Schrödinger functional formalism with Ginsparg-Wilson fermion

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§1 Introduction

- Schrödinger functional (Lüscher et al,,,)

$$Z = \langle C'; T | C; 0 \rangle = \int \mathcal{D}\Phi e^{-S[\Phi]}$$

  - renormalization scheme
  - finite box $L^3 \times T \sim L^4$
  - Dirichlet boundary condition

$$A_k|_{x_0=0} = C_k, \quad A_k|_{x_0=T} = C'_k,$$
$$P_+\psi|_{x_0=0} = \rho, \quad P_-\psi|_{x_0=T} = \rho', \quad P_\pm = \frac{1 \pm \gamma_0}{2}$$
$$\Downarrow$$

Renormalizable (Lüscher et al, Sint)

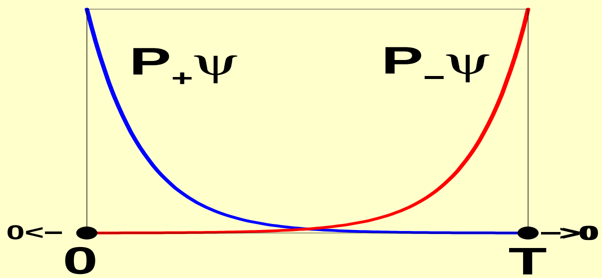
  - finite mass gap $\sim 1/T$

- A potential problem with Dirichlet BC $\Rightarrow$ zero mode
  - zero eigenvalue equation

$$\mathcal{L} = \overline{\psi}\left(\gamma_\mu \partial_\mu + m\right)\psi \quad \Rightarrow \quad \left(\gamma_0 \partial_0 + m\right)\psi = 0$$

  - under boundary condition

$$P_- \psi|_{x_0=0} = 0, \quad P_+ \psi|_{x_0=T} = 0$$



$$\psi = P_+ e^{-m x_0} + P_- e^{-m(T-x_0)}$$

SF BC : $P_+ \psi|_{x_0=0} = 0$, $P_- \psi|_{x_0=T} = 0$ $\Rightarrow$ forbids zero mode

- Wilson fermion (Sint)
  - DBC $\Leftrightarrow$ Wilson parameter $r = \pm 1$

$$
\begin{aligned}
D_W &= \gamma_\mu \nabla_\mu - \frac{r}{2}\Delta + M \\
&= -\frac{1-\gamma_0}{2}\delta_{m_0,n_0+1} - \frac{1+\gamma_0}{2}\delta_{m_0,n_0-1} + D_W^{(3)} + (M+1) \\
&= \begin{pmatrix} D_W^{(3)} + M + 1 & -P_- & \\ -P_+ & D_W^{(3)} + M + 1 & -P_- \\ & -P_+ & D_W^{(3)} + M + 1 \end{pmatrix}
\end{aligned}
$$

$$\Downarrow$$

SF BC : $P_+\psi|_{x_0=0} = 0,\ P_-\psi|_{x_0=T} = 0$

  - $M \geq 0$ to forbid zero mode solution

- Problem may become fatal in overlap Dirac operator

- Overlap Dirac operator (Neuberger)

$$aD = 1 + D_W \frac{1}{\sqrt{D_W^\dagger D_W}}$$

$$D_W = -\frac{1-\gamma_\mu}{2}\delta_{m,n+\mu} - \frac{1+\gamma_\mu}{2}\delta_{m,n-\mu} + (-M+4)$$

Signature of r, M fixed to eliminate doubler

(for $r = 1$; $-1 < 1 - M < 1$)

⇒ zero mode solution is allowed in D_W

$$\text{SF BC} : P_+\psi|_{x_0=0} = 0,\ P_-\psi|_{x_0=T} = 0$$

$$\psi = P_-(1-M)^{x_0} + P_+(1-M)^{(T-x_0)}$$

⇒ non-locality ⇒ Dirichlet BC in D_W does not work!

- Domain-wall fermion (Kaplan, Shamir)

$$D = \gamma_M \nabla_M - \frac{1}{2}\Delta^2 - M$$

Dirichlet BC in 5-th direction

$$P_R\psi|_{x_5=0} = 0, \quad P_L\psi|_{x_5=N_5} = 0, \quad P_{R/L} = \frac{1 \pm \gamma_5}{2}$$

$\Rightarrow$ zero mode solution in 5-th direction

$$\psi = P_L(1-M)^{x_5} + P_R(1-M)^{(N_5-x_5)}$$

$M \sim$ smallest eigenvalue of $\gamma_5 D_W^{(4)}$

SF Dirichlet BC in temporal direction

$$P_+\psi|_{x_0=0} = 0, \quad P_-\psi|_{x_0=T} = 0, \quad P_\pm = \frac{1 \pm \gamma_0}{2}$$

$\Rightarrow$ zero mode solution in $D_W^{(4)}$

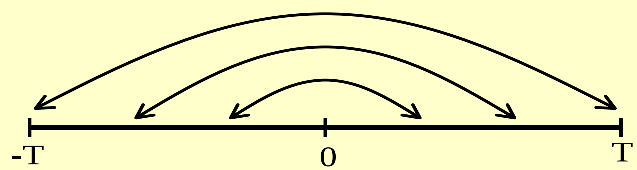
§2 Orbifolding

- A criterion to introduce Dirichlet BC in $D_{\rm OV}$, $D_{\rm dwf}$
  - chiral symmetry breaking by BC : $(1+\gamma_0)\psi|_{x_0=0}$
    - $\Rightarrow$ generate mass gap

- field theory with BC $\Leftrightarrow$ orbifolded field theory

orbifolding: S^1/Z_2

$x_0 \leftrightarrow -x_0$



- Identification of fields by symmetry (projection)
  - $\Rightarrow$ to break chiral symmetry
  - $\Rightarrow$ to produce SF Dirichlet BC at fixed points

- Symmetry for field orbifolding
  - time reversal symmetry
    $$\psi \to i\gamma_0\gamma_5 R\psi, \quad \overline{\psi} \to \overline{\psi} i\gamma_0\gamma_5 R, \quad R\psi(x_0) = \psi(-x_0)$$
  - chiral symmetry (massless theory)
    $$\psi \to i\gamma_5\psi, \quad \overline{\psi} \to \overline{\psi} i\gamma_5$$
  - anti-periodicity
    $$\psi(x_0 + 2T) = -\psi(x_0), \quad R\psi(0) = \psi(0),\ R\psi(T) = -\psi(T)$$

- Orbifolding $\quad \psi(x) = -\gamma_0 R\psi(x), \quad \overline{\psi}(x) = \overline{\psi}(x)\gamma_0 R$

- SF Dirichlet BC at fixed points
  $$(1+\gamma_0)\psi(0) = 0, \quad (1-\gamma_0)\psi(T) = 0$$
  $$\overline{\psi}(0)(1-\gamma_0) = 0, \quad \overline{\psi}(T)(1+\gamma_0) = 0$$

- Orbifolded action
  $$S = \sum \overline{\psi} D_{\mathrm{SF}} \psi, \quad D_{\mathrm{SF}} = \frac{1+\Gamma}{2} \not{D} \frac{1-\Gamma}{2}, \quad \Gamma = \gamma_0 R$$

- Comments
  - Gauge fields (external fields) $\Leftarrow$ SF YM is well defined

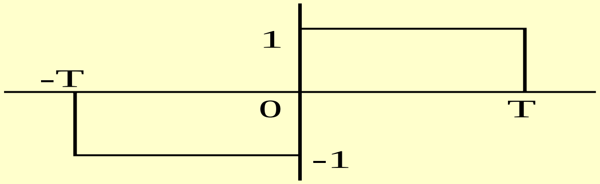
    orbifolding $A_k(x_0) = A_k(-x_0), \quad A_0(x_0) = -A_0(-x_0)$

    SF DBC $\quad A_k(0) = C_k, \quad A_k(T) = C'_k, \quad (\partial_0 A_0(0) = 0)$

  - Mass term $\Rightarrow$ should be consistent with orbifolding

    $\{M(x), \Gamma\} = 0$

    $M(x) = m\eta(x)$



- SF Dirac operator

$$D_{\rm SF} = \frac{1+\Gamma}{2}\left(\not{D} + m\eta(x)\right)\frac{1-\Gamma}{2}$$

$$D_{\rm SF} : \mathcal{H}_+ \to \mathcal{H}_-, \quad D_{\rm SF}^\dagger : \mathcal{H}_- \to \mathcal{H}_+$$

$$\mathcal{H}_\pm = \{\psi | (1 \pm \Gamma)\psi = 0\}$$

- Eigenvalue problem (free theory) (Sint)

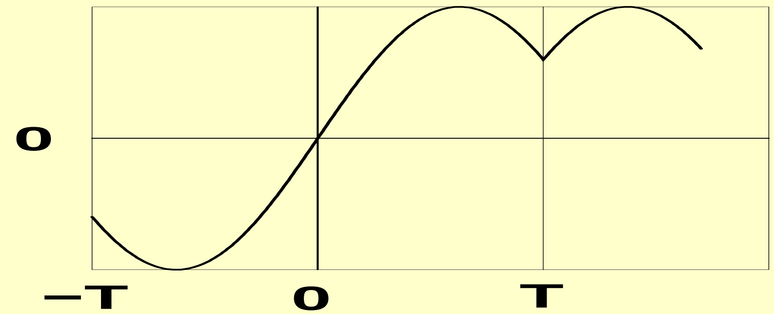
$$D_{\rm SF}^{\dagger} D_{\rm SF} = \frac{1-\Gamma}{2}\left(-\partial^2 + m^2 - 2m\gamma_0\left(\delta(x_0) - \delta(x_0 - T)\right)\right)\frac{1-\Gamma}{2}$$

$$D_{\rm SF}^{\dagger} D_{\rm SF}\psi = \lambda^2\psi, \quad (1+\Gamma)\psi = 0, \quad \psi(x_0+T) = -\psi(x_0-T)$$

- Define $\psi_{\pm} \equiv \frac{1\pm\gamma_0}{2}\psi$

$$\psi_+(x_0) = -\psi_+(-x_0), \quad \psi_+(T+x_0) = \psi_+(T-x_0)$$

$$\psi_+ = A(\sin p_0 x_0) e^{i\vec{p}\vec{x}}$$
$$\lambda^2 = p_0^2 + \vec{p}^2 + m^2$$



- continuity at $x_0 = 0, T$
- matching of $\partial_0\psi$ at $x_0 = 0, T$

$$-\partial_0\psi_+\Big|_{T-\epsilon}^{T+\epsilon} = -2m\psi_+\Big|_{T-\epsilon}^{T+\epsilon} \quad \Rightarrow \quad \tan p_0 T = -\frac{p_0}{m}$$

$p_0 = \frac{2n+1}{2T}\pi$ for $m = 0$

- Propagator

$$G_{\rm SF} = 2\frac{1-\Gamma}{2}\frac{1}{\not{D}+m\eta(x)}\frac{1+\Gamma}{2}$$

$$D_{\rm SF}G_{\rm SF} = (1+\Gamma)\,\delta(x_0-y_0) = \delta(x_0-y_0) \qquad 0 < x_0, y_0 < T$$

$$(1+\gamma_0)G_{\rm SF}|_{x_0=0} = 0, \qquad (1-\gamma_0)G_{\rm SF}|_{x_0=T} = 0$$

- Massless free propagator

$$G_{\rm SF} = 2\not{D}^\dagger\frac{1+\Gamma}{2}\frac{1}{\not{D}\not{D}^\dagger}\frac{1+\Gamma}{2} = \not{D}^\dagger\left(P_+G_L + P_-G_R\right)$$

$$G_{\rm R}(x_0,y_0) = \frac{1}{2T}\sum_{n=-\infty}^{\infty}\frac{1}{p^2}\left(e^{ip_0(x_0-y_0)} - e^{ip_0(x_0+y_0)}\right)$$

$$G_{\rm L}(x_0,y_0) = \frac{1}{2T}\sum_{n=-\infty}^{\infty}\frac{1}{p^2}\left(e^{ip_0(x_0-y_0)} + e^{ip_0(x_0+y_0)}\right)$$

$$p_0 = \frac{2n+1}{2T}\pi$$

$\Rightarrow$ Coincides with that given by Lüscher-Weisz

§3 Overlap Dirac operator

$$aD = M\left(1 + D_W \frac{1}{\sqrt{D_W^\dagger D_W}}\right) \qquad \text{(Neuberger)}$$

- **time reversal symmetry**

$$\psi \to i\gamma_0\gamma_5 R\psi, \quad \overline{\psi} \to \overline{\psi} i\gamma_0\gamma_5 R$$
$$U_k(x_0) = U_k(-x_0), \quad U_0(x_0) = U_0^\dagger(-x_0 - 1) \quad \text{(reflection)}$$
$$U_k(0) = W_k, \quad U_k(T) = W_k' \quad \text{(SF Dirichlet BC)}$$

- **chiral symmetry (Lüscher)**

$$\psi \to i\hat{\gamma}_5\psi, \quad \overline{\psi} \to \overline{\psi} i\gamma_5, \quad \hat{\gamma}_5 = \gamma_5(1 - aD)$$

- **anti-periodicity** $\psi(x_0 + 2T) = -\psi(x_0)$

- **Orbifolding projection**

$$\psi(x) = -\hat{\Gamma}\psi(x), \quad \overline{\psi}(x) = \overline{\psi}(x)\Gamma, \quad \Gamma = \gamma_0 R, \quad \hat{\Gamma} = \Gamma(1 - aD)$$

- Projection property

$$[i\gamma_5\Gamma, D] = 0, \quad \gamma_5 D + D\gamma_5 = aD\gamma_5 D$$

$$\Rightarrow \Gamma D + D\Gamma = aD\Gamma D, \quad \Gamma^2 = \hat{\Gamma}^2 = 1$$

- Orbifolding

$$(1+\hat{\Gamma})\psi = 0, \quad \overline{\psi}(1-\Gamma) = 0 \quad \Rightarrow \quad \text{SF BC in } a \to 0$$

  - Physical quark $\quad q = (1 - \frac{a}{2}D)\psi, \quad \overline{q} = \overline{\psi}$

$$(1+\Gamma)q = 0, \quad \overline{q}(1-\Gamma) = 0$$

- Orbifolded action

$$S = \sum \overline{\psi} D_{\mathrm{SF}} \psi, \quad D_{\mathrm{SF}} = \frac{1+\Gamma}{2}\left(D + m\eta(x_0)\left(1 - \frac{a}{2}D\right)\right)\frac{1-\hat{\Gamma}}{2}$$

  - No index

$$\mathrm{tr}\Gamma = \mathrm{tr}\hat{\Gamma} = 0 \quad \Leftarrow \quad [i\gamma_5\Gamma, D] = 0$$

$$\frac{1\pm\Gamma}{2} \Leftrightarrow \frac{1\pm\hat{\Gamma}}{2} \quad : \quad \hat{\Gamma} = u^\dagger \Gamma u \quad (u : \text{local, unitary})$$

  - D_{SF} is local

- Eigenvalue problem

$$D_{\rm SF} = \frac{1+\Gamma}{2}(D + m\eta u)\frac{1-\hat{\Gamma}}{2} \quad : \quad \widehat{\mathcal{H}}_+ \to \mathcal{H}_-$$

$$\mathcal{D} = \begin{pmatrix} & D_{\rm SF}^\dagger \\ D_{\rm SF} & \end{pmatrix} \quad : \quad \widehat{\mathcal{H}}_+ \oplus \mathcal{H}_- \to \widehat{\mathcal{H}}_+ \oplus \mathcal{H}_-$$

$$\text{on} \quad \Psi = \begin{pmatrix} \frac{1-\hat{\Gamma}}{2} & \\ & \frac{1+\Gamma}{2} \end{pmatrix} \psi_E \quad \in \widehat{\mathcal{H}}_+ \oplus \mathcal{H}_-$$

  - Eigenvalue

$$\lambda^2 = A_\mu^2 + B^2 + m^2, \; A_\mu = M\frac{\sin p_\mu}{\sqrt{\lambda_W^2}}, \; B = M\left(1 + \frac{W}{\sqrt{\lambda_W^2}}\right)$$

$$\lambda_W^2 = \sin^2 p_\mu + W^2, \quad W = -M + \sum(1 - \cos p_\mu)$$

$$\tan p_0 T = -\frac{A_0}{m}$$

  - In $a \to 0$ $\qquad \lambda^2 \to p_0^2 + \vec{p}^2 + m^2, \quad \tan p_0 T = -\frac{p_0}{m}$

- Free propagator

$$G_{\mathrm{SF}} = 2\frac{1-\hat{\Gamma}}{2}\frac{1}{D}\frac{1+\Gamma}{2} = 2D^{\dagger}\frac{1+\Gamma}{2}\frac{1}{DD^{\dagger}}\frac{1+\Gamma}{2}$$
$$= D^{\dagger}\left(P_{+}G_{L} + P_{-}G_{R}\right)$$

$$G_{\mathrm{R}}(x_0, y_0) = \frac{1}{2N_T}\sum_{n}\frac{1}{DD^{\dagger}(p)}\left(e^{ip_0(x_0-y_0)} - e^{ip_0(x_0+y_0)}\right)$$

$$G_{\mathrm{L}}(x_0, y_0) = \frac{1}{2N_T}\sum_{n}\frac{1}{DD^{\dagger}(p)}\left(e^{ip_0(x_0-y_0)} + e^{ip_0(x_0+y_0)}\right)$$

$$DD^{\dagger}(p) = M^2\frac{\sin^2 p_\mu}{\lambda_W^2} + M^2\left(1 - \frac{W}{\sqrt{\lambda_W^2}}\right)^2$$

$$p_0 = \frac{2n+1}{2N_T}\pi, \quad n = -N_T+1, \cdots, N_T$$

- $a \to 0$ limit

$$G_{\mathrm{SF}} \to G_{\mathrm{SF}}^{(\mathrm{cont.})}$$

- Phase of the determinant
  - No γ_5 Hermiticity $\quad (D_{\rm SF}^\dagger \neq \gamma_5 D_{\rm SF} \gamma_5)$

$$D_{\rm SF} = \frac{1+\Gamma}{2} D \frac{1-\hat{\Gamma}}{2}, \qquad D_{\rm SF}^\dagger = \frac{1-\hat{\Gamma}}{2} D^\dagger \frac{1+\Gamma}{2}$$

$$D_{\rm SF}^\dagger = \gamma_5 u D_{\rm SF} u^\dagger \gamma_5$$

- Hilbert space

$$D_{\rm SF} \; : \; \widehat{\mathcal{H}}_+ \to \mathcal{H}_- \quad \Rightarrow \quad D_{\rm SF} u^\dagger \gamma_5 \; : \; \mathcal{H}_- \to \mathcal{H}_-$$

$$u = \frac{1+i\gamma_5\Gamma}{2}(1-aD) + \frac{1-i\gamma_5\Gamma}{2} \quad : \quad \text{local, unitary}$$

$$D_{\rm SF} u^\dagger \gamma_5 = \frac{1+\Gamma}{2} D u^\dagger \gamma_5 \frac{1+\Gamma}{2}$$

- Determinant in $\mathcal{H}_-$ subspace

$$\det_{\{\mathcal{H}_-\}} \left(D_{\rm SF} u^\dagger \gamma_5 \right) = \det \left(\frac{1+\Gamma}{2} D u^\dagger \gamma_5 \frac{1+\Gamma}{2} + \frac{1-\Gamma}{2} \right)$$

- **Phase of the determinant**

$$D_{\mathrm{SF}}^{\dagger} = \gamma_5 u D_{\mathrm{SF}} u^{\dagger} \gamma_5 \Rightarrow \left(D_{\mathrm{SF}} u^{\dagger} \gamma_5\right)^{\dagger} = (\gamma_5 u)^2 \left(D_{\mathrm{SF}} u^{\dagger} \gamma_5\right)$$

$$e^{i\phi} = \sqrt{\det u} = \prod_{n \in \{+i\gamma_5\Gamma\}} (1 - a\lambda_n)^{1/2}$$

$$u = \frac{1 + i\gamma_5\Gamma}{2}(1 - aD) + \frac{1 - i\gamma_5\Gamma}{2}$$

$$D\psi_n^{(+)} = \lambda_n \psi_n^{(+)}, \qquad (i\gamma_5\Gamma)\psi_n^{(+)} = +\psi_n^{(+)}$$

$$D\gamma_5\psi_n^{(+)} = \lambda_n^* \gamma_5 \psi_n^{(+)}, \qquad (i\gamma_5\Gamma)\gamma_5\psi_n^{(+)} = -\gamma_5\psi_n^{(+)}$$

(a) determinant is complex

(b) however phase is irrelevant $O(a)$

localized at the boundary $(\delta \ln \det u)$

(c) $N_f = 2 \Rightarrow$ divide the det by $\det u \Rightarrow$ real & positive

§4 Domain-wall fermion

$$S = \sum \overline{\psi} \left(\gamma_M D_M - \frac{1}{2} D^2 - M \right) \psi + m_f \overline{q} q$$

$$q = \frac{1 - \gamma_5}{2} \psi_1 + \frac{1 + \gamma_5}{2} \psi_{N_5}$$

- time reversal symmetry

$$\psi \to i\gamma_0\gamma_5 RP\psi, \quad \overline{\psi} \to \overline{\psi} i\gamma_0\gamma_5 RP, \quad P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

- chiral symmetry (Furman-Shamir)

$$\psi \to -iQ\psi, \quad \overline{\psi} \to \overline{\psi} iQ, \quad Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- anti-periodicity $\psi(x_0 + 2T) = -\psi(x_0)$

- Orbifolding projection

$$\psi(x) = A\psi(x), \quad \overline{\psi}(x) = \overline{\psi}(x)A, \quad A = \gamma_0\gamma_5 PQR$$
$$q(x) = -\Gamma q(x), \quad \overline{q}(x) = \overline{q}(x)\Gamma, \quad \Gamma = \gamma_0 R$$

- Orbifolded action

$$S = \sum \overline{\psi} D_{\mathrm{SF}} \psi, \quad D_{\mathrm{SF}} = \frac{1+A}{2} D_{\mathrm{dwf}} \frac{1+A}{2}$$

  - γ_5 Hermiticity

$$D_{\mathrm{SF}}^{\dagger} = \frac{1+A}{2} D_{\mathrm{dwf}}^{\dagger} \frac{1+A}{2} = P\gamma_5 D_{\mathrm{SF}} \gamma_5 P$$

- Quark propagator

$$\langle q\overline{q}\rangle_{\mathrm{SF}} = \frac{1-\Gamma}{2} \langle q\overline{q}\rangle \frac{1+\Gamma}{2} \Rightarrow (N_5 \to \infty) \Rightarrow \frac{1-\Gamma}{2} \frac{2}{D_{\mathrm{OV}}} \frac{1+\Gamma}{2}$$

- Overlap Dirac fermion propagator

$$G_{\mathrm{SF}}^{\mathrm{OD}} = \frac{1-\Gamma}{2} \frac{2}{D_{\mathrm{OV}}} \frac{1+\Gamma}{2} + a\frac{1+\Gamma}{2} = 2\frac{1-\hat{\Gamma}}{2} \frac{1}{D_{\mathrm{OV}}} \frac{1+\Gamma}{2}$$

§5 Conclusion

- A potential problem of zero mode with Dirichlet BC.
- Dirichlet BC in D_W does not work for OD and DWF.
- A criterion to introduce Dirichlet BC in $D_{\rm OV}$, $D_{\rm dwf}$
  - chiral symmetry (GW relation) breaking

$$(1+\gamma_0)\psi|_0 = 0, \quad (1-\gamma_0)\psi|_T = 0$$

- field theory with BC ⇔ orbifolded field theory
  - time reversal symmetry
  - chiral symmetry (massless theory)
  - anti-periodicity

- Orbifolded action

$$S = \sum \overline{\psi} D_{\rm SF}\psi, \quad D_{\rm SF} = \frac{1+\Gamma}{2}\left(D + m\eta(x_0)\left(1-\frac{a}{2}D\right)\right)\frac{1-\hat{\Gamma}}{2}$$

  - SF Dirichlet BC
  - Classical level, eigenvalue and propagator

- (technical) Problem
  - **(a)** determinant is complex
  - **(b)** however phase is irrelevant $O(a)$
    localized at the boundary $(\delta \ln \det u)$
  - **(c)** $N_f = 2 \Rightarrow$ divide the det by $\det u \Rightarrow$ real & positive