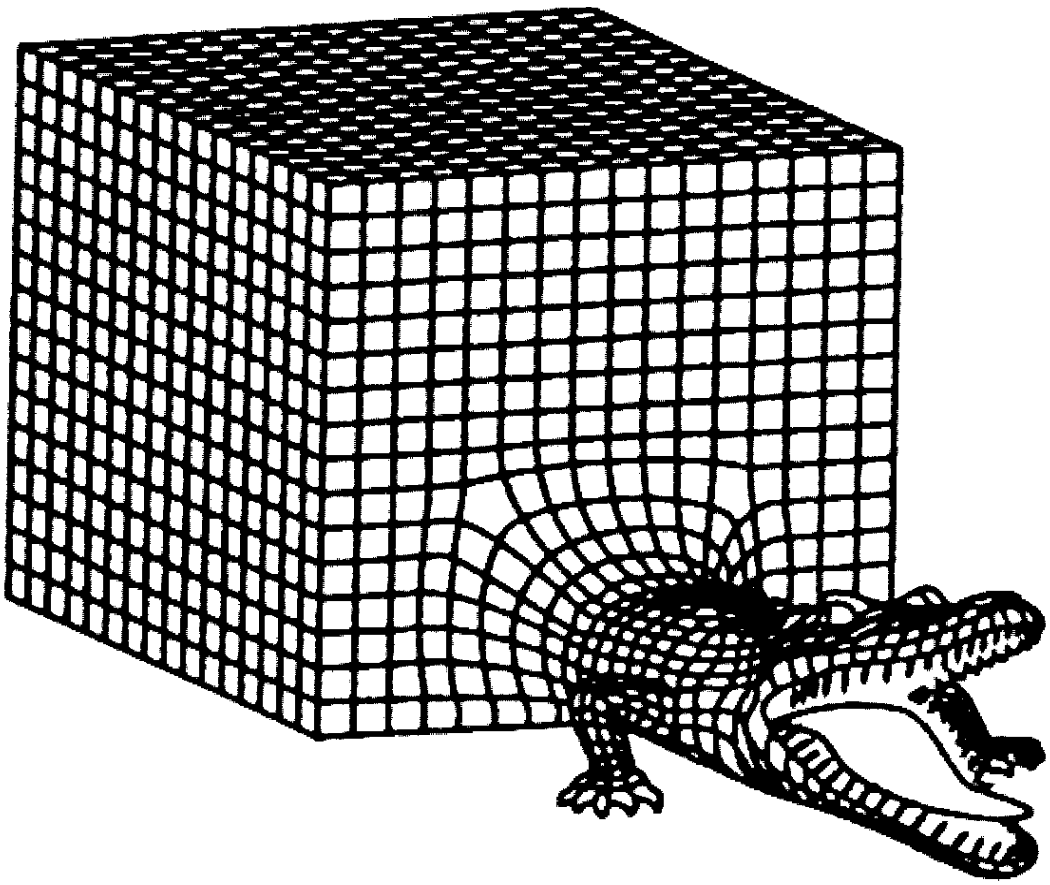
Going chiral:
overlap versus Wilson twisted mass fermions

Karl Jansen

- **Why new lattice actions**
- **Scaling with Wilson twisted mass fermions**
  extended scaling test → A. Shindler
- **Comparison of overlap and Wilson twisted mass fermions**
  - approach to the chiral limit
  - simulation cost
- **Conclusions**

There are dangerous lattice animals



→ discretization errors

→ chiral symmetry

→ computational cost

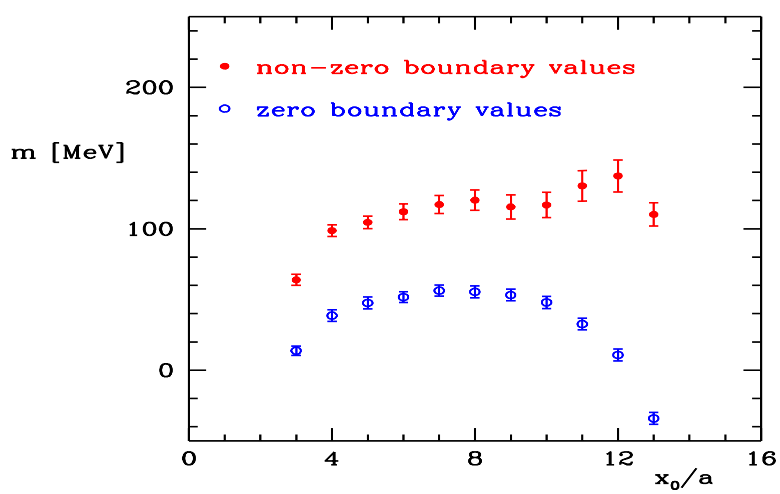
Discretization Errors ↔ violation of chiral symmetry

Wilson-Dirac operator

$$D_{\mathrm{W}} = m_q + \tfrac{1}{2}\gamma_\mu \left[\nabla_\mu + \nabla^*_\mu\right] - ar\tfrac{1}{2}\nabla^*_\mu\nabla_\mu$$

$$\nabla_\mu\psi(x) = \tfrac{1}{a}\left[U(x,\mu)\psi(x+a\hat{\mu}) - \psi(x)\right]\ ,\ U = e^{iag_0A_\mu(x)}$$

$$\nabla^*_\mu\psi(x) = \tfrac{1}{a}\left[\psi(x) - U(x-a\hat{\mu},\mu)\psi(x-a\hat{\mu})\right]$$



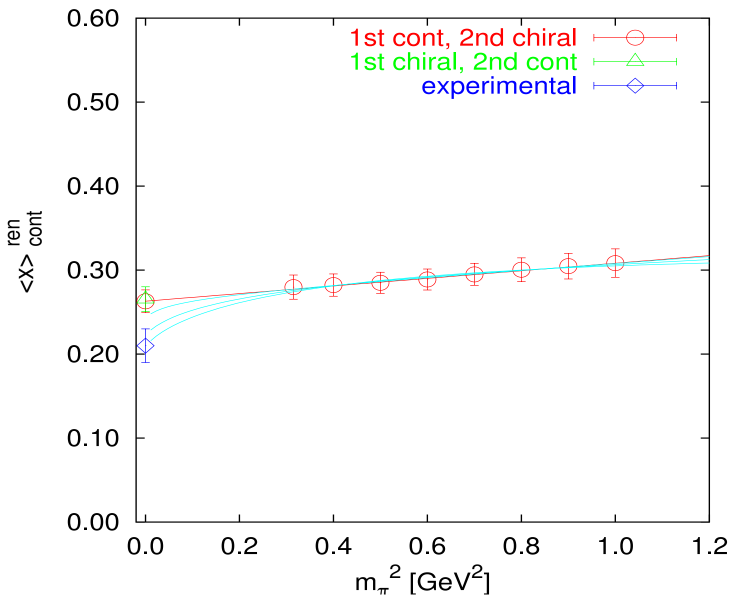
ALPHA Collaboration

→ two definitions of quark mass

→ large discretization effect linear in the lattice spacing

Chiral extrapolation

ZeRo collaboration (Guagnelli, K.J., Palombi, Petronzio, Shindler, Wetzorke)
perform continuum extrapolation of a twist-2, non-singlet pion operator



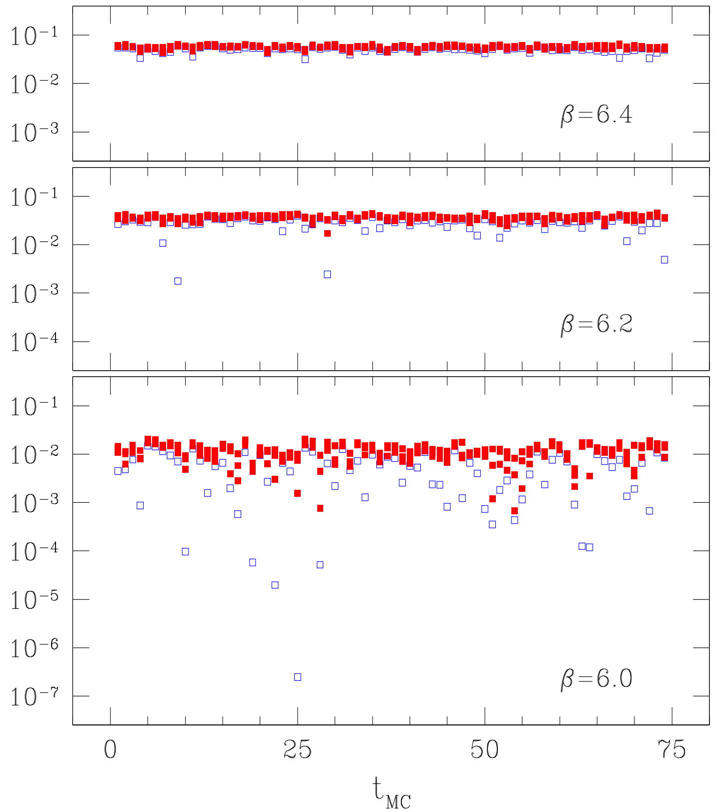
linear continuum extrapolation $a \rightarrow 0$
combined fit of Wilson and
Clover data
4 lattice spacings at fixed m_π^2

chiral extrapolation $m \rightarrow 0$

- linear: $\langle x \rangle_{\overline{\rm MS}} = 0.263(14)$
- non-linear: $\langle x \rangle_{\overline{\rm MS}} = 0.221^{+10+21}_{-\ 9-13}$

$\Rightarrow$ no data in interesting region of pion mass

Why are we not able to reach realistic pion mass?



in addition to
natural slowing down as $m_\pi \longrightarrow 0$

- very small eigenvalues
- slow down algorithms
- spoil signal

$\longrightarrow$ problem appears for Wilson fermions

$\longrightarrow$ similar (maybe worse) for O(a)-improved Wilson fermions

(staggered fermions: size of systematic error) solution: **Ginsparg-Wilson relation**

$$\boxed{\gamma_5 D + D\gamma_5 = 2aD\gamma_5 D}$$

$$\Rightarrow D^{-1}\gamma_5 + \gamma_5 D^{-1} = 2a\gamma_5$$

D^{-1} anti-commutes with γ_5 at all non-zero distances

$\longrightarrow$ only mild (i.e. local) violation of chiral symmetry

Ginsparg and Wilson arrived at this expression already
in the early days of lattice gauge theories from a completely different path
$\Leftarrow$ block spinning from the continuum

one solution of GW relation: overlap operator D_{ov} (Neuberger)
(alternatives: domain wall fermions and perfect actions)

$$\boxed{D_{\mathrm{ov}} = \left[1 - A(A^\dagger A)^{-1/2}\right]}$$

with $A = 1 + s - D_{\mathrm{w}}(m_q = 0)$; s a tunable parameter, $0 < s < 1$

Adding a mass term (naive definition)

$$D_{\rm ov} = m_q + \left[1 - A(A^{\dagger}A)^{-1/2}\right]$$

$\longrightarrow$ exact (lattice) chiral symmetry at $m_q = 0$

$\longrightarrow$ infrared safe: quark mass m_q
$\Leftarrow$ can reach very small quark masses

$\longrightarrow$ O(a)-improved

$\longleftarrow$ computationally very demanding,

O(10-100) more expensive than standard Wilson fermions

Nevertheless: exist problems for overlap fermions: ϵ-regime of chiral perturbation theory, complicated operator mixings

Wilson (Frezzotti, Rossi) twisted mass QCD (Frezzotti, Grassi, Sint, Weisz)

equivalent Wilson-Dirac operator

$$D_{\rm W} = m_q e^{i\omega\gamma_5\tau^3} + \frac{\gamma_\mu}{2}\left[\nabla_\mu + \nabla_\mu^*\right] - a\frac{r}{2}\nabla_\mu^*\nabla_\mu + M_{\rm cr} \equiv m_q e^{i\omega\gamma_5\tau^3} + D_{\rm crit} + D_{\rm r}$$

$\omega = 0 \rightarrow$ usual Wilson-Dirac operator

$\omega = \pi/2 \rightarrow i\gamma_5\tau_3 m_q$: twisted mass term at zero quark mass

under parity, $R_5 = e^{i\omega\gamma_5\tau^3}$

- $D_{\rm crit}$ invariant
- $m_q e^{i\omega\gamma_5\tau^3} \rightarrow -m_q e^{i\omega\gamma_5\tau^3}$
- $D_{\rm r} \rightarrow -D_{\rm r}$

$\Rightarrow D_{\rm W}$ invariant under $R_5 \times (r \rightarrow -r) \times (m_q \rightarrow -m_q)$

Wilson twisted mass QCD

it can be shown than

$$\langle \mathcal{O} \rangle|_{(m_q,r)} = (-1)^P \, \langle \mathcal{O} \rangle|_{(-m_q,-r)}$$

- action invariant under R_5
- change of integration measure compensated by the parity transformation of operator

Wilson twisted mass QCD

Symanzik expansion

$$\langle \mathcal{O} \rangle|_{(m_q,r)} = [\xi(r) + am_q\eta(r)] \langle \mathcal{O} \rangle|^{\rm cont}_{m_q} + a\chi(r) \langle \mathcal{O}' \rangle|^{\rm cont}_{m_q}$$

$$\langle \mathcal{O} \rangle|_{(-m_q,-r)} = [\xi(-r) - am_q\eta(-r)] \langle \mathcal{O} \rangle|^{\rm cont}_{m_q} + a\chi(-r) \langle \mathcal{O}' \rangle|^{\rm cont}_{-m_q}$$

in addition

$$\langle \mathcal{O} \rangle|^{\rm cont}_{m_q} = (-1)^P \langle \mathcal{O} \rangle|^{\rm cont}_{-m_q}$$

$$\Rightarrow \quad \xi(r) = +\xi(-r)\ ,\ am_q\eta(r) = -am_q\eta(-r)\ ,\ a\chi(r) = -a\chi(-r)$$

$$\Rightarrow \quad \frac{1}{2}\left[\langle \mathcal{O} \rangle|_{m_q,r} + \langle \mathcal{O} \rangle|_{-m_q,-r}\right] = \xi(r) \langle \mathcal{O} \rangle|^{\rm cont}_{m_q} + O(a^2)$$

$\Rightarrow$ at $m_q = m_0 - M_{\rm cr} = 0$ the Wilson average is O(a) improved

Wilson twisted mass QCD

$$\langle \mathcal{O} \rangle|_{m_q,r,\omega=\pi/2} = \frac{1}{2}\left[\langle \mathcal{O} \rangle|_{m_q,r,\omega=\pi/2} + \langle \mathcal{O} \rangle|_{m_q,-r,\omega=\pi/2}\right]$$

$\Rightarrow$ choose $\omega = \pm\pi/2$ and bare quark mass the critical quark mass

$\Rightarrow$ all quantities even in $\omega = \pm\pi/2$ are automatic O(a) improved

Examples:

- hadron masses
- matrix elements
- form factors
- decay constants

Hopping paramter representation

$$\chi \longrightarrow \frac{\sqrt{2\kappa}}{a^{3/2}}\chi, \quad \bar{\chi} \longrightarrow \frac{\sqrt{2\kappa}}{a^{3/2}}\bar{\chi}, \quad \kappa = \frac{1}{2am_0+8r}$$

$$\begin{aligned} S[\chi,\overline{\chi},U] &= \sum_x \Big\{\overline{\chi}(x)\Big(1+2ia\mu\kappa\gamma_5\tau_3\Big)\chi(x) \\ &- \kappa\overline{\chi}(x)\sum_{\mu=1}^{4}\Big(U(x,\mu)(r+\gamma_\mu)\chi(x+a\hat{\mu}) \\ &+ U^\dagger(x-a\hat{\mu},\mu)(r-\gamma_\mu)\chi(x-a\hat{\mu})\Big)\Big\} \end{aligned}$$

$\longrightarrow$ maximal twist: $\kappa \longrightarrow \kappa_{\mathrm{crit}}$

Frezzotti, Rossi: κ_{crit} to be known to O(a)

we take κ_{crit} from pion mass intercept

Aoki, Bär criticism $\longrightarrow$ talk by S. Aoki

$\longrightarrow$ talk by A. Shindler

Field rotations and bilinears

field rotations

$$\psi(x) \equiv e^{i\frac{\omega}{2}\gamma_5\tau_3}\chi(x) = \left(\cos\tfrac{\omega}{2} + i\gamma_5\tau_3\sin\tfrac{\omega}{2}\right)\chi(x)$$

$$\overline{\psi}(x) \equiv \overline{\chi}(x)e^{i\frac{\omega}{2}\gamma_5\tau_3} = \overline{\chi}(x)\left(\cos\tfrac{\omega}{2} + i\gamma_5\tau_3\sin\tfrac{\omega}{2}\right)$$

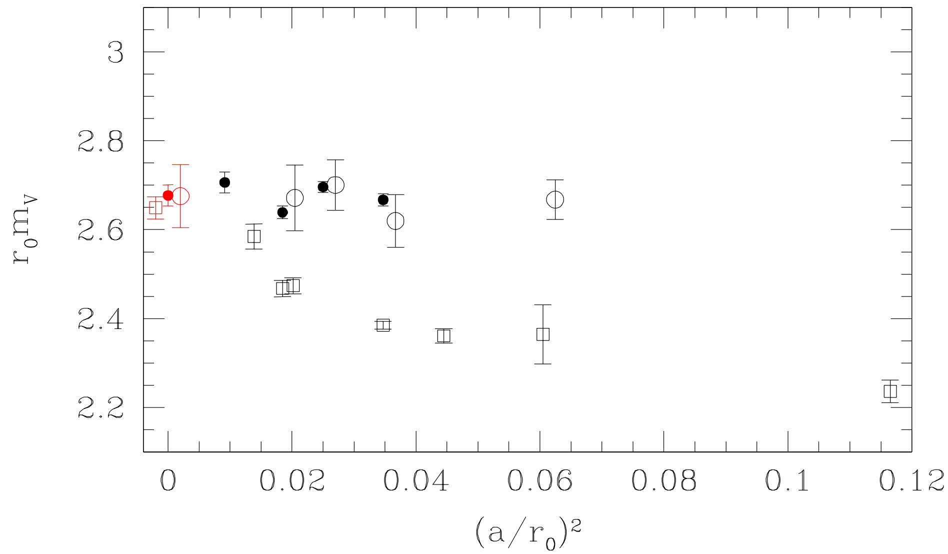
bilinears example: axial and vector currents
"physical basis" (unprimed quantities)
"twisted basis" (primed quantities)

$$\begin{aligned} A'^{\alpha}_{\mu} &= \cos(\omega)A^{\alpha}_{\mu} + \epsilon^{3\alpha\beta}\sin(\omega)V^{\beta}_{\mu}(\alpha = 1,2) \\ &= A^{3}_{\mu}(\alpha = 3) \end{aligned}$$

$$\begin{aligned} V'^{\alpha}_{\mu} &= \cos(\omega)V^{\alpha}_{\mu} + \epsilon^{3\alpha\beta}\sin(\omega)A^{\beta}_{\mu}(\alpha = 1,2) \\ &= V^{3}_{\mu}(\alpha = 3) \end{aligned}$$

Scaling test at heavy mass

K.J., A. Shindler, C. Urbach, I. Wetzorke



$D_{\rm crit}$ ($\kappa_{\rm crit}$) determined with pure Wilson fermions correlation function

$$f_P^\alpha(t) = \sum_{\vec{x}} \langle P^\alpha(x) P^\alpha(0) \rangle$$

$$r_0 m_\pi = 1.79 \approx 750\mathrm{MeV}$$

open circles: twisted mass
open squares: standard Wilson
filled symbols: O(a) improved Wilson

Pion decay constant

definition of F_{PS}: $\langle 0|A_0^1|PS\rangle = m_{\mathrm{PS}}F_{\mathrm{PS}}$

twisted basis, at $\omega = \pi/2$, the axial current is related to the vector current

$$\partial_\mu \langle 0|V_\mu^2|\mathrm{PS}\rangle = F_{\mathrm{PS}}m_{\mathrm{PS}}^2 \ .$$

vector ward identity implies $Z_P = Z_\mu^{-1}$

$$F_{\mathrm{PS}}m_{\mathrm{PS}}^2 = \partial_\mu \langle 0|V_\mu^2|\mathrm{PS}\rangle = 2\mu_q \langle 0|P^1|PS\rangle$$

For asymptotic Euclidean times, the pseudoscalar correlation function f_P^1 assumes the form

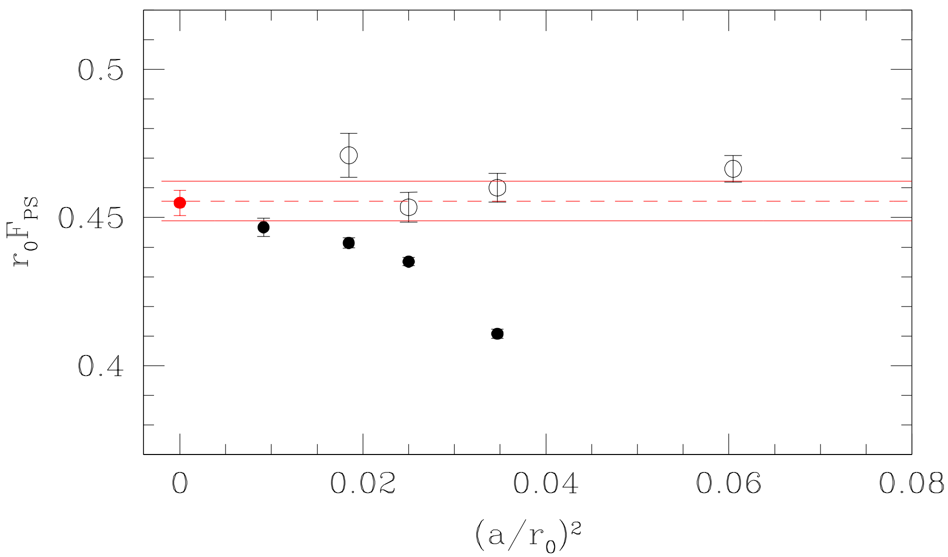
$$f_P^1(t) = \frac{|\langle 0|P^1|\mathrm{PS}\rangle|^2}{2m_{\mathrm{PS}}} \cdot \left(e^{-m_{\mathrm{PS}}t} + e^{-m_{\mathrm{PS}}(T-t)}\right) \ , a \ll t \ll T \ .$$

fitting the pseudo scalar correlation function for large time separation, obtain the pseudoscalar mass and
$|\langle 0|P^1|\mathrm{PS}\rangle|^2/m_{\mathrm{PS}}$

from which we get $|\langle 0|P^1|\mathrm{PS}\rangle|$.

Pion decay constant

scaling result and comparison to O(a) improved Wilson fermions



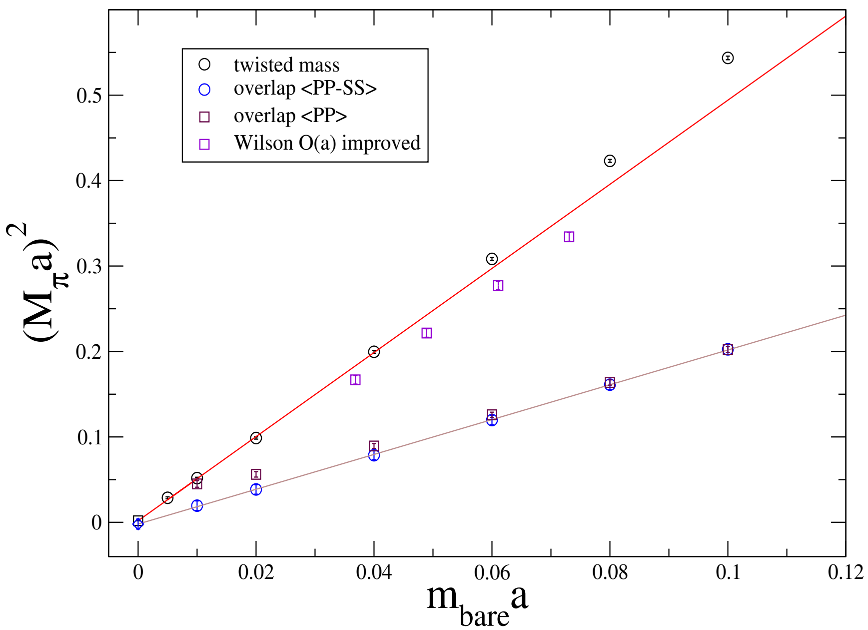
open symbols: twisted mass

filled symbols: O(a) improved Wilson

twisted mass against overlap fermions: pion mass

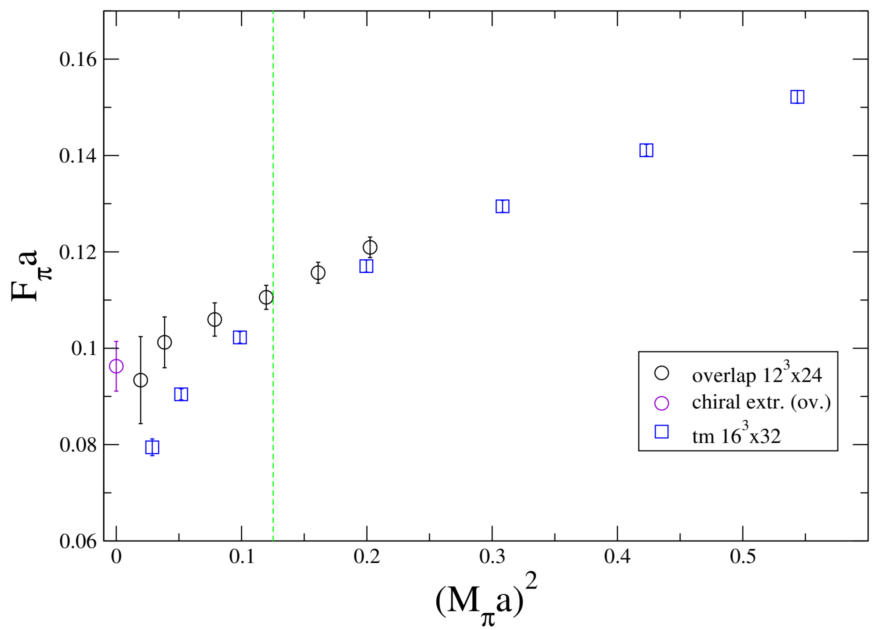
W. Bietenholz, T. Chiarappa, M. Hasenbusch, K.J., K. Nagai, M. Papinutto, L. Scorzato, S. Shcheredin, A. Shindler, C. Urbach, U. Wenger, I. Wetzorke

here: only one $\beta = 5.85$



$\Rightarrow$ twisted mass simulations can reach quarks masses as small as overlap substantially smaller than O(a)-improved Wilson fermions

twisted mass against overlap fermions: pseudoscalar decay constant



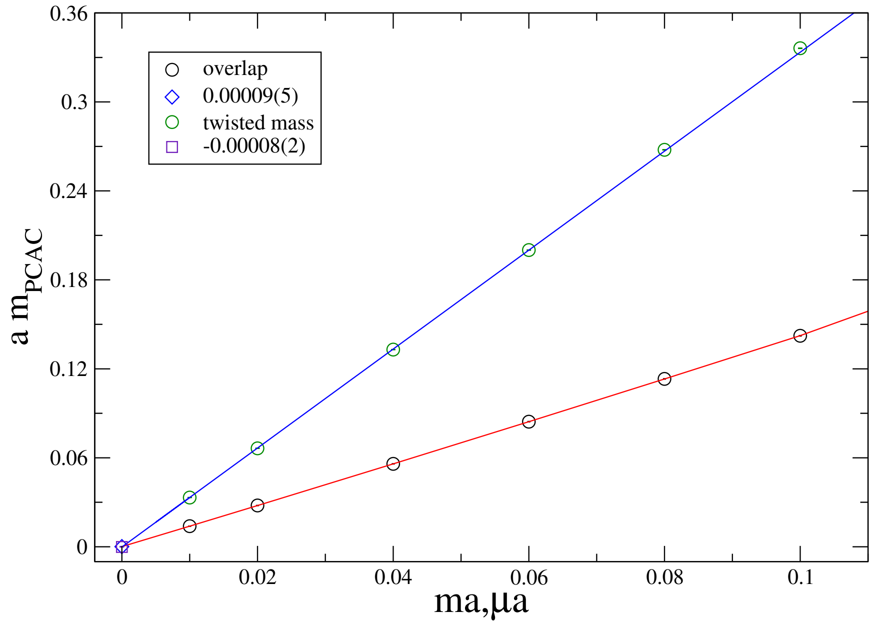
⇒ competition of Wilson term and twisted mass term

$$m_q \Lambda_{\mathrm{QCD}}^{-1} \gg a^2 \Lambda_{\mathrm{QCD}}^2$$

⇒ check continuum limit

PCAC quark masses

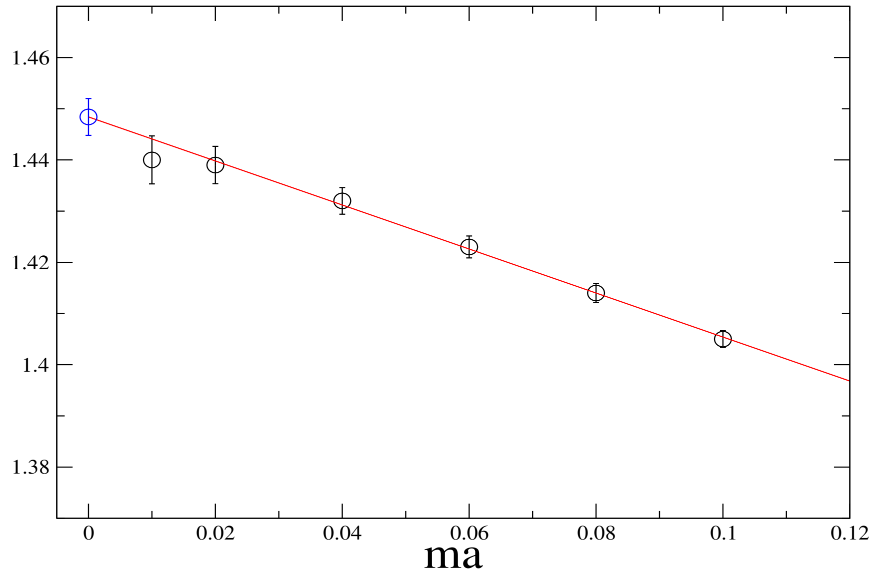
$$m_{\rm PCAC}^{\rm ov} = \frac{\sum_{\mathbf{x}}\langle \partial_0 A_0^a(x)\ P^a(0)\rangle}{\sum_{\mathbf{x}}\langle P^a(x) P^a(0)\rangle}\ ,\ m_{\rm PCAC}^{\rm tm} = \frac{\epsilon^{3ab}\sum_{\mathbf{x}}\langle \partial_0 V_0^b(x)\ P^a(0)\rangle}{\sum_{\mathbf{x}}\langle P^a(x) P^a(0)\rangle} \quad a = 1,2$$



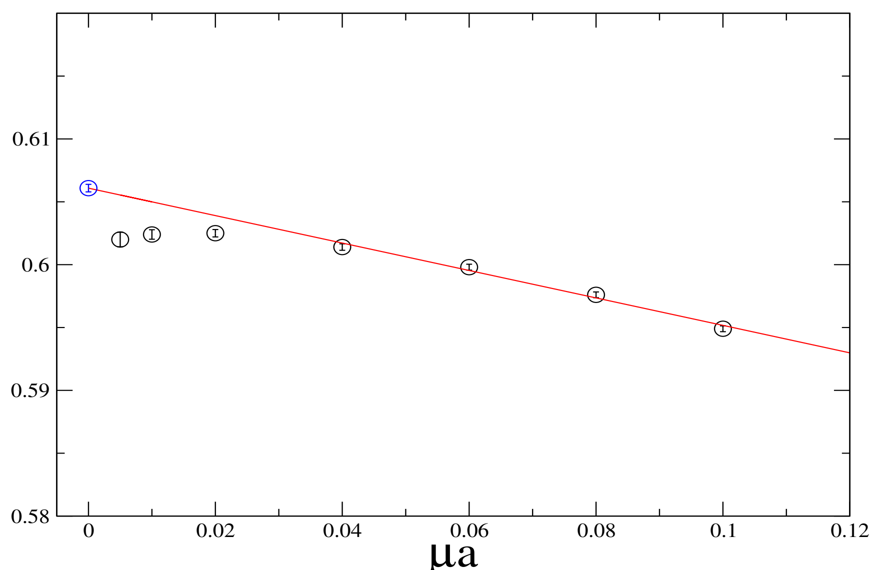
$\Rightarrow$ perfectly linear behaviour (?)

Renormalization constants

compute $m_{\mathrm{PCAC}}/m_{\mathrm{bare}}$

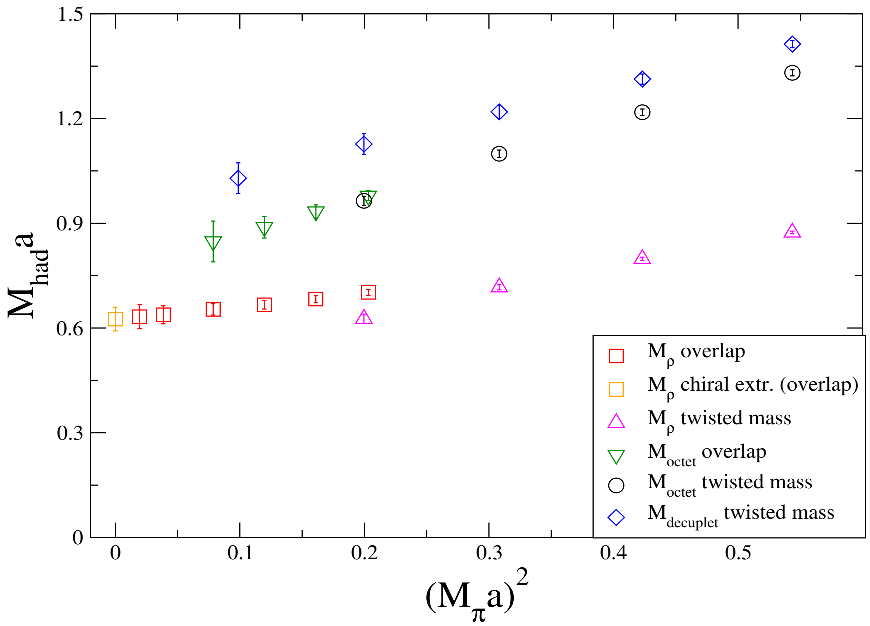


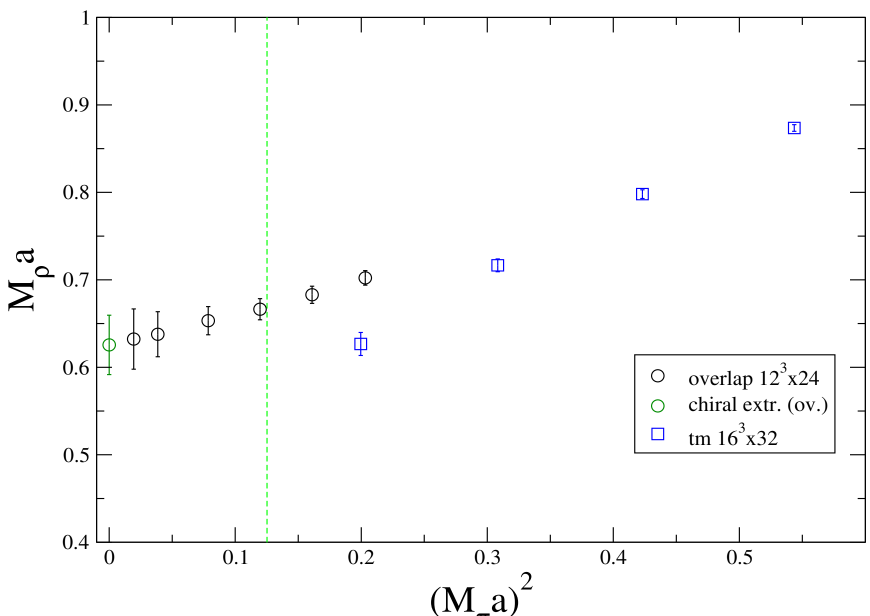
overlap data



twisted mass data

twisted mass against overlap fermions: m_ρ and baryon masses





- for too small $(m_\pi a)$:

– signal noisy
– difficult to extract ground state

Cost comparison

T. Chiarappa, K.J., K. Nagai, M. Papinutto, L. Scorzato,
A. Shindler, C. Urbach, U. Wenger, I. Wetzorke

- testing different solvers
  CG, MR, CGS, GMRES, SUMR

- testing pion masses $m_\pi = 720$MeV, $m_\pi = 390$MeV and $m_\pi = 250$Mev
  also one larger mass

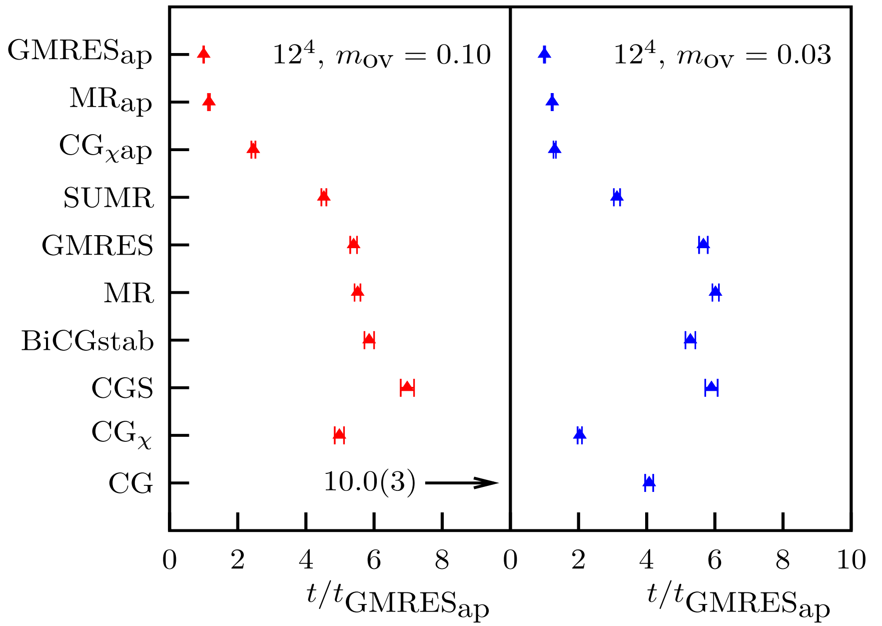
- lattices 12^4, 16^4

- variety of algorithmic improvements
  – adaptive precision
  – chiral projection
  – eigenvalue deflation
  – multiple mass solver

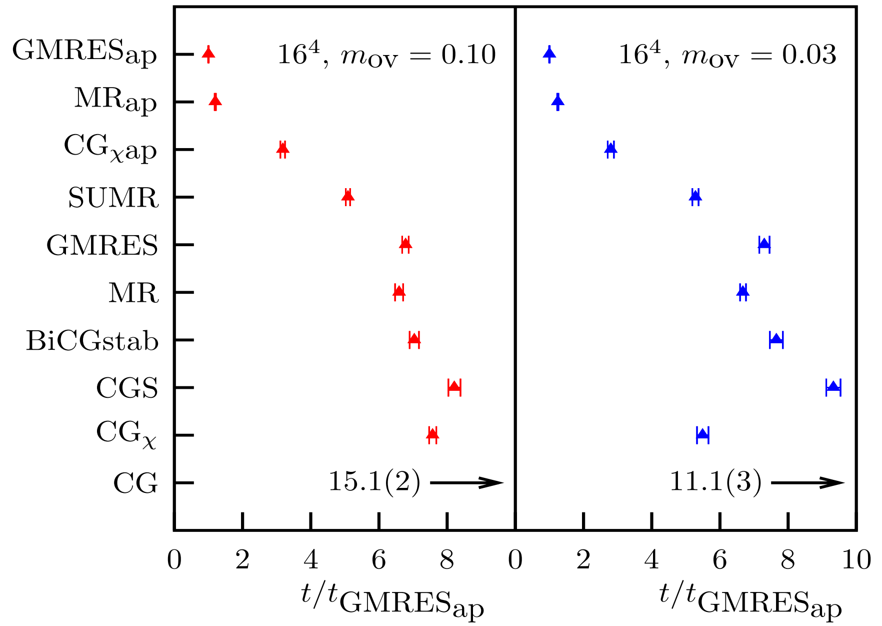
- missing: preconditioning (running)

- tested also eigenvalue computation

Best algorithm for overlap fermions

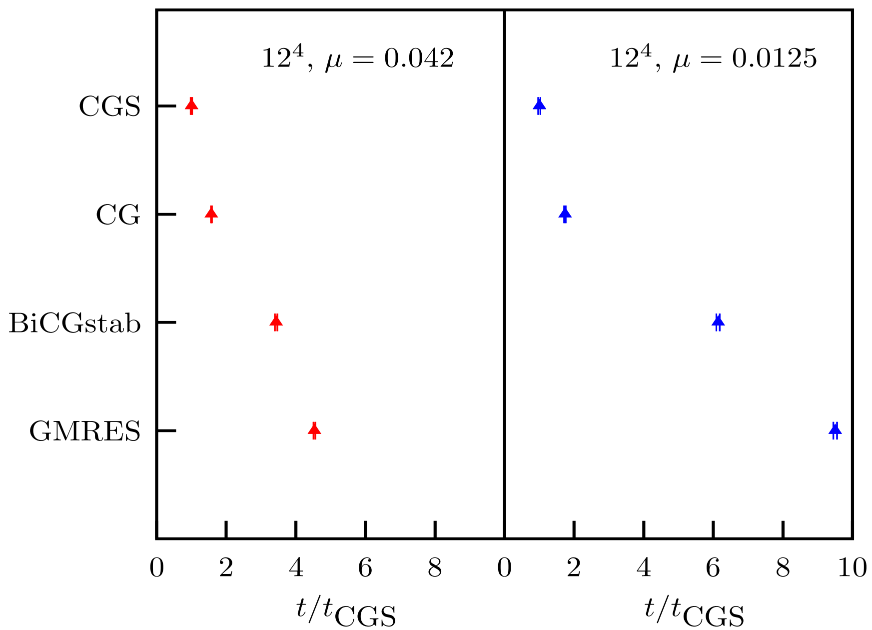


12^4

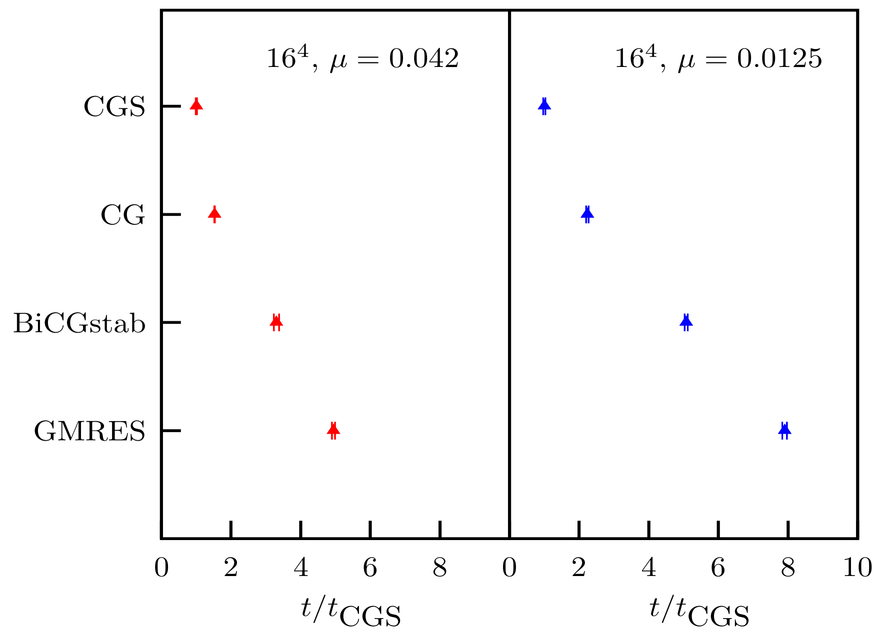


16^4

Best algorithm for twisted mass fermions

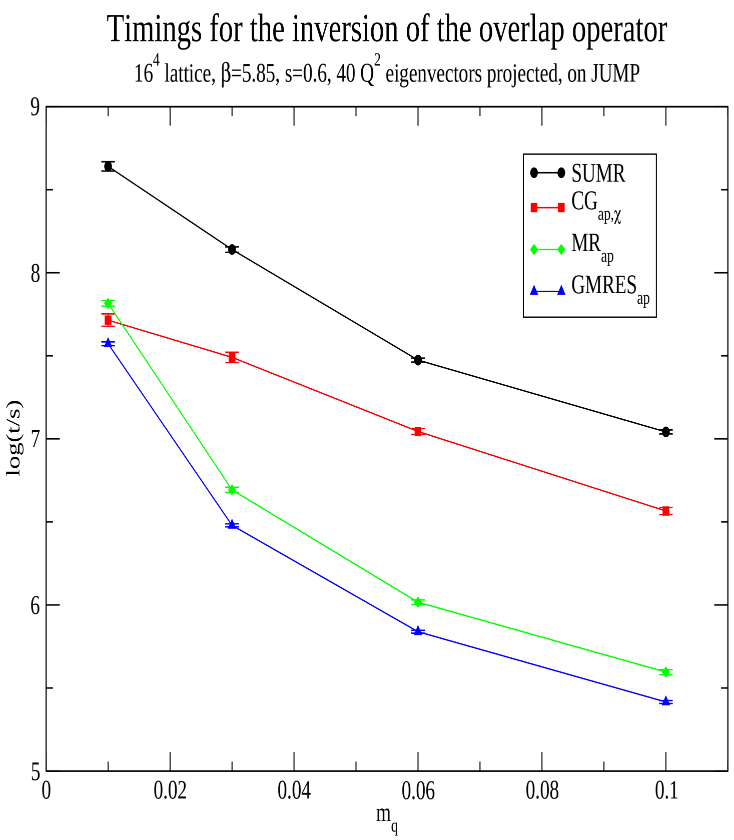


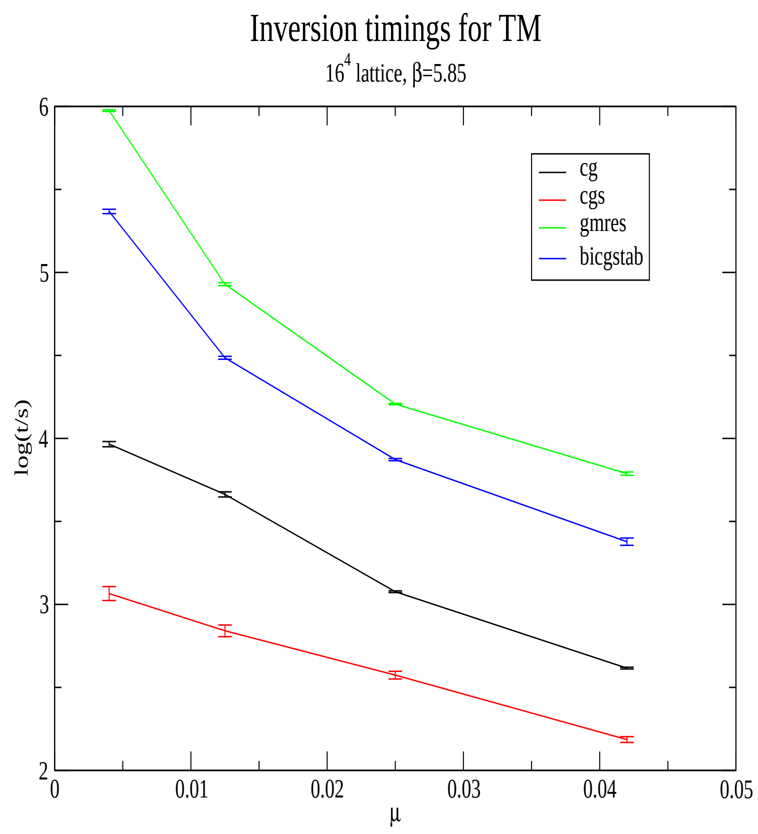
12^4



16^4

Timings as function of quark mass





Cost comparison

V, m_π	Overlap	TM	rel. factor
$12^4, 720$Mev	48.8(6)	2.6(1)	18.8
$12^4, 390$Mev	142(2)	4.0(1)	35.4
$16^4, 720$Mev	225(2)	9.0(2)	25.0
$16^4, 390$Mev	653(6)	17.5(6)	37.3
$16^4, 250$Mev	1949(22)	22.1(8)	88.6

Best absolute timings in seconds on one node of JUelich MultiProzessor (JUMP) IBM p690 Regatta in Juelich

find: Wilson twisted mass fermions are **20-80** cheaper than overlap fermions

Conclusion

★ overlap and Wilson twisted mass fermions

$\longrightarrow$ O(a)-improved

$\longrightarrow$ safe against exceptionally small eigenvalues
$\Rightarrow$ allow for simulations at small quark masses

- pion mass as small as 230 MeV
- twisted mass: unexpected behaviour of, e.g., f_π at very small quark mass

$\Rightarrow$ conceptual versus practical advantages