$I = 2$  Pion Scattering Length
from Two - Pion Wave Function

at ILFT04 2004/09/22
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with CP-PACS Collaboration

We find following :

1. Lattice calculation of two-pion wave function for ground state.
   POSSIBLE

2. Calculation of the scattering length from wave function.
   POSSIBLE with small statistical error

Report was presented at Lat04(FNAL).
1. Introduction

Why do we calculate scattering length in lattice?

1. First step of study of hadron decay based on QCD.

\[ \text{ex}) \quad \rho \to \pi\pi, \quad \sigma \to \pi\pi \]

Lattice calc.

Why do we calculate scattering length in lattice?

1. First step of study of hadron decay based on QCD.

\[ \text{ex}) \quad \rho \to \pi\pi, \quad \sigma \to \pi\pi \]

$L=1$ pion SC. phase shift

\[ m_\rho = 762.4 \pm 1.8 \text{ MeV} \]
\[ \Gamma_\rho = 145.2 \pm 2.8 \text{ MeV} \]

Hyams et al.
Protopopescu et al.

phase of the form factor

Lattice calc.?
2. Preparation of calculation of $K \rightarrow \pi\pi$ matrix elements
by Lellouch - Lüscher formula.

amp. on finite volume : $A^L(m_K, p_\pi)$
amp. in infinite volume : $A(m_K, p_\pi)$

$$|A(m_K, p_\pi)|^2 = 32\pi^2 \cdot \left(\frac{m_K^2}{p_\pi}\right) \cdot \rho(p_\pi) \cdot |A^L(m_K, p_\pi)|^2$$

$$\rho(p_\pi) = \left(\frac{\partial \delta_0(p_\pi)}{\partial p_\pi} + \frac{\partial \phi(p_\pi)}{\partial p_\pi}\right) \cdot \frac{2\sqrt{m_\pi^2 + p_\pi^2}}{4\pi p_\pi}$$

$\delta_0(p_\pi)$ : S-wave pion SC. phase shift
$\phi(p_\pi)$ : spherical zeta function

: Lellouch - Lüscher formula

This was derived from Lüscher’s quantization condition.
See also C.-J.D. Lin, et.al. NPB619(01)465.

Investigation of final state interaction in two-pion system
is necessary before calc. of weak matrix elements.
Previous works of $I=2$ pion scattering:

Sharpe, Gupta and Kilcup ‘92
Gupta, Patel and Sharpe ‘93
Kuramashi ‘93
JLQCD ‘99
Liu, Zhang, Chen, Ma ‘01
CP-PACS ‘02 (phase shift)
Juge ‘03
Kim ‘03 (phase shift)
CP-PACS ‘03 (phase shift, Nf=2)
BGR ‘04

Method: Finite size method by Lüscher
Results of $I=2$ SC. length


**Quench approximation**

<table>
<thead>
<tr>
<th>$a_0 / m_\pi$ ($1/\text{GeV}^2$)</th>
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<tbody>
<tr>
<td>-3.0</td>
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**Legend:**
- JLQCD (LIN)
- JLQCD (EXP)
- Liu et.al. (Schem I)
- Liu et.al. (Schem II)

**Note:**
- CHPT Prediction
- Std. Wilson
- Std. PL
- Imp. action on anisotropic lattice
Results of $I=2$ SC. phase shift

Yamazaki et.al., CP-PACS collaboration, hep-lat/0402025
Imp. Wilson with perturbative CSW + RG imp. gauge

Full QCD Nf = 2, in continuum limit
**Finite size method**

In $L \times L \times L$ periodic box

Energy of **one-pion**:

$$E = \sqrt{m^2 + k^2} \quad k^2 = \left(\frac{2\pi}{L}\right)^2 \cdot n, \quad n \in \mathbb{Z}$$

Energy of **two-pion**:

$$E = 2 \cdot \sqrt{m^2 + k^2} \quad k^2 = \left(\frac{2\pi}{L}\right)^2 \cdot n, \quad n \notin \mathbb{Z}$$

$$\frac{1}{\tan \delta_0(k)} = \frac{4\pi}{k} \cdot \frac{1}{L^3} \sum_{n \in \mathbb{Z}^3} \frac{1}{p_n^2 - k^2} \quad p_n = n \cdot \left(\frac{2\pi}{L}\right) \quad : \text{SC. length in infinite volume}

: Lüscher’s formula

for ground state $k^2 \sim 0$

$$\frac{\tan \delta_0(k)}{k} = a_0 + O(k^2) \quad : \text{SC. length in infinite volume}$$
In previous works, we obtained energy of two-pion state from

\[ \langle 0 | (\pi\pi)(t) (\pi\pi)(0) | 0 \rangle \sim e^{-E \cdot t} \]

in large time region

: time correlation function
Derivation of Lüscher’s formula

In infinite volume

Let’s consider two-pion wave function defined by

\[ \phi_\infty(x) = \langle 0 | \pi(x/2) \pi(-x/2) | \pi(k), \pi(-k) \rangle \]

\[ |\pi(k), \pi(-k)\rangle : \text{asymptotic state} \]

\[ : \text{Behten-Salpeter amplitude} \]

\[ \phi_\infty(x) = \int \frac{d^3p}{(2\pi)^3 2E_p} e^{ip \cdot x/2} \langle \pi(p) | \pi(-x/2) | \pi(k), \pi(-k) \rangle \]

\[ + \langle \pi \pi \pi | \pi(-x/2) | \pi(k), \pi(-k) \rangle + \cdots \]

\[ E_p = \sqrt{m_\pi^2 + p^2} \]

We consider only case \[ 2 \cdot \sqrt{k^2 + m_\pi^2} \ll 4 \cdot m_\pi \]

and neglect contribution from in-elastic scattering.
\[ e^{-i \mathbf{p} \cdot \mathbf{x}/2} \cdot \langle \pi(p) | \pi(-\mathbf{x}/2) | \pi(k) \pi(-k) \rangle = 2E_p (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{k}) + \frac{H(p; k)}{p^2 - k^2 - i\epsilon} \]

Disconnected part

Connected part

where we assume

\[ \delta_l \sim 0 \quad \text{with} \quad l \geq 1 \]

so \( H(p; k) \) is depend only on

\[ p = |\mathbf{p}| \quad \text{and} \quad k = |\mathbf{k}| \]
\[
\phi_\infty(x) = e^{ik \cdot x} + \int \frac{d^3p}{(2\pi)^3} \frac{H(p; k)}{p^2 - k^2 - i\epsilon} e^{ip \cdot x}
\]

\[
= e^{ik \cdot x} + \frac{ik}{4\pi} H(k; k) j_0(kx) + \mathbf{P} \int \frac{d^3p}{(2\pi)^3} \frac{H(p; k)}{p^2 - k^2} j_0(px)
\]

\[j_0(kx) : \text{spherical Bessel}\]

\[
H(k; k) = \frac{4\pi}{k} \sin \delta_0(k) e^{i\delta_0(k)}
\]

: on shell two-pion scattering amplitude

Here we assume

\[
h(x; k) \equiv \int \frac{d^3p}{(2\pi)^3} H(p; k) e^{-ip \cdot x} = -(\nabla^2 + k^2)\phi_\infty(x) = 0
\]

for \(R < x\)

\[
R : \text{two-pion interaction range}
\]
In free region $R < x$

$$\phi_{\infty}(x) = e^{i k \cdot x} + \frac{ik}{4\pi} H(k; k) j_0(kx) + P \int \frac{d^3 p}{(2\pi)^3} \frac{H(p; k) - H(k; k)}{p^2 - k^2} j_0(px)$$

$$E(x; k) \equiv P \int \frac{d^3 p}{(2\pi)^3} \frac{H(p; k) - H(k; k)}{p^2 - k^2} e^{i p \cdot x}$$

$$= \int_x^\infty dz (4\pi z^2) \cdot h(z; k) \cdot \frac{k}{4\pi} \left( j_0(kx)n_0(kz) - n_0(kx)j_0(kz) \right)$$

$$= 0 \quad \text{for} \quad R < x \leq z$$

**formula:**

$$\frac{k}{4\pi} n_0(kx) = P \int \frac{d^3 p}{(2\pi)^3} \frac{j_0(px)}{p^2 - k^2}$$

$$= P \int \frac{d^3 p}{(2\pi)^3} \frac{e^{i p \cdot x}}{p^2 - k^2}$$
in free region \( R < x \)

\[
\phi_\infty(x) = e^{ik \cdot x} + \frac{ik}{4\pi} H(k; k) j_0(kx) + \mathcal{P} \int \frac{d^3 p}{(2\pi)^3} \frac{H(p; k)}{p^2 - k^2} j_0(px)
\]

\[
= e^{ik \cdot x} + \frac{ik}{4\pi} H(k; k) j_0(kx) + H(k; k) \mathcal{P} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{p^2 - k^2} j_0(px)
\]

\[
= e^{ik \cdot x} + \frac{ik}{4\pi} H(k; k) j_0(kx) + \frac{k}{4\pi} H(k; k) n_0(kx)
\]

\[
H(k; k) = \frac{4\pi}{k} \sin \delta_0(k) e^{i\delta_0(k)}
\]

: on shell two-pion scattering amplitude
On finite volume

Let’s consider two-pion wave function defined by

\[ \phi(x) = \langle 0 | \pi(x/2)\pi(-x/2) | \pi\pi; k \rangle \]

\[ |\pi\pi; k \rangle : \text{energy eigen state with} \]

\[ E = 2 \cdot \sqrt{m_\pi^2 + k^2} \]

Assumption: Two-pion interaction range: \( R < L/2 \)

In free region: \( R < x < L/2 \)

\[ (\nabla^2 + k^2)\phi(x) = 0 \quad : \text{Helmholtz Eq.} \]
Solution of Eq. :  \( \text{ ( neglecting } \delta_l \text{ with } l \geq 1 ) \)

\[
\phi(x) = C \cdot G(x; k)
\]

\[
G(x; k) = \sum_{n \in \mathbb{Z}^3} (p_n^2 - k^2)^{-1} e^{i p_n \cdot x} \quad p_n = n(2\pi)/L
\]

: periodic Green's function

\[
\phi(x) \text{ can be written by linear combination of }
\]

\[
\phi(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} C_{lm} \cdot \phi_{lm}(x)
\]

\[
\phi_{lm}(x) = \int d\Omega \ Y_{lm}(\Omega) \phi_\infty(x)
\]

\begin{align*}
\text{In free region} & \quad R < x < L/2 \\
\text{wave function on finite volume} & \quad \phi(x) \\
\text{wave function in infinite volume} & \quad \phi_\infty(x)
\end{align*}
In free region $R < x < L/2$

- $\phi(x) = C \cdot G(x; k)$

$$= C \cdot \left[ \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sqrt{4\pi} Y_{lm}(\Omega_x) \cdot g_{lm}(k) j_l(kx) + \frac{k}{4\pi} n_0(kx) \right]$$

- $\phi_\infty(x) = e^{ik \cdot x} + \frac{ik}{4\pi} H(k; k) j_0(kx) + \frac{k}{4\pi} H(k; k) n_0(kx)$

By comparing S-wave comp., we get

$$C \cdot \left[ g_{00}(k) j_0(kx) + \frac{k}{4\pi} n_0(kx) \right] = C_{00} \cdot \left[ (1 + \frac{ik}{4\pi} H(k; k)) j_0(kx) + \frac{k}{4\pi} H(k; k) n_0(kx) \right]$$

$$g_{00}(k) \cdot H(k; k) = (1 + \frac{ik}{4\pi} H(k; k)) \quad \left( H(k; k) = \frac{4\pi}{k} \sin \delta_0(k) e^{i\delta_0(k)} \right)$$

: Lüscher's formula
Now we consider:

1. Two-pion wave function plays an important role.
   But, it does not appear in traditional method.

   Calculation of wave function in the lattice is possible?

2. Lüscher’s formula can be applied,
   if two-pion interaction is localized,
   so that interaction range: \( R < L/2 \)

   Estimation of \( R \) is possible?
   How large?

3. Scattering length must be calculated from the wave function.

   Is it possible in the lattice?

Similar work has been carried for 2-dim XY-model
by J. Balog, et.al. NPB618(01)315.

Same idea has been discussed
by C.-J. Lin, et.al. NPB619(01)465.
2. Method of calculation of wave function

In this work,
we restrict to ground state of \( I=2 \) two-pion

\[ k^2 \sim 0 \quad \text{ie. scattering length} \]

Definition of wave function

\[
\phi(x) = \sum_X \sum_R \langle 0 | \pi(R[x] + X), t \rangle \pi(X), t \rangle |\pi\pi; k\rangle \times e^{2E_k \cdot t} \]

\( R \) : element of cubic group

\( |\pi\pi; k\rangle \) : ground state of two-pion with energy

\[
2E_k = 2 \cdot \sqrt{m^2_{\pi} + k^2}
\]

\[ \sum_X \] : zero total momentum projection

\[ \sum_R \] : projection to irreducible rep. \( A_1^+ \) of cubic group

\[
A_1^+ \sim S\text{-wave upto } l \geq 4
\]
Calculation method of wave function

\[ G(x, t) = \sum_X \sum_R \langle 0 | \pi(R[x] + X, t) \pi(X, t) \ W(t_0) \ W(t_0 + 1) | 0 \rangle \]

\[ W(t_0) = \sum_x \bar{q}(x, t_0) \gamma_5 \sum_y q(y, t_0) \ : \ "\text{wall source}" \]

in large \( t \) region

\[ G(x, t) \sim \phi(x) \times \langle \pi\pi; k | W((t_0)W(t_0 + 1)|0) \times e^{-2E_k \cdot t} \]

so \[ \phi(x) = \frac{G(x, t)}{G(x_0, t)} \]

up to over all cont.
3. Simulation parameters

- Quench approximation
- RG improve gauge action
- Improve Wilson fermion action with perturbative $C_{SW}$

\[
\frac{1}{a} = 1.207 \text{ GeV} \\
a = 0.1632 \text{ fm}
\]

- Lattice size and # of conf.:

  Time extent: $T = 80$

  \[
  \begin{array}{cccc}
  L & 16 & 20 & 24 \\
  La & 2.61 \text{ fm} & 3.26 \text{ fm} & 3.92 \text{ fm} \\
  #C & 1200 & 1000 & 506 \\
  \end{array}
  \]

- Quak masses

  \[
  m_{\pi}^2 \text{ (GeV}^2) = 0.27, 0.35, 0.44, 0.59, 0.74
  \]

- Wall source position: $t_0 = 12$

  Coulomb gauge fix
4. Results of two-pion wave function

\[ \phi(x) \text{ at } t = 52, \; z = 0 \; \quad 24^3, \; m_\pi^2 = 0.27 \text{ GeV} \]
There are some “branches”. Why?

If \( \phi(x) = \phi(x = |x|) \)

then \( \phi(x) \) does not satisfy periodic BC.,

because \( \phi(x) \neq 0 \) at boundary.
$24^3$, $m_\pi^2 = 0.27 \text{ GeV}$

\[ \nabla^2 \phi(x)/\phi(x) \sim -k^2 \]

Signal is very clear!!

Very strong repulsive potential at origin
$24^3$, $m_\pi^2 = 0.27$ GeV $\nabla^2 \phi(x)/\phi(x) \sim -k^2$

stable for $t \geq 44$
We can apply the formula for all quark masses.

\[ m_p^2 = 0.736 \text{ GeV}^2 \]

\[ m_p^2 = 0.444 \text{ GeV}^2 \]

\[ m_p^2 = 0.273 \text{ GeV}^2 \]

Interaction range \( R \) is larger for larger quark mass.

Worst case (largest quark mass)

\[ R \sim 9.8 \text{ (1.6 fm)} < (L/2 = 12) \]

Criterion of \( R \):

\[ U(x) = 0 \text{ within Stat. Err. for } R < |x| < L/2 \]

without any assumptions for two-pion interaction

\[ U(x) \equiv \frac{(\nabla^2 + k^2)\phi(x)}{k^2\phi(x)} \sim 0 \]

from two-pion time correlation function
The validity of the necessary condition of the formula is not supported for these masses. We can not clearly show a zero region in $|x| < L/2$ for larger quark masses.

$U(x) \equiv \left( \nabla^2 + k^2 \right) \phi(x) \approx 0$

$m^2_\pi (\text{GeV}^2) = 0.27, 0.35, 0.44, 0.59, 0.74$
\[ m^2_{\pi} (\text{GeV}^2) = 0.27, 0.35, 0.44, 0.59, 0.74 \]

We can not clearly show zero region in \(|x| < L/2\) for larger quark masses.

Validity of necessary condition of the formula is not supported for these masses.
4. Scattering length from wave function

Procedure:

(1) Extracting \( k^2 \) by

(a) from

\[-k^2 = \nabla^2 \phi(x)/\phi(x)\]

(b) from fitting wave function with periodic Green’s function

\[\phi(x) = A \cdot G(x; k)\]

- fitting parameters: \( A, k^2 \)

fitting range: \( R \leq x \leq \{\max x = \sqrt{3L/2}\} \)

(2) Substituting \( k^2 \) into Lüscher’s formula

\[\frac{\tan \delta_0(k)}{k} = a_0 + O(k^2)\]
Results of fitting wave function

\[ \phi(x) \text{ at } t = 52^{24} \; 3, \; m^{2}_{\pi} = 0.27 \text{ GeV} \]

\[ \phi(x) \text{ at } t = 52 \]

\[ m^{2}_{\pi} = 0.27 \text{ GeV} \]

Fitted very well!!
Results of SC. length

We can not clearly show zero region of $U(x)$ for these masses

- $-1.8$
- $-1.6$
- $-1.4$
- $-1.2$
- $-1.0$
- $-0.8$

from time correlation function (by traditional method)

- from $-k^2 = \nabla^2 \phi(x)/\phi(x)$ ($t = 52$)

- from $\phi(x) = A \cdot G(x; k)$ ($t = 52$)

Consistent with results given by traditional method.

Statistical error is smaller!!
5. Summary

We calculate two-pion wave function of ground state of \( I=2 \) two-pion.

We find:

(1) Calculation is possible and not difficult.

(2) Scattering length can be obtained from wave function. Statistical error is small.

(3) Two-pion interaction range can be estimated from wave function. Interaction range is larger for larger quark mass.

For \( m^2_\pi \leq 0.74 \text{ GeV}^2 \)

Necessary volume: \( L > 24 \) (4fm)

As shown, necessary volume is smaller for lighter quark mass.

The two-pion interaction does not exactly equals to zero in the quantum field theory. There is exponential tail in potential at large distance. But we do not consider it.

But ...... , How can we estimate it?
(4) We find no significant volume dependence for all quark masses on $L > 16$ (2.6 fm), including data for which are not satisfied with necessary condition of Lüscher’s formula.

We can not clearly show zero region of $U(x)$ for these masses.

Accident? Is deformation of two-pion interaction negligible?
Extensions of this works:

(1) Scattering phase shift for $I=2$ two-pion system.
   Wave function has “node” in this case.

(2) $I=0$ and $I=1$ two-pion system.
   What happens at resonance point?

(3) Estimation of contribution from in-elastic scattering.
   If it is not small, Lüscher’s formula can not be applied.
   
   $$
   \phi(x) = \langle 0|\pi(0)|\pi \rangle \langle \pi|\pi(x)|\pi \pi \rangle + \langle 0|\pi(0)|\pi\pi\pi \rangle \langle \pi\pi\pi|\pi(x)|\pi\pi \rangle + \cdots
   $$
   $\pi\pi \rightarrow \pi\pi$
   $3\pi \rightarrow 3\pi, 4\pi \rightarrow 2\pi$
   
   We can access this question by estimating each terms.

(4) Two Nucleon system.
   
   $$\alpha^{(1S_0)} = -23.714 \text{ fm}$$
   
   Is lattice calculation possible?
   We can access this question from wave function.

(5) Pentaquark?
   Using wave function to separate
   Pentaquak and $K+N$ scattering states.
\( \frac{a_0}{m_\pi} \) (1/GeV\(^2\)) on \( 24^3 \)

\[
(a_0/m_\pi) = A + B m_\pi^2 \log(m_\pi^2) + C m_\pi^2 \\
A = -2.39(16)
\]

\[
(a_0/m_\pi) = A + B m_\pi^2 + C m_\pi^4 \\
A = -2.117(83)
\]

CHPT = -2.265(51)