{C}_{\mu\nu} \) satisfies the followings:

\[ |\mathcal{C}_{\mu\nu}| \leq \kappa L^{\sigma} e^{-L/\rho}, \quad \int_{0}^{2\pi} dt_{\mu} \int_{0}^{2\pi} dt_{\nu} \mathcal{C}_{\mu\nu} = 0 \]
the measure term for the Wilson lines

\[ \mathcal{L}_{\eta(4)} = \frac{1}{2\pi} \int_0^{2\pi} dt_4 \int_0^{(t_1, t_2, t_3)} \{ dr_1 C_{14} + dr_2 C_{24} + dr_3 C_{34} \}, \]

\[ \mathcal{L}_{\eta(3)} = \int_0^{t_4} dr_4 C_{43} - \frac{t_4}{2\pi} \int_0^{2\pi} dr_4 C_{43} + \left[ \frac{1}{2\pi} \int_0^{2\pi} dr_3 \int_0^{(t_1, t_2)} \{ dr_1 C_{13} + dr_2 C_{23} \} \right]_{t_4=0}, \]

\[ \mathcal{L}_{\eta(2)} = \int_0^{t_4} dr_4 C_{42} - \frac{t_4}{2\pi} \int_0^{2\pi} dr_4 C_{42} \]
\[ + \left[ \int_0^{t_3} dr_3 C_{32} - \frac{t_3}{2\pi} \int_0^{2\pi} dr_3 C_{32} \right]_{t_4=0} + \left[ \frac{1}{2\pi} \int_0^{2\pi} dr_2 \int_0^{(t_1)} \{ dr_1 C_{12} \} \right]_{t_4=t_3=0}, \]

\[ \mathcal{L}_{\eta(1)} = \int_0^{t_4} dr_4 C_{41} - \frac{t_4}{2\pi} \int_0^{2\pi} dr_4 C_{41} \]
\[ + \left[ \int_0^{t_3} dr_3 C_{31} - \frac{t_3}{2\pi} \int_0^{2\pi} dr_3 C_{31} \right]_{t_4=0} + \left[ \int_0^{t_2} dr_2 C_{21} - \frac{t_2}{2\pi} \int_0^{2\pi} dr_2 C_{21} \right]_{t_4=t_3=0}. \]

- smooth, local and periodic w.r.t. the Wilson lines \( t_\mu \)
- gauge invariant
- integrability condition

\[ \delta_{\eta(\mu)} \mathcal{L}_{\eta(\nu)} - \delta_{\eta(\nu)} \mathcal{L}_{\eta(\mu)} = \mathcal{C}_{\mu\nu}, \quad (U = U[w] V[m]). \]
Lattice symmetries

$j_\mu(x), k_\mu(x)$ should transform as axial vector currents

- current $k_\mu(x)$

$$\bar{k}_\mu(x)|_U = \frac{1}{2^4 4!} \sum_{\Lambda \in O(4,\mathbb{Z})} \det \Lambda \Lambda_{\mu\nu} k_\nu(\Lambda x)|_U \to U^\Lambda.$$

- measure term for the Wilson lines $\mathcal{L}_{\eta}|U = U[w]V[m]$}

$$\bar{\mathcal{L}}_{\eta(\nu)}|U = U[w]V[m] = \frac{1}{2^4 4!} \sum_{\Lambda \in O(4,\mathbb{Z})} \det \Lambda \mathcal{L}_{\eta(\nu)}|U \to U^\Lambda, \eta \to \eta^\Lambda,$$
Proof(1): Gauge-invariance, Smoothness of $j_\mu(x)$

Under the gauge variation of the vector potential

$$\tilde{A}_\mu(x) \rightarrow \tilde{A}_\mu(x) + \partial_\mu \omega(x),$$

$$\delta_\omega \mathcal{L}_\eta = \int_0^1 dt \, \text{Tr}_L \left\{ \hat{P}_L[[\omega, \hat{P}_L], \delta_\eta \hat{P}_L] \right\} + \int_0^1 dt \sum_{x \in \Gamma_4} \left\{ \partial_\mu \omega(x) \delta_\eta \bar{k}_\mu(x) \right\}$$

$$= - \int_0^1 dt \, \text{Tr}_L \left\{ \omega \delta_\eta \hat{P}_L \right\} - \int_0^1 dt \sum_{x \in \Gamma_4} \left\{ \omega(x) \delta_\eta \partial^*_\mu \bar{k}_\mu(x) \right\}$$

$$= - \int_0^1 dt \sum_{x \in \Gamma_4} \omega(x) \delta_\eta \left\{ \partial^*_\mu \bar{k}_\mu(x) - \text{tr} \left\{ \gamma_5 (1 - aD)(x, x) \right\} \right\}$$

$$= 0$$

Obviously, $j_\mu(x)$ is smooth w.r.t. to the vector potential $\tilde{A}_\mu(x)$. Since $j_\mu(x)$ is gauge-invariant under the gauge variation of the vector potential $\tilde{A}_\mu(x) \rightarrow \tilde{A}_\mu(x) + \partial_\mu \omega(x)$, it is also smooth w.r.t. to the link variables.
Proof(2): Anomalous conservation law of $j_\mu(x)$

For the gauge variation, we set $\eta_\mu(x) = -\partial_\mu \omega(x)$ in the L.H.S.

$$\text{L.H.S.} = \sum_{x \in \Gamma_4} \omega(x) \partial^*_\mu j_\mu(x)$$

On the other hand, in the R.H.S.

$$\delta_\eta \hat{P}_L = it[\omega, \hat{P}_L], \quad \delta_\eta \bar{k}_\mu(x) = 0$$

Then

$$\text{R.H.S.} = - \int_0^1 dt \ t \ Tr_L \left\{ \omega \partial_t \hat{P}_L \right\} + \int_0^1 dt \ \sum_{x \in \Gamma_4} \omega(x) \partial^*_\mu \bar{k}_\mu(x)$$

$$= \sum_{x \in \Gamma_4} \omega(x) \ \text{tr}\{\gamma_5 (1 - aD)(x, x)\}|_{t=1}$$
Proof(3): Local integrability condition of $j_\mu(x)$

Noting
\[ [\delta_\eta, \delta_\zeta] = 0, \quad \text{Tr} \left\{ \delta_1 \hat{P}_L \delta_2 \hat{P}_L \delta_3 \hat{P}_L \right\} = 0 \]

and

second term of R.H.S. = $\delta_\eta \int_0^1 dt \sum_{x \in \Gamma_4} \left\{ \tilde{A}_\mu(x) \tilde{k}_\mu(x) \right\}$

\[
\delta_\eta \mathcal{L}_\zeta - \delta_\zeta \mathcal{L}_\eta = i \int_0^1 dt \text{Tr}_L \left\{ \hat{P}_L[\delta_\eta \partial_t \hat{P}_L, \delta_\zeta \hat{P}_L] - \hat{P}_L[\delta_\zeta \partial_t \hat{P}_L, \delta_\eta \hat{P}_L] \right\} \\
+ \{\delta_\eta \mathcal{L}_\zeta - \delta_\zeta \mathcal{L}_\eta\} |_{U=U[w] V[m]} \\
= i \int_0^1 dt \partial_t \text{Tr}_L \left\{ \hat{P}_L[\delta_\eta \hat{P}_L, \delta_\zeta \hat{P}_L] \right\} \\
+ \{\delta_\eta \mathcal{L}_\zeta - \delta_\zeta \mathcal{L}_\eta\} |_{U=U[w] V[m]} \\
= \left. i \text{Tr}_L \left\{ \hat{P}_L[\delta_\eta \hat{P}_L, \delta_\zeta \hat{P}_L] \right\} \right|_{t=1}
\]
Proof(4): Global integrability condition of $j_\mu(x)$

- Recall the topology of the space of U(1) gauge fields

$$\mathcal{U}[m] \cong U(1)^4 \times U(1)^L^4 \times \mathcal{A}[m]$$

- Non-contractible loops ($0 \leq t \leq 2\pi$)
  - Gauge loops

$$U_t(x, \mu) = \Lambda_t(x) V[m](x, \mu) \Lambda_t(x + \hat{\mu})^{-1}, \quad \Lambda_t(x) = \exp\{it\delta_{\tilde{x}\tilde{y}}\}$$

  - Non-gauge loops

$$U_t(x, \mu) = V[m](x, \mu) \exp\{it\delta_{\mu\nu}\delta_{\tilde{x}0}\}$$

- Computations of $W$ and $\det\{1 - P_0 + P_0 Q_1\}$ ($t_k = 2\pi k/n$)

$$W = \exp\{i \int_0^{2\pi} dt \mathcal{L}_\eta\}, \quad i\eta_\mu(x) = \partial_t U_t(x, \mu) U_t(x, \mu)^{-1}$$

$$\det\{1 - P_0 + P_0 Q_1\} = \lim_{n \to \infty} \det\{1 - P_{t_0} + P_{t_n} P_{t_{n-1}} \cdots P_{t_0}\}$$
Gauge loops

$$\mathcal{L}_\eta = \text{tr}\{\gamma_5(1 - aD)(y, y)\}$$

$$\therefore i\eta_\mu(x) = U_t(x, \mu)^{-1}\partial_t U_t(x, \mu) = -i\partial \delta \tilde{x}\tilde{y}$$

$$W = \exp\{i2\pi\text{tr}\{\gamma_5(1 - aD)(y, y)\}|_{t=0}\}$$

$$\text{det}\{1 - P_0 + P_0 Q_1\}$$

$$= \lim_{n \to \infty} \text{det}\left\{1 - P_0 + (P_0 \Lambda_{\Delta t}^{-1} P_0)^n\right\} \quad (\Delta t = \frac{2\pi}{n})$$

$$= \exp\{-i2\pi\text{Tr}[\omega P_0]\} \quad (\omega(x) = \delta \tilde{x}\tilde{y})$$

$$= W$$
Non-gauge loops

Under $U(x, \mu) \rightarrow U(x, \mu)' = U(-x - \hat{\mu}, \mu)^{-1}$

$$U_t(x, \mu)' = \Omega_t(x) U_{2\pi - t}(x, \mu) \Omega_t(x + \hat{\mu})^{-1}$$

where $\Omega_t(x) = \Omega_0(x) \exp\{it\delta\tilde{x}_0\}$, $\Omega(0) = 1$

$\Delta$

$$j_\mu(x)\big|_{t \rightarrow 2\pi - t} = -j_\mu(-x - \hat{\mu})$$

$\Delta$

$$P_t Q_t = Q_t P_{2\pi - t}, \quad Q_t Q_{2\pi - t} = 1$$

where $Q_t$ is a unitary operator s.t.

$$Q_t \psi(x) = \Omega_t(-x) \gamma_5 \psi(-x)$$
Non-gauge loops (cont'd)

For $i\eta_\mu(x) = U_t(x, \mu)^{-1} \partial_t U_t(x, \mu) = i\delta_{\mu\nu}\delta_{\bar{x}0}$

$$W = \exp \left\{ i \int_0^{2\pi} dt \mathcal{L}_\eta \right\} = \exp \left\{ i \int_0^\pi dt \sum_{x \in \Gamma_4} \delta_{\bar{x}0} \partial_\mu^* j_\mu(x) \right\}$$

$$= \exp \left\{ i \int_0^\pi dt \sum_{x \in \Gamma_4} \delta_{\bar{x}0} \text{tr} \{ \gamma_5 (1 - aD)(x, x) \} \right\}$$

$(n = 2r)$

$$\det \{ 1 - P_0 + P_0 Q_1 \}$$

$$= \lim_{n \to \infty} \det \left\{ 1 - P_0 + P_0 (Q_{t_1})^{-1} P_{t_1} Q_{t_1} \cdots (Q_{t_r})^{-1} P_{t_r} Q_{t_r} \times P_{t_{r-1}} P_{t_{r-2}} \cdots P_{t_1} P_0 \right\}$$

$$= W \det \left\{ 1 - P_0 + P_0 (Q_0)^{-1} P_0 \right\} \det \left\{ 1 - P_\pi + P_\pi (Q_\pi)^{-1} P_\pi \right\}$$

Extra factor is unity for the fermion with an even charge and for the pair of fermions with $(e, +e)$ or $(e, -e)$
An explicit formula of Weyl fermion measure

- One-parameter interpolation
  \[ U^t_\mu(x) = e^{it\tilde{A}_\mu(x)} U_{[w]}(x, \mu) V_{[m]}(x, \mu) \quad t \in [0, 1] \]

- Choice of basis vectors
  \[ v_i(x) = \begin{cases} Q_1 v^0_{1[w]}(x) W^{-1} & \text{if } i = 1 \\ Q_1 v^0_{i[w]}(x) & \text{otherwise} \end{cases} \]

- \( Q_t \): Evolution op.
  \[ P_t = Q_t P_0 Q_t^{-1}, \quad P_t = \hat{P}_- |_{U=U_t} \]

- \( W \): “Local counter term”
  \[ \ln W = i \int_0^1 dt \sum_x \tilde{A}_\mu(x) \left\{ \tilde{k}_\mu(x) \right\} |_{A \rightarrow tA} \]
Summary

- Weyl fermion on the lattice based on the GW relation
- Lattice formulation of U(1) chiral gauge theories
  Locality, Gauge-invariance, Integrability, lattice symmetries
  An explicit (simple) formula for the Weyl fermion measure
- Fermion zero modes / Index theorem on the lattice

\[ Z_{\text{Weyl}} = \det \{ M_{ki} \}, \quad M_{ki} = a^4 \sum_x \bar{\nu}_k(x) D\nu_i(x) \]

- \( i = 1, \cdots, N_-, \quad k = 1, \cdots, N_0 \) \( (N_0 = 2N_cL^4) \)
  \( N_- \) (the shape of the matrix \( M_{ki} \)) is a dynamical variable
- Fermion zero modes, Fermion number nonconservation

⇒ Practical Implementation of U(1) chiral gauge theories
SU(2) × U(1) Electroweak theory on the lattice

- SU(2) × U(1) Electroweak theory

\[
\begin{pmatrix}
\nu_L(x) \\
e_L(x)
\end{pmatrix}
\bigg|_{Y = -\frac{1}{2}},
\begin{pmatrix}
e_R(x) \\
u_L(x)
\end{pmatrix}
\bigg|_{Y = -1},
\begin{pmatrix}
d_L(x) \\
u_L(x)
\end{pmatrix}
\bigg|_{Y = \frac{1}{6}},
\begin{pmatrix}
d_R(x) \\
\frac{1}{3}
\end{pmatrix}
\bigg|_{Y = -\frac{1}{3}}
\]

where \( i \) is the color index \((i = 1, 2, 3)\)

- Anomaly cancellation conditions

\[
\sum_L Y^3 - \sum_R Y^3 = 0,
\sum_{\text{doublet}(L)} Y = 0, \quad \sum_{\text{singlet}(R)} Y = 0
\]
On a finite lattice of the size $L$ with P.B.C.

$$\Gamma_4 = \{ x = (x_0, \cdots, x_3) \in \mathbb{Z}^4 \mid 0 \leq x_\mu < L \} = \mathbb{L}^4$$

Admissible SU(2)$\times$U(1) link variables

$$U^{(2)}(x, \mu), \quad U^{(1)}(x, \mu)$$

Admissibility condition

$$\| 1 - U^{(2)}_\Box \| \leq \varepsilon, \quad \| 1 - \{U^{(1)}_\Box \}^Y \| \leq \varepsilon, \quad \varepsilon < \frac{1}{30}$$

topological invariants

$$m_{\mu\nu} = \frac{1}{2\pi} \sum_{s,t} F^{(1)}(x + s\hat{\mu} + t\hat{\nu})$$

$$Q = \sum_{x \in \Gamma_4} \text{tr}\{\gamma_5(1 - aD)(x, x)\} \bigg|_{U^{(2)}, \cdots}$$
Construction of $\mathcal{L}_\eta$ in SU(2) $\times$ U(1) EW theory

Topological sectors with $m_{\mu\nu} = 0$

- One-parameter family of the admissible SU(2) $\times$ U(1) fields

$$U_t(x, \mu) = e^{it\tilde{A}_\mu(x)} U_{[W]}(x, \mu), \quad U^{(2)}(x, \mu) \quad t \in [0, 1]$$

- Cohomological analysis of the U(1) gauge anomaly

$$q(x) = \text{tr} \left\{ \gamma_5 (1 - aD)(x, x) \right\} \big|_{U^{(2)}, U^{(1)}}$$

$$q(x) = q(x)|_{U^{(2)}} + \frac{1}{32\pi^2} \epsilon_{\mu\nu\lambda\rho} F_{\mu\nu}(x) F_{\lambda\rho}(x + \hat{\mu} + \hat{\nu}) + \partial^* k_\mu(x)$$

$$\sum_\alpha Y_\alpha q(x)^\alpha = \partial^* \bar{k}_\mu(x), \quad \bar{k}_\mu(x) = \sum_\alpha Y_\alpha \{ k_\mu(x) \} \big|_{U \rightarrow U Y_\alpha}$$
Explicit formula of the measure term:
\[
\eta_\mu(x) = \eta_\mu^{(1)}(x) \oplus \eta_\mu^{(2)}(x)
\]

\[
\mathcal{L}_\eta = i \int_0^1 dt \text{Tr}_L \left\{ \hat{P}_- [\partial_t \hat{P}_-, \delta_\eta \hat{P}_-] \right\}
\]
\[
+ \delta_\eta \int_0^1 dt \sum_{x \in \Gamma_4} \left\{ \tilde{A}_\mu(x) \bar{k}_\mu(x) \right\} + \mathcal{L}_\eta |_{U^{(1)}=U_{[w]} V_{[m]}, U^{(2)}}
\]

It is possible to prove explicitly that \( j_\mu(x) \) is a local current which satisfies the following properties

- \( j_\mu(x) \) is defined for all admissible \( \text{SU}(2) \times \text{U}(1) \) gauge fields with \( m_{\mu\nu} = 0 \) and depends smoothly on the link variables

- \( j_\mu(x) \) is gauge-covariant and transforms as an axial vector current under the lattice symmetries

- \( j_\mu(x) \) satisfies the anomalous conservation law

- The linear functional \( \mathcal{L}_\eta = \sum_{x \in \Gamma_4} \eta_\mu(x) \cdot j_\mu(x) \) satisfies the local integrability condition (global integrability condition ?)
Reconstruction of the measure

- One-parameter interpolation \((m_{\mu\nu} = 0)\)
  \[
  U^{(2)}(x, \mu), \quad U^{(1)}_t(x, \mu) = e^{it\tilde{A}_\mu(x)} \quad t \in [0, 1]
  \]
- Choice of basis vectors
  \[
  v_i(x) = \begin{cases} 
  Q_1 v^0_1(x) W^{-1} & \text{if } i = 1 \\
  Q_1 v^0_i(x) & \text{otherwise}
  \end{cases}
  \]
  - \(v^0_i\) is chosen for a pair of SU(2) doublets, (a) and (b), as
    \[
    v^{0(a)}_j(x) = v^0_j(x), \quad v^{0(b)}_j(x) = (\gamma_5 C^{-1} \otimes i\sigma_2) [v^0_j(x)]^* 
    \]
- \(Q_t\): Evolution op.
  \[
  P_t = Q_t P_0 Q_t^{-1}, \quad P_t = \hat{P} \big|_{U=U_t}
  \]
- \(W\): “Local counter term”
  \[
  \ln W = i \int_0^1 dt \sum_x \tilde{A}_\mu(x) \{ \tilde{k}_\mu(x) \} \big|_{A\to tA}
  \]
\[ \text{SU}(2) \quad \text{m} \quad \text{[U}(1)\text{]} \quad \text{Q} \quad \text{[SU}(2) \]
SU(2) × U(1) EW theory on the lattice based on GW rel.

- Local cohomology problem in 4+2 dim. is solved for $L = \infty$
  ⇒ Exact cancellations of the gauge anomalies
  $U(1)^3$, $SU(2)^2 \times U(1)$-mixed type

- Measure term is constructed for $L < \infty$, $m_{\mu\nu} = 0$
  Construction similar to the U(1) case
  ⇒ Practical implementation

- Global integrability ? ⇐ Numerically check-able!
Possible applications

- Rate of Baryon number non-conservation at finite $T$
  - $T < T_c$ (Broken phase)
    Sphaleron contribution in SU($x$) × U(1) Electroweak theory
    \[
    \Gamma \simeq \frac{\omega}{\pi} \beta \text{Im} F \simeq \frac{\omega}{\pi} \frac{\text{Im} Z_{\text{barrier}}}{Z_0}
    \]
    Fermion contribution? (Large) Yukawa coupling to Higgs
  - $T > T_c$ (Symmetric phase)
    "Lattice" simulation by Bodeker et al., (PRD61, 056003)

- Two-dim. model: Chiral Abelian Higgs model
  \[
  \mathcal{L} = -\frac{1}{4} F_{\mu \nu}^2 + |D_\mu \phi|^2 - \lambda (|\phi|^2 - c^2/2)^2 \\
  + i \bar{\psi} \gamma_\mu (\partial_\mu - ig \gamma_5 A_\mu) \psi + iy \bar{\psi}_L \psi_R \phi^* - iy \bar{\psi}_R \psi_L \phi
  \]

- Recent work: Burnier, Shaposhnikov \textit{hep-ph/0507130}
  Fermion determinant with Yukawa coupling around Instanton background ($T = 0$)
Local cohomology problem in SU(2) × U(1) EW theory

- Infinite volume part of the 4+2 dim. topological field

\[ q(z) \left[ U^{(2)}_\mu, U^{(1)}_\mu \right] = q_\infty(z) + \Delta q(z) \]

- Cohomological analysis w.r.t. \( U^{(1)}_\mu \) in the infinite volume

\[ q_\infty(z) = \alpha \left[ U^{(2)}_\mu \right] + \beta_{\mu\nu} \left[ U^{(2)}_\mu \right] F^{(1)}_{\mu\nu}(z) \]
\[ + \gamma_{\mu\nu\rho\sigma} \left[ U^{(2)}_\mu \right] F^{(1)}_{\mu\nu}(z) F^{(1)}_{\rho\sigma}(z + \hat{\mu} + \hat{\nu}) \]
\[ + \delta \epsilon_{\mu\nu\rho\sigma\lambda\tau} F^{(1)}_{\mu\nu}(z) F^{(1)}_{\rho\sigma}(z + \hat{\mu} + \hat{\nu}) \times \]
\[ F^{(1)}_{\lambda\tau}(z + \hat{\mu} + \hat{\nu} + \hat{\rho} + \hat{\sigma}) \]
\[ + \partial^* k_\mu(z) + \partial_t k_s(z) - \partial_s k_t(z) \]

where

\[ \partial^*_\mu \beta_{\mu\nu}(z) = 0, \quad \partial^*_\mu \gamma_{\mu\nu\rho\sigma}(z) = 0, \quad \partial^*_\mu \delta = 0 \text{ (const.)} \]
Using the pseudo reality of SU(2):

\[ q_\infty(z)[U_\mu^{(2)}, A_\mu] = -q_\infty(z)[(U_\mu^{(2)})^*, -A_\mu] = -q_\infty(z)[U_\mu^{(2)}, -A_\mu] \]

in particular

\[ q_\infty(z)[U_\mu^{(2)}, 0] = 0 \]

then

\[ q_\infty(z) = \beta_{\mu\nu} \left[ U_\mu^{(2)} \right] F_{\mu\nu}^{(1)}(z) \]

\[ + \delta\epsilon_{\mu\nu\rho\sigma\lambda\tau} F_{\mu\nu}^{(1)}(z) F_{\rho\sigma}^{(1)}(z + \hat{\mu} + \hat{\nu}) \times F_{\lambda\tau}^{(1)}(z + \hat{\mu} + \hat{\nu} + \hat{\rho} + \hat{\sigma}) \]

\[ + \partial^* k_\mu(z) + \partial_t k_s(z) - \partial_s k_t(z) \]

Imposing the anomaly cancellation conditions:

\[ \sum_\alpha q_\infty(z)^\alpha = \partial^* k_\mu(z) + \partial_t k_s(z) - \partial_s k_t(z) \]
Anomaly cancellation conditions in the electroweak theory

\[
\sum_{L} Y^{3} - \sum_{R} Y^{3} = 0, \quad \sum_{\text{doublet}(L)} Y = 0, \quad \sum_{\text{singlet}(R)} Y = 0
\]

then

\[
\sum_{\alpha} q_{\infty}(z)^{\alpha} = \partial_{\mu} k_{\mu}(z) + \partial_{t} k_{s}(z) - \partial_{s} k_{t}(z)
\]
An explicit formula of Weyl fermion measure

- One-parameter interpolation

\[ U^t_\mu(x) = e^{it\tilde{A}_\mu(x)} U_{[w]}(x, \mu) V_{[m]}(x, \mu) \quad t \in [0, 1] \]

- Choice of basis vectors

\[ v_i(x) = \begin{cases} 
Q_1 v^0_{1[w]}(x) W^{-1} & \text{if } i = 1 \\
Q_1 v^0_{i[w]}(x) & \text{otherwise}
\end{cases} \]

- \( Q_t \): Evolution op.

\[ P_t = Q_t P_0 Q_t^{-1}, \quad P_t = \hat{P}_{-|U=U_t} \]

- \( W \): “Local counter term”

\[ \ln W = i \int_0^1 dt \sum_x \tilde{A}_\mu(x) \{ \bar{k}_\mu(x) \} |_{A \rightarrow tA} \]
Practical Implementation of U(1) chiral lattice gauge theories

- Problems from the practical point of view
  - Cohomological analysis in the infinite lattice for $k_\mu(x)$
  - Continuous interpolation with $t$ in the space of gauge fields
  - Evolution operator $Q_t$

- A practical implementation
  - Solve the local cohomology problem within the finite-volume lattice
  - Perform the integration of the continuous parameter numerically

  $\implies$ Numerical computation of $k_\mu(x)$

  - Discretization of the continuous interpolation with DWF

  $\implies$ Numerical computation of chiral determinant
  $\implies$ Check the independence on the path of the interpolation
Cohomological analysis within $V = L^4$

- Vector potential representation of the link variables

\[ U(x, \mu) = e^{i \tilde{A}_\mu(x)} U[w](x, \mu) V[m](x, \mu) \]

\[ F_{\mu \nu}(x) = \partial_\mu \tilde{A}_\nu(x) - \partial_\nu \tilde{A}_\mu(x) + \frac{2\pi m_{\mu \nu}}{L^2} \]

\[
\begin{cases}
|\tilde{A}_\mu(x)| \leq \pi (1 + 4 \| x \|) & (\| x \| \leq L/2) \\
|\tilde{A}_\mu(x)| \leq \pi (1 + 2L + 2(n - 1)L^2) & (\text{otherwise})
\end{cases}
\]

- One-parameter family of the admissible U(1) gauge fields

\[ U_t(x, \mu) = e^{it\tilde{A}_\mu(x)} U[w](x, \mu) V[m](x, \mu) \quad t \in [0, 1] \]
Topological property of $q(x) \Rightarrow$ properties of $j_\mu(x, y)$

$$q(x) = q_{[m,w]}(x) + \sum_y j_\mu(x, y)\tilde{A}_\mu(y)$$

where

$$j_\mu(x, y) = \int_0^1 dt \frac{\partial q(x)}{\partial \tilde{A}_\mu(y)} \bigg|_{\tilde{A} \rightarrow t\tilde{A}}$$

$$\sum_x j_\mu(x, y) = 0, \quad j_\mu(x, y)\partial_{\mu}^* = 0$$
Cohomological analysis within $V = L^4$ (cont’ d)

- Poincaré lemma on the lattice (the finite lattice)

**Lemma a** Let $f$ be a $k$-form which satisfies

$$d^* f = 0 \quad \text{and} \quad \sum_{x \in \Gamma_n} f(x) = 0 \quad \text{if} \quad k = 0$$

Then there exist a form $g \in \Omega_{k+1}$ and a form $\Delta f \in \Omega_k$ s.t.

$$f = d^* g + \Delta f, \quad |\Delta f_{\mu_1 \ldots \mu_k}(x)| < cL^{\sigma}e^{-L/2\rho}$$

**Lemma b** Let $f$ be a $k$-form which satisfies

$$d^* f = 0 \quad \text{and} \quad \sum_{x \in \Gamma_n} f(x) = 0$$

Then there exist a form $g \in \Omega_{k+1}$ such that

$$f = d^* g$$

- (a lattice counter part of the corollary of de Rham theorem)
Cohomological analysis within $V = L^4$ (cont' d)

- Solve the local cohomology problem within $V = L^4$
  
  $$q(x) = \text{tr} \{ \gamma_5 (1 - aD)(x, x) \}$$

  $$q(x) = q_{[m,w]}(x) + \phi_{[m,w] \mu \nu}(x) \tilde{F}_{\mu \nu}(x)$$
  
  $$+ \gamma_{[m,w]} \epsilon_{\mu \nu \lambda \rho} \tilde{F}_{\mu \nu}(x) \tilde{F}_{\lambda \rho}(x + \hat{\mu} + \hat{\nu}) + \partial^* k_\mu(x)$$

- Proof of exact gauge anomaly cancellation at $V = L^4$

  $$\sum_{\alpha} e_\alpha A(x)|_{U \to U e_\alpha} = \Delta q(x), \quad |\Delta q(x)| \leq cL^\sigma e^{-L/2 \rho}$$

  Since $\sum_{x \in \Gamma_4} \Delta q(x) = 0$

  $$\sum_{\alpha} e_\alpha q(x)|_{U \to U e_\alpha} = \partial^* k_\mu(x)$$
Numerical computation of $k_{\mu}(x)$ in 2 dim.

- Numerical computation of the bi-local current $j_{\mu}(x, y)$
  - Rational approximation
    Optimized coefficients $c_k$ and $b_k$ (Zolotarev optimization)

$$ q(x) = -\frac{1}{2} \text{tr} \left\{ \frac{H_w}{\sqrt{H_w^2}}(x, x) \right\} $$

$$ \approx -\frac{1}{2} \text{tr} \left\{ h_w \sum_{k=1}^{N_r} \frac{b_k}{h_w^2 + c_{2k-1}}(x, x) \right\} $$

where $h_w = H_w/\lambda_{\text{min}}$

$$ \frac{\partial q(x)}{\partial \tilde{A}_{\nu}(y)} \approx -\frac{1}{2} \text{tr} \left\{ \sum_{k=1}^{N_r} b_k \frac{1}{h_w^2 + c_{2k-1}} \times \right. $$

$$ (c_{2k-1} v_{\mu}(y) - h_w v_{\mu}(y) h_w) \frac{1}{h_w^2 + c_{2k-1}}(x, x) \right\} $$
Parameter integration by Gaussian Quadrature formula

\[ j_\nu(x, y) \equiv \sum_{i=1}^{N_g} w_i \left( \frac{\partial q(x)}{\partial \tilde{A}_\nu(y)} \right) \tilde{A} \rightarrow t_i \tilde{A} \]

\{(t_i, w_i) | i = 1, \cdots, N_g(\approx 20)\} : the set of the abscissas and weights

\[ \sum_j j_\mu(x, y) = O(10^{-15}), \quad j_\mu(x, y) \overset{\text{exact}}{\rightarrow} 0 \]

Numerical cohomological analysis in two dimensions

\[ q(x) = q_{[m,w]}(x) + \gamma_{[m,w]} \epsilon_{\mu\nu} \tilde{F}_{\mu\nu}(x) + \partial_{\mu}^* h_{\mu}(x) + \Delta q(x) \]

Check of locality
apply a small variation: \( \eta_\mu(x) = 0.05 \times 2\pi \delta_{x,x_0} \delta_{\mu,1} \)

\[ \delta_\eta f(r) = \max \{ |\delta_\eta f(x)| | r = \| x - x_0 \| \} \]
Locality of $q(x)$

$L = 12$

$q(x)$

$|x - x_0|$
Locality of $h_1(x)$, $h_2(x)$
$L = 12$

Plot of $h_1$, $h_2$ vs $|x-x_0|$
Gauge anomaly cancellation

\[ \sum_{\alpha} e_{\alpha} q^{\alpha}(x) = \partial^\ast \bar{k}_{\mu}(x) \]

\[ \bar{k}_{\mu}(x) \equiv \sum_{\alpha} e_{\alpha} \{ h_{\mu}(x) + \Delta h_{\mu}(x) \}^\alpha + \Delta k_{\mu}(x) \]

Check of gauge anomaly cancellation
anomaly part: \( A(x) = q_{[m]}(x) + \gamma_{[m,w]} \epsilon_{\mu\nu} \tilde{F}_{\mu\nu}(x) \)

11112 model: \( \sum_{i=1}^{4} e^2 - (2e)^2 = 0, \ A^\alpha(x) = A(x)|_{U \rightarrow U^{e\alpha}} \)

- \( 4 \times A^1(x); A^2(x) \); \( \sum_{\alpha} e_{\alpha} A^\alpha(x) = 4 \times A^1(x) - A^2(x) \)
Anomaly cancellation (11112)
Anomaly cancellation (11112) on the finite lattice (L=10)
How to compute $Q_1 v_j^0$

Discretization of the continuous evolution of chiral basis

- **Naive approach**

$$\det \left( v_i^1, Q_1 v_i^0 \right) \iff \det \left( v_i^1, \prod_{t \in c} \hat{P}_t v_i^0 \right) / \det \left( v_i^1, \prod_{t \in c} \hat{P}_t v_i^0 \right)$$

- **Use of Domain Wall Fermion**

$$\det \left( v_i^1, Q_1 v_i^0 \right) \iff \det \left( v_i^1, \prod_{t \in c} \hat{T}_t v_i^0 \right) / \det \left( v_i^1, \prod_{t \in c} \hat{T}_t v_i^0 \right)$$

- $\hat{T}$: Transfer matrix of D+1 dim. Wilson fermion
Domain wall fermion (D+1 dim. lattice Implementation)

\[
\begin{align*}
(t = -N + 1) & \quad \psi_L(x) \quad -m_0 \\
(t = N) & \quad \psi_R(x) \quad (+m_0)
\end{align*}
\]

Figure: Domain wall fermion in the simpler vector-like setup
D+1 dim. lattice Implementation for chiral gauge theories

Figure: Interpolating five-dimensional gauge field on the lattice
Explicit formula of Weyl fermion measure via Domain wall fermion

- **basis vectors**

\[ v_i(x) = \begin{cases} 
  v_i^1(x) \langle v^1, Q_1 v^0 \rangle W^{-1} & \text{if } i = 1 \\
  v_i^1(x) & \text{otherwise}
\end{cases} \]

- **\( Q_t \) (a representation) : Product of the transfer-matrix**

\[ \langle v^1, Q_1 v^0 \rangle \equiv \frac{\det \left( v_i^1, \prod_{t \in c} \hat{T}_t v_i^0 \right)}{\det \left( v_i^1, \prod_{t \in c} \hat{T}_t v_i^0 \right)} \]

- **\( W \) : Chern-Simons term (assure the path-independence)**

\[
\ln W = \sum_{x, t \in c_1} c(x, t) \\
= - \sum_{x, t \in c_1} \left\{ \int_0^1 ds \frac{1}{2} \bar{\chi}_{\nu\mu}(z) |\tilde{A} \rightarrow s\tilde{A}\rangle \tilde{F}_{\nu\mu}(z) \right\}
\]
Local cohomology problem in D+1 dim. lattice

- Dependence on the interpolations

\[ \text{Im} \ln \det D_{d+1}^{[c_1]} - \text{Im} \ln \det D_{d+1}^{[c_2]} = \int_0^1 ds \sum_{x, t \in c_1 + (-c_2)} J_\mu(x, t)^s \tilde{A}_\mu(x, t) \]

- A continuous deformation of the path

\[ U_\mu(x, t)^s = V_{[m]}(x, \mu) \exp(is\tilde{A}_\mu(x, t)) \]

- Chern-Simons current of D+1 dim. Wilson fermion (with mass \(-m_0\))

\[ J_\mu(x, t) = \text{ImTr} \left\{ \frac{\delta D_{d+1}}{\delta A_\mu(x, t)} D_{d+1}^{-1} \right\} \quad \partial^*_\mu J_\mu(x, t) = 0 \]
Cohomological analysis in d+1 dim. lattice ($L^d \times 4N$)

\[ J_\mu(z) = \gamma[m,w] \varepsilon_{\mu\nu\rho\lambda\sigma} \tilde{F}_{\nu\rho}(z) \tilde{F}_{\lambda\sigma}(z + \hat{\nu} + \hat{\rho}) + \partial^*_\nu \chi_{\nu\mu}(z) + \Delta J_\mu(z) \quad (d = 4) \]

"Local counter term" which assures the path-independence

\[
\text{Im } \ln \det D_{d+1}^{[c_1]} - \text{Im } \ln \det D_{d+1}^{[c_2]}
\]

\[
= \sum_{x,t \in c_1 + (-c_2)} \frac{1}{2} \gamma[m,w] \varepsilon_{\mu\nu\rho\lambda\sigma} \tilde{A}_\mu(z) \tilde{F}_{\nu\rho}(z) \tilde{F}_{\lambda\sigma}(z + \hat{\nu} + \hat{\rho})
\]

\[
- \sum_{x,t \in c_1 + (-c_2)} \left\{ \int_0^1 ds \frac{1}{2} \tilde{\chi}_{\nu\mu}(z) |_{\tilde{A} \to s\tilde{A}} \right\} \tilde{F}_{\nu\mu}(z)
\]

\[
= \sum_{x,t \in c_1} c(z) - \sum_{x,t \in c_2} c(z) \quad \left( \text{if } \sum_\alpha e_\alpha^3 = 0 \right)
\]
Numerical computations in 2+1 dim.

- Cohomological analysis in 3 dim. (check of locality)
  \[ J_\mu(z) = \gamma_{[m,w]} \epsilon_{\mu\nu\rho} \tilde{F}_{\nu\rho}(x + \hat{\mu}) + \partial^*_\nu \chi_{\nu\mu}(z) + \Delta J_\mu(z) \]

- Gauge anomaly cancellation (11112 model)
  \[ \sum_\alpha e_\alpha J_\mu(z)^\alpha = \partial^*_\nu \bar{\chi}_{\nu\mu}(z) + \delta J_\mu(z) \]

where
\[
\bar{\chi}_{\nu\mu}(z) = \sum_\alpha e_\alpha \chi_{\nu\mu}(z)^\alpha + \Delta \chi_{\nu\mu}(z)
\]

\[ \delta J_3(z) = 0, \quad \partial_k^* \delta J_k(z) = 0 \]

- Reconstruction of Chern-Simons term (Gauss quadrature)
  \[ c(z) = \int_0^1 ds (1 - s) \left\{ -\left(1/2\right) \bar{\chi}_{\nu\mu}(z)|_{\tilde{A} \to s\tilde{A}} \tilde{F}_{\nu\mu}(z) \right. \]
  \[ \left. + \delta j_k(z)|_{\tilde{A} \to s\tilde{A}} \tilde{A}_k(z) \right\} \]
Independence on the interpolation path
Two different interpolation-paths

\[ U(x, t) = V[m](x) \exp(is(t)\tilde{A}(x)) \]

Path 1: \( s(t) = \frac{1 - \cos(\pi t)}{2} \)
Path 0: \( s(t) = t \)

<table>
<thead>
<tr>
<th>phase</th>
<th>(Path 1)</th>
<th>(Path 0)</th>
<th>(Path 1) - (Path 0)</th>
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<td>( \text{det}(v, Qv^0) )</td>
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<td>( W^{-1} )</td>
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<td>-0.1870420</td>
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<td>( \text{det}(DP^R) )</td>
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<td>-0.2408034</td>
<td>-0.0000068</td>
</tr>
</tbody>
</table>

Table: Numerical values of the complex phases of Twist, Chern-Simons term, Chiral determinant for two different interpolation-paths, L=8, Nt=20, Nt=6, Ng=10
Monte Carlo history of the complex phase of chiral determinant (quench simulation)

Complex Phase of Chiral Determinant
L=6, beta=25.0 (s_eff=0.04)
11112 model (m0=-0.9, Nt=20, Nlt=6, Ng=10)

# Iteration

Phase

Path 1
Path 0
Summary

▶ A Practical formulation of U(1) chiral gauge theories
  ▶ Cohomological analysis performed directly in the finite volume lattice with modified Poincaré lemma
  ▶ Numerical computation of the integral of the continuous parameter
  ▶ Discretization of the continuous evolution of chiral basis using DWF (use of the transfer-matrix \( \hat{T} \))
  ▶ Applicable to SU(2) \( \times \) U(1) \( (m_{\mu\nu} = 0) \)

▶ Numerical cohomological analysis in 2, 3 dimensions
  ▶ the Gaussian quadrature formula for the parameter-integral
  ▶ Locality and anomaly cancellation, checked

▶ Explicit computation of the Weyl fermion measure – possible!
  ▶ Check of the gauge invariance
  ▶ Behavior of the complex phase \( \Leftrightarrow \) sign problem
  ▶ Evaluation of some observables like ’t Hooft vertex
Possible applications

- Two-dim. model: Chiral Abelian Higgs model

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + |D_\mu \phi|^2 - \lambda (|\phi|^2 - c^2/2)^2 \\
+ i \bar{\psi} \gamma_\mu (\partial_\mu - ig\gamma_5 A_\mu) \psi + iy \bar{\psi}_L \psi_R \phi^* - iy \bar{\psi}_R \psi_L \phi \]

- Recent work: Burnier, Shaposhnikov *hep-ph/0507130*
  Fermion determinant with Yukawa coupling around Instanton background \((T = 0)\)

- Rate of Baryon number non-conservation at finite \(T\)
  - \(T < T_c\) (Broken phase)
    Sphaleron contribution in SU(2) × U(1) Electroweak theory

\[ \Gamma \simeq \frac{\omega^-}{\pi} \beta \text{Im} F \simeq \frac{\omega^-}{\pi} \frac{\text{Im} Z_{\text{barrier}}}{Z_0} \]

Fermion contribution? (Large) Yukawa coupling to Higgs

- \(T > T_c\) (Symmetric phase)
  "Lattice" simulation by Bodeker et al., (PRD61, 056003)