$I = 2$ Pion Scattering Length from Two - Pion Wave Function

at ILFT04 2004/09/22

Naruhito Ishizuka (University of Tsukuba)
with CP-PACS Collaboration

We find following :

1. Lattice calculation of two-pion wave function for ground state.

   POSSIBLE

2. Calculation of the scattering length from wave function.

   POSSIBLE with small statistical error

Report was presented at Lat04(FNAL).

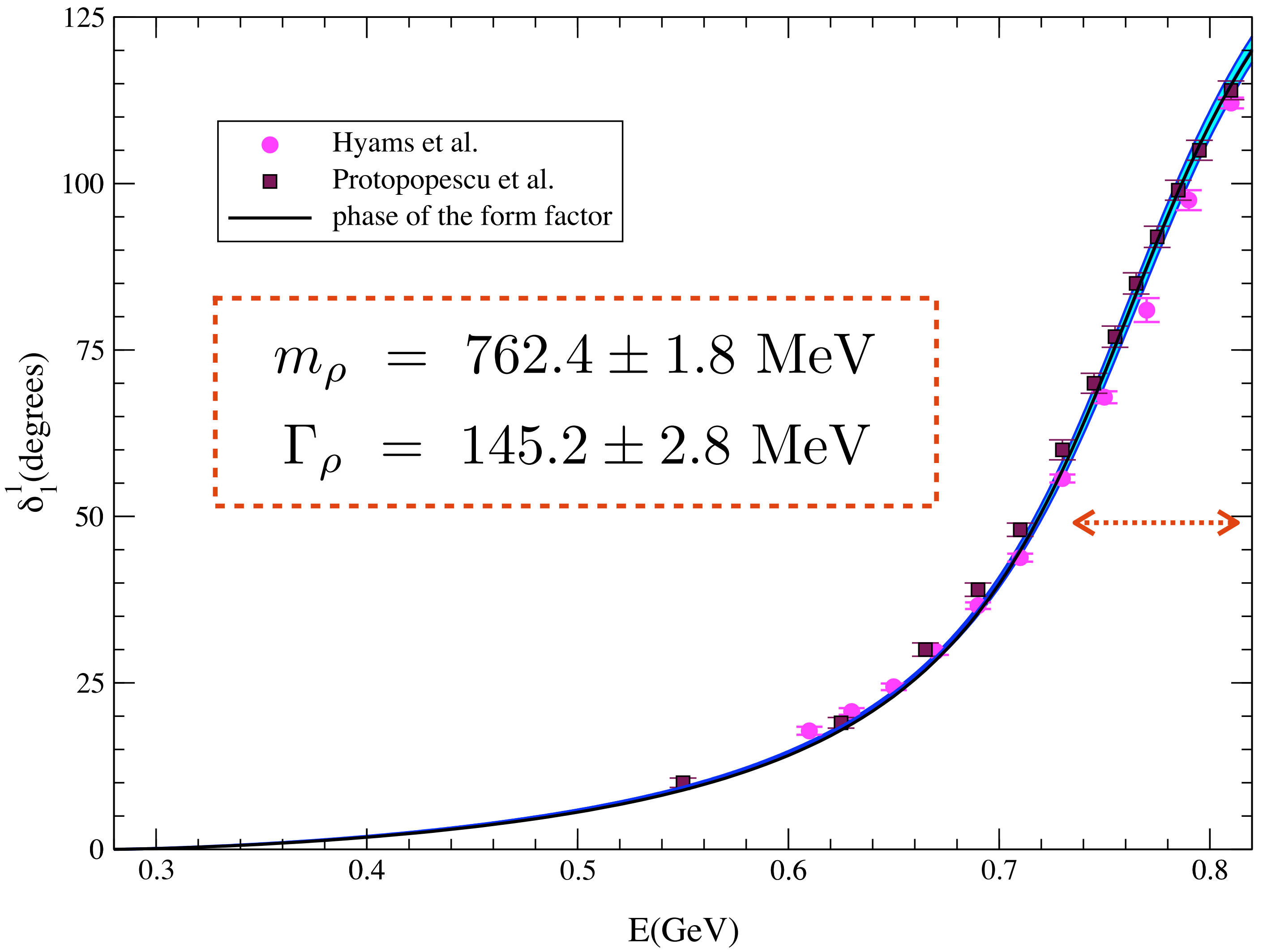
1. Introduction

Why do we calculate scattering length in lattice ?

1. First step of study of hadron decay based on QCD.

ex) $\rho \longrightarrow \pi\pi \quad , \quad \sigma \longrightarrow \pi\pi$

I=1 pion SC. phase shift



Lattice calc. ?

2. Preparation of calculation of $K \to \pi\pi$ matrix elements by Lellouch - Lüscher formula.

amp. on finite volume : $A^L(m_K, p_\pi)$

amp. in infinite volume : $A(m_K, p_\pi)$

$$|A(m_K, p_\pi)|^2 = 32\pi^2 \cdot \left(\frac{m_K^2}{p_\pi}\right) \cdot \rho(p_\pi) \cdot |A^L(m_K, p_\pi)|^2$$

$$\rho(p_\pi) = \left(\frac{\partial \delta_0(p_\pi)}{\partial p_\pi} + \frac{\partial \phi(p_\pi)}{\partial p_\pi}\right) \cdot \frac{2\sqrt{m_\pi^2 + p_\pi^2}}{4\pi p_\pi}$$

$\delta_0(p_\pi)$: S-wave pion SC. phase shift

$\phi(p_\pi)$: spherical zeta function

: Lellouch - Lüscher formula

This was derived from Lüscher's quantization condition.
Lellouch and Lüscher CMP219(2001)31.
See also C.-J.D. Lin, et.al. NPB619(01)465.

Investigation of final state interaction in two-pion system is necessary before calc. of weak matrix elements.

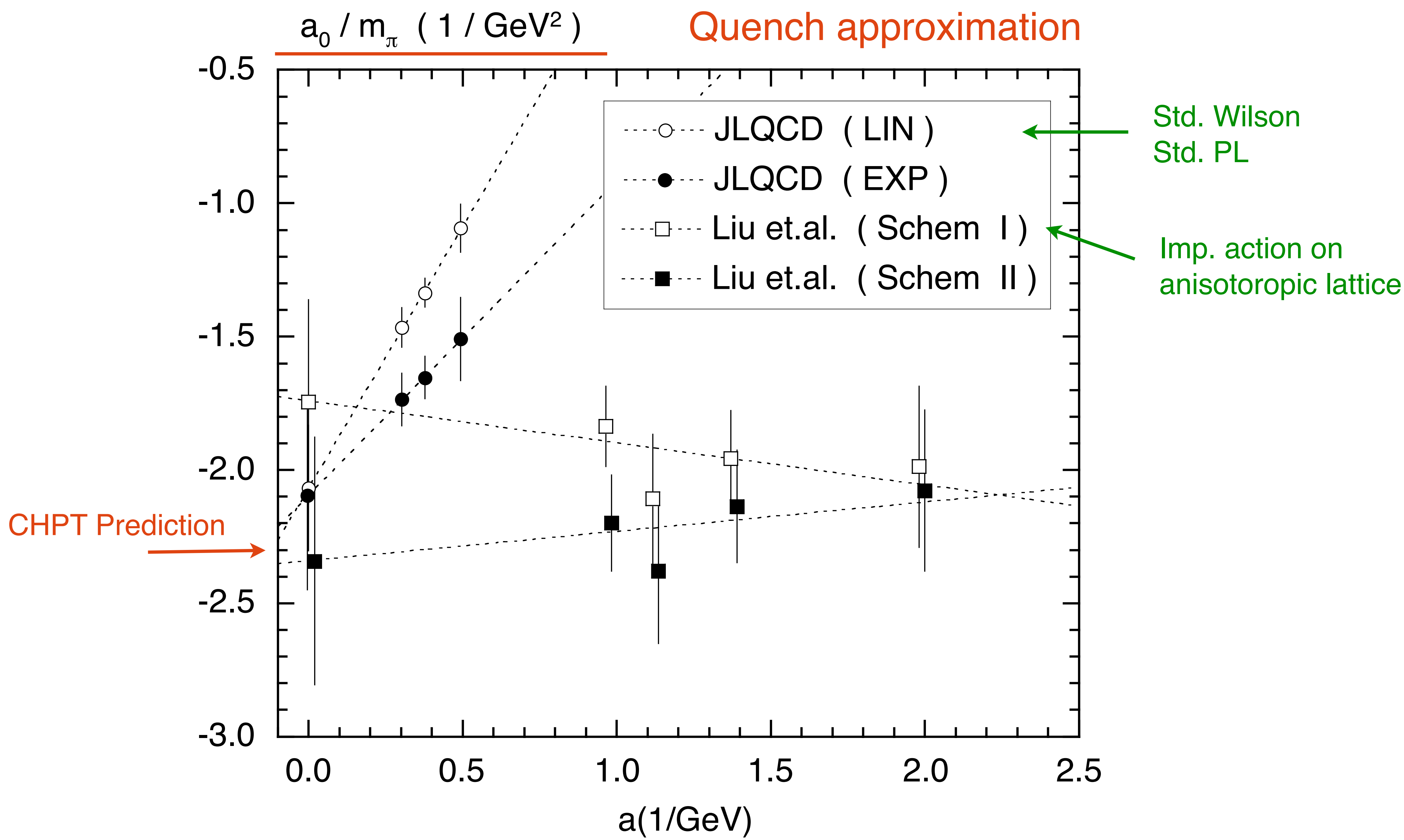
Previous works of $I=2$ pion scattering :

Sharpe, Gupta and Kilcup ‘92
Gupta, Patel and Sharpe ‘93
Kuramashi ‘93
JLQCD ‘99
Liu, Zhang, Chen, Ma ‘01
CP-PACS ‘02 (phase shift)
Juge ‘03
Kim ‘03 (phase shift)
CP-PACS ‘03 (phase shift, Nf=2)
BGR ‘04

Method : Finite size method by Lüscher

Results of $I=2$ SC. length

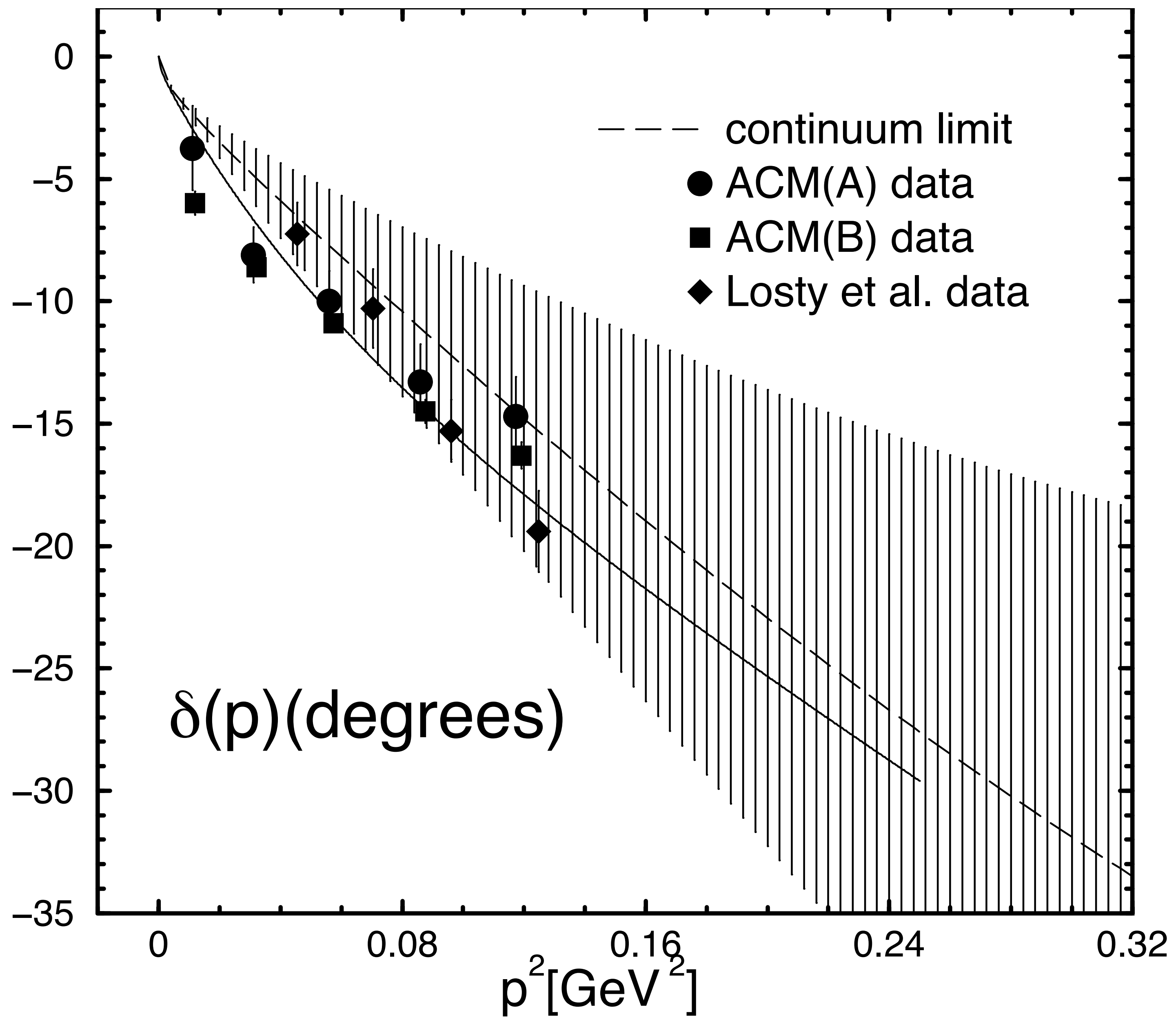
JLQCD collaboration, PRD66(2002)077501.
Liu et.al., NPB624(2002)360.



Results of $I=2$ SC. phase shift

Yamazaki et.al. , CP-PACS collaboration, hep-lat/0402025
Imp. Wilson with perturbative CSW + RG imp. gauge

Full QCD Nf = 2 , in continuum limit



Finite size method

Lüscher CMP105(86)153, NPB354(91)531

In L x L x L periodic box

Energy of one-pion :

$$E = \sqrt{m_\pi^2 + k^2} \qquad k^2 = (2\pi/L)^2 \cdot n \quad , \quad n \in \mathcal{Z}$$

Energy of two-pion :

$$E = 2 \cdot \sqrt{m_\pi^2 + k^2} \qquad k^2 = (2\pi/L)^2 \cdot n \quad , \quad n \notin \mathcal{Z}$$

$$\frac{1}{\tan \delta_0(k)} = \frac{4\pi}{k} \cdot \frac{1}{L^3} \sum_{\mathbf{n} \in \mathcal{Z}^3} \frac{1}{p_n^2 - k^2} \qquad \mathbf{p}_n = \mathbf{n} \cdot (2\pi)/L$$

: SC. phase shift in infinite volume

: Lüscher's formula

for ground state $k^2 \sim 0$

$$\frac{\tan \delta_0(k)}{k} = a_0 + O(k^2)$$

: SC. length in infinite volume

SC. phase shift or SC. length

<== Energy eigenvalue of two pion state on finite volume

In previous works,
we obtained energy of two-pion state from

$$\langle\ 0 \mid (\pi\pi)(t)\ (\pi\pi)(0) \mid 0\rangle \ \sim\ \mathrm{e}^{-E\cdot t}$$

in large time region

: time correlation function

Derivation of Lüscher's formula

Lüscher CMP105(86)153, NPB354(91)531
See also C.-J.D. Lin, et.al. NPB619(01)465.

In infinite volume

Let's consider two-pion wave function defined by

$$\phi_\infty(\mathbf{x}) \;=\; \langle 0|\pi(\mathbf{x}/2)\pi(-\mathbf{x}/2)\;|\pi(\mathbf{k}),\pi(-\mathbf{k})\rangle$$

$|\pi(\mathbf{k}),\pi(-\mathbf{k})\rangle$: asymptotic state

: Behte - Salpeter amplitude

$$\phi_\infty(\mathbf{x}) \;=\; \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_p}\; \mathrm{e}^{i\mathbf{p}\cdot\mathbf{x}/2}\; \underbrace{\langle\pi(\mathbf{p})|\pi(-\mathbf{x}/2)|\pi(\mathbf{k})\pi(-\mathbf{k})\rangle}_{\text{elastic scattering}}$$

$$+\; \underbrace{\langle\pi\pi\pi|\pi(-\mathbf{x}/2)|\pi(\mathbf{k})\pi(-\mathbf{k})\rangle \;+\; \cdots}_{\text{in-elastic scattering}}$$

$$E_p = \sqrt{m_\pi^2 + p^2}$$

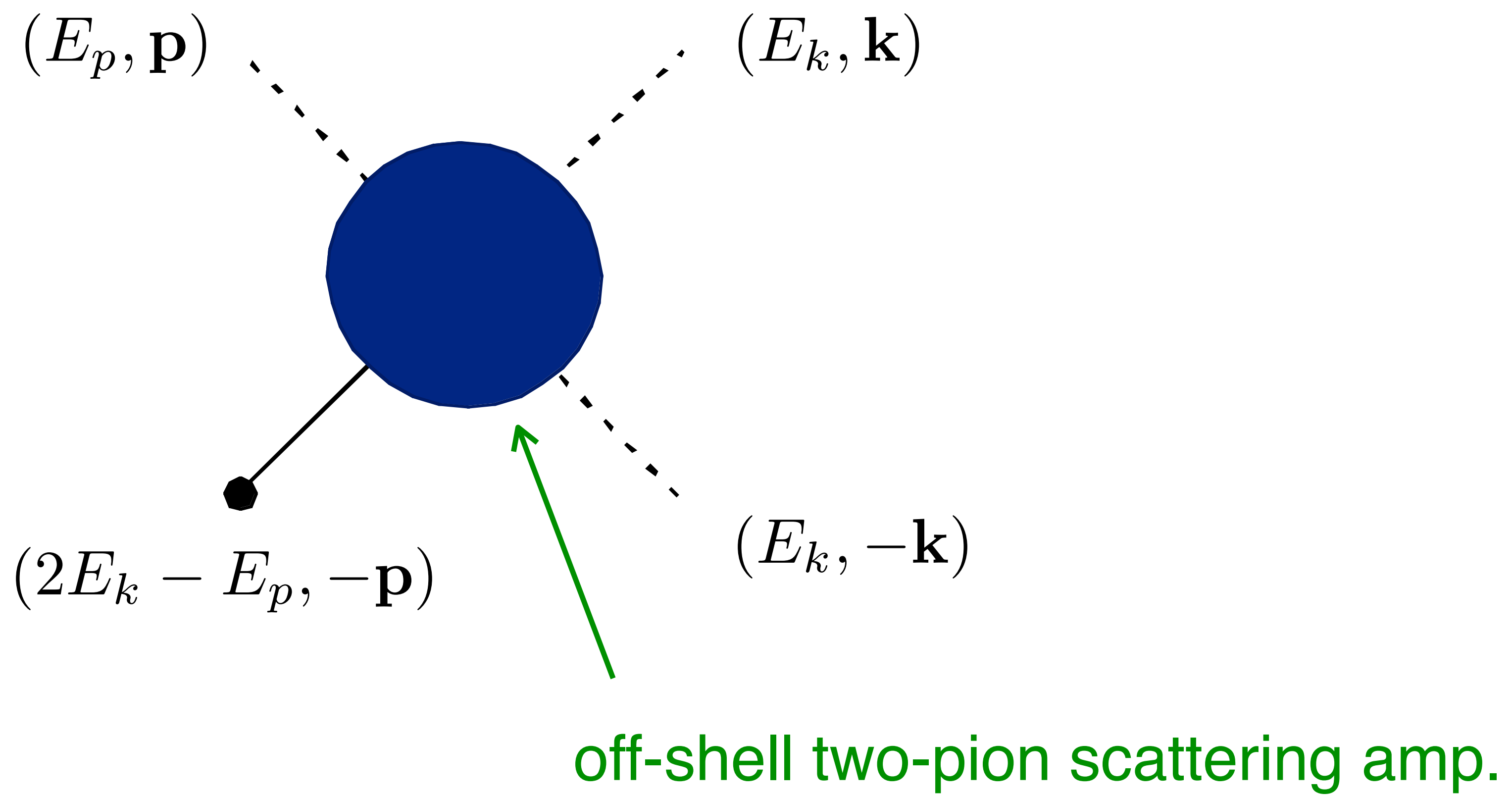
We consider only case $2\cdot\sqrt{k^2+m_\pi^2} \;<<\; 4\cdot m_\pi$
and neglect contribution from in-elastic scattering.

$$e^{-i\mathbf{p}\cdot\mathbf{x}/2} \cdot \langle \pi(\mathbf{p}) | \pi(-\mathbf{x}/2) | \pi(\mathbf{k})\pi(-\mathbf{k}) \rangle \;=\; \underbrace{2E_p(2\pi)^3\delta^3(\mathbf{p}-\mathbf{k})}_{\text{Disconnected part}} \;+\; \underbrace{\frac{H(p;k)}{p^2-k^2-i\epsilon}}_{\text{Connected part}}$$

Disconnected part

Connected part

$$\frac{H(p;k)}{p^2-k^2-i\epsilon} \;=$$



where we assumme

$$\delta_l \;\sim\; 0 \qquad \text{with } l \geq 1$$

so $H(p;k)$ is depend only on

$$p = |\mathbf{p}| \quad \text{and} \quad k = |\mathbf{k}|$$

$$\phi_\infty(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}} + \int \frac{d^3p}{(2\pi)^3} \frac{H(p;k)}{p^2 - k^2 - i\epsilon} e^{i\mathbf{p}\cdot\mathbf{x}}$$

$$= e^{i\mathbf{k}\cdot\mathbf{x}} + \frac{ik}{4\pi} H(k;k) j_0(kx) + \mathbf{P}\int \frac{d^3p}{(2\pi)^3} \frac{H(p;k)}{p^2 - k^2} j_0(px)$$

$j_0(kx)$: spherical Bessel

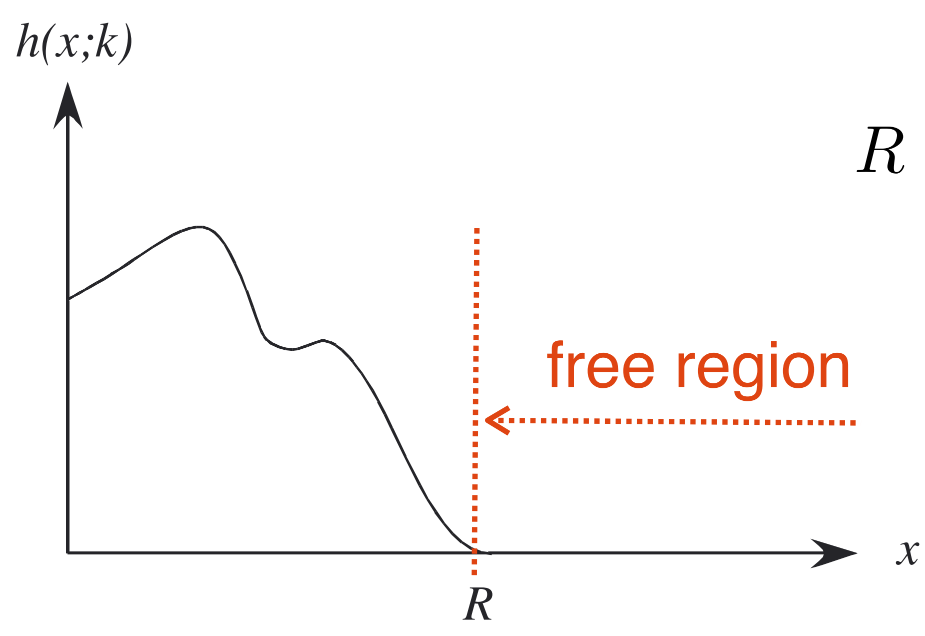
$$H(k;k) = \frac{4\pi}{k} \sin\delta_0(k) e^{i\delta_0(k)}$$

: on shell two-pion scattering amplitude

Here we assume

$$h(x;k) \equiv \int \frac{d^3p}{(2\pi)^3} H(p;k)\, e^{-i\mathbf{p}\cdot\mathbf{x}} = -(\nabla^2 + k^2)\phi_\infty(\mathbf{x}) = 0$$

for $R < x$



R : two-pion interaction range

In free region $R < x$

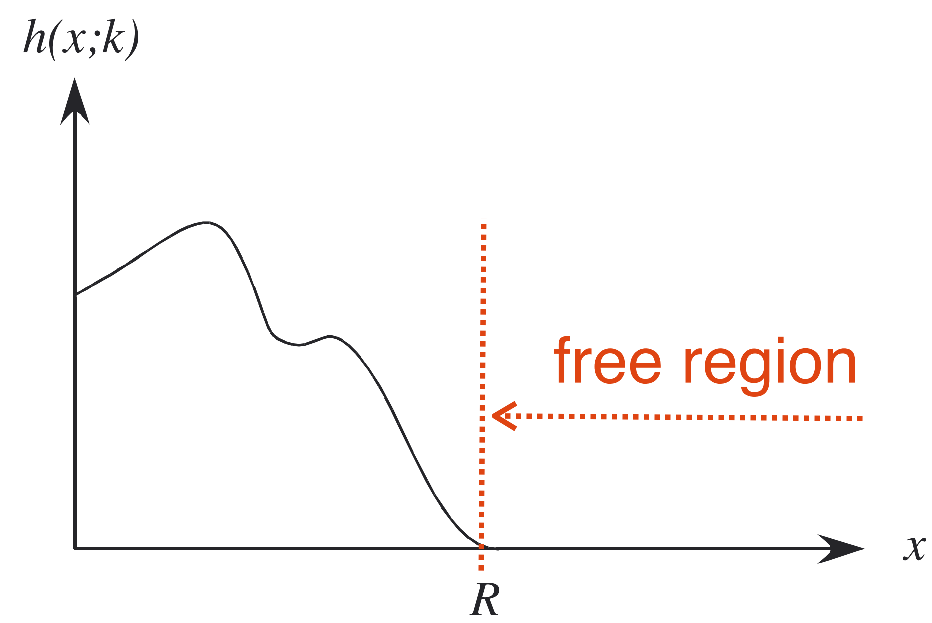
$$\phi_\infty(\mathbf{x}) = \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} + \frac{ik}{4\pi} H(k;k) j_0(kx) + \underline{\mathbf{P}\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{H(p;k)}{p^2 - k^2} j_0(px)}$$

$$E(x;k) \equiv \mathbf{P}\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{H(p;k) - H(k;k)}{p^2 - k^2} \mathrm{e}^{i\mathbf{p}\cdot\mathbf{x}}$$

$$= \int_x^\infty \mathrm{d}z (4\pi z^2) \cdot \underline{h(z;k)} \cdot \frac{k}{4\pi} \Big(j_0(kx) n_0(kz) - n_0(kx) j_0(kz) \Big)$$

$$= 0$$

$= 0$ for $R < x \leq z$



formula :

$$\frac{k}{4\pi} n_0(kx) = \mathbf{P}\int \frac{\mathrm{d}^3}{(2\pi)^3} \frac{j_0(px)}{p^2 - k^2}$$

$$= \mathbf{P}\int \frac{\mathrm{d}^3}{(2\pi)^3} \frac{\mathrm{e}^{i\mathbf{p}\cdot\mathbf{x}}}{p^2 - k^2}$$

in free region $R < x$

$$\phi_\infty(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}} + \frac{ik}{4\pi} H(k;k) j_0(kx) + \mathbf{P}\int \frac{d^3p}{(2\pi)^3} \frac{H(p;k)}{p^2-k^2} j_0(px)$$

$$= e^{i\mathbf{k}\cdot\mathbf{x}} + \frac{ik}{4\pi} H(k;k) j_0(kx) + \underline{H(k;k)}\ \mathbf{P}\int \frac{d^3p}{(2\pi)^3} \frac{1}{p^2-k^2} j_0(px)$$

$$= e^{i\mathbf{k}\cdot\mathbf{x}} + \frac{ik}{4\pi} \underline{H(k;k)} j_0(kx) + \frac{k}{4\pi} \underline{H(k;k)} n_0(kx)$$

$$H(k;k) = \frac{4\pi}{k} \sin\delta_0(k) e^{i\delta_0(k)}$$

: on shell two-pion scattering amplitude

On finite volume

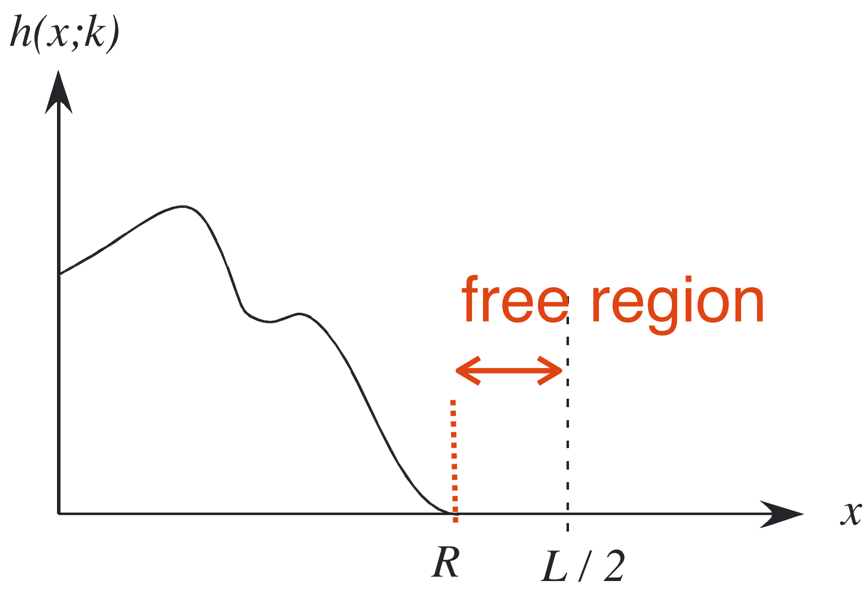
Let's consider two-pion wave function defined by

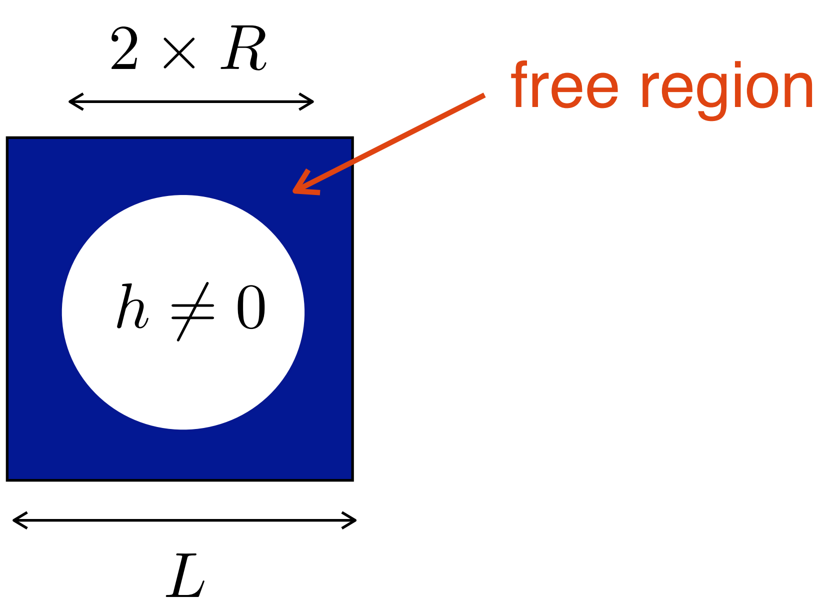
$$\phi(\mathbf{x}) \;=\; \langle 0|\pi(\mathbf{x}/2)\pi(-\mathbf{x}/2)|\pi\pi;k\rangle$$

$|\pi\pi;k\rangle$: energy eigen state with

$$E = 2\cdot\sqrt{m_\pi^2 + k^2}$$

Assumption : Two-pion interaction range : $R \;<\; L/2$





In free region : $R < x < L/2$

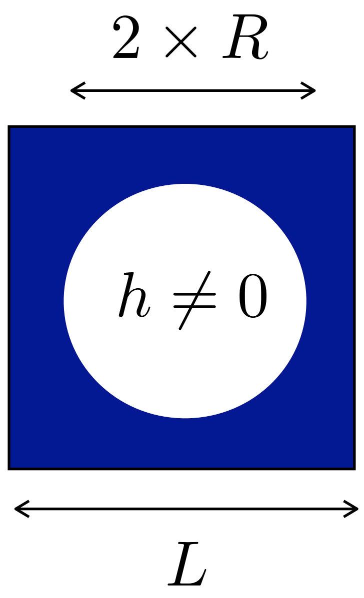
$$(\;\nabla^2 \;+\; k^2\;)\phi(\mathbf{x}) \;=\; 0$$: Helmholtz Eq.

Solution of Eq. : (neglecting δ_l with $l \geq 1$)

$$\phi(\mathbf{x}) \;=\; C \cdot G(\mathbf{x};k)$$

$$G(\mathbf{x};k) = \sum_{\mathbf{n}\in\mathcal{Z}^3} (p_n^2 - k^2)^{-1} \mathrm{e}^{i\mathbf{p}_n\cdot\mathbf{x}} \qquad \mathbf{p}_n = \mathbf{n}(2\pi)/L$$

: periodic Green's function



In free region $R < x < L/2$

wave function on finite volume : $\phi(\mathbf{x})$

wave function in infinite volume : $\phi_\infty(\mathbf{x})$

$\phi(\mathbf{x})$ can be written by linear combination of

$$\phi(\mathbf{x}) \;=\; \sum_{l=0}^{\infty} \sum_{m=-l}^{l} C_{lm} \cdot \phi_\infty^{lm}(x)$$

$$\phi_\infty^{lm}(x) \;=\; \int \mathrm{d}\Omega \; Y_{lm}(\Omega)\phi_\infty(\mathbf{x})$$

In free region $R < x < L/2$

- $$\begin{aligned}\phi(\mathbf{x}) &= C \cdot G(\mathbf{x};k) \\ &= C \cdot \left[\sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sqrt{4\pi} Y_{lm}(\Omega_x) \cdot g_{lm}(k) j_l(kx) \;+\; \frac{k}{4\pi} n_0(kx) \right]\end{aligned}$$

$$g_{lm}(k) = \left(\frac{i}{k}\right)^l \frac{1}{L^3} \sum_{\mathbf{n}\in\mathcal{Z}^3} \frac{\sqrt{4\pi} Y_{lm}(\Omega_{p_n})}{p_n^2 - k^2}$$

- $$\phi_\infty(\mathbf{x}) = \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} + \frac{ik}{4\pi} H(k;k) j_0(kx) + \frac{k}{4\pi} H(k;k) n_0(kx)$$

By comparing S-wave comp., we get

$$C \cdot \left[g_{00}(k) j_0(kx) + \frac{k}{4\pi} n_0(kx) \right] = C_{00} \cdot \left[\left(1 + \frac{ik}{4\pi} H(k;k)\right) j_0(kx) + \frac{k}{4\pi} H(k;k) n_0(kx) \right]$$

$$g_{00}(k) \cdot H(k;k) = (1 + \frac{ik}{4\pi} H(k;k)) \qquad \left(H(k;k) = \frac{4\pi}{k} \sin\delta_0(k) \mathrm{e}^{i\delta_0(k)} \right)$$

: Lüscher’s formula

Now we consider :

1. Two-pion wave function plays an important role.

   But, it does not appear in traditional method.

   Calculation of wave function in the lattice is possible ?

2. Lüscher's formula can be applied,

   if two-pion interaction is localized,
   so that interaction range : $R < L/2$

   Estimation of R is possible ?
   How large ?

3. Scattering length must be calculated from the wave function.

   Is it possible in the lattice ?

Similar work has been carried for 2-dim XY-model
by J. Balog, et.al. NPB618(01)315.

Same idea has been discussed
by C.-J. Lin, et.al. NPB619(01)465.

2. Method of calculation of wave function

In this work,
we restrict to ground state of $I=2$ two-pion

ie. $k^2 \sim 0$ ie. scattering length

Definition of wave function

$$\phi(\mathbf{x}) = \sum_{\mathbf{X}} \sum_{\mathbf{R}} \langle 0| \, \pi(\mathbf{R}[\mathbf{x}] + \mathbf{X}), t) \, \pi(\mathbf{X}), t) \, |\pi\pi; k\rangle \times \mathrm{e}^{2E_k \cdot t}$$

$\mathbf{R}$: element of cubic group

$|\pi\pi; k\rangle$: ground state of two-pion with energy $2E_k = 2 \cdot \sqrt{m_\pi^2 + k^2}$

$\sum_{\mathbf{X}}$: zero total momentum projection

$\sum_{\mathbf{R}}$: projection to irreducible rep. $\mathbf{A}_1^+$ of cubic group

$$\mathbf{A}_1^+ \sim \text{S-wave upto } l \geq 4$$

Calculation method of wave function

$$G(\mathbf{x},t) = \sum_{\mathbf{X}} \sum_{\mathbf{R}} \langle 0|\ \pi(\mathbf{R}[\mathbf{x}]+\mathbf{X},t)\ \pi(\mathbf{X},t)\ \underline{W(t_0)\ W(t_0+1)}\ |0\rangle$$

$$W(t_0) = \sum_{\mathbf{x}} \bar{q}(\mathbf{x},t_0)\ \gamma_5 \sum_{\mathbf{y}} q(\mathbf{y},t_0)$$: " wall source "

in large t region

$$G(\mathbf{x},t) \sim \underline{\phi(\mathbf{x})} \times \langle \pi\pi; k|W((t_0)W(t_0+1)|0\rangle \times \mathrm{e}^{-2E_k \cdot t}$$

so

$$\phi(\mathbf{x}) = \frac{G(\mathbf{x},t)}{G(\mathbf{x_0},t)}$$

up to over all cont.

3. Simulation parameters

- Quench approximation
- RG improve gauge action
- Improve Wilson fermion action with perturbative C_{SW}

$$1/a = 1.207 \text{ GeV}$$
$$a = 0.1632 \text{ fm}$$

- Lattice size and # of conf. :

Time extent : $T = 80$

L	16	20	24
La	2.61 fm	3.26 fm	3.92 fm
$\#C$	1200	1000	506

- Quak masses

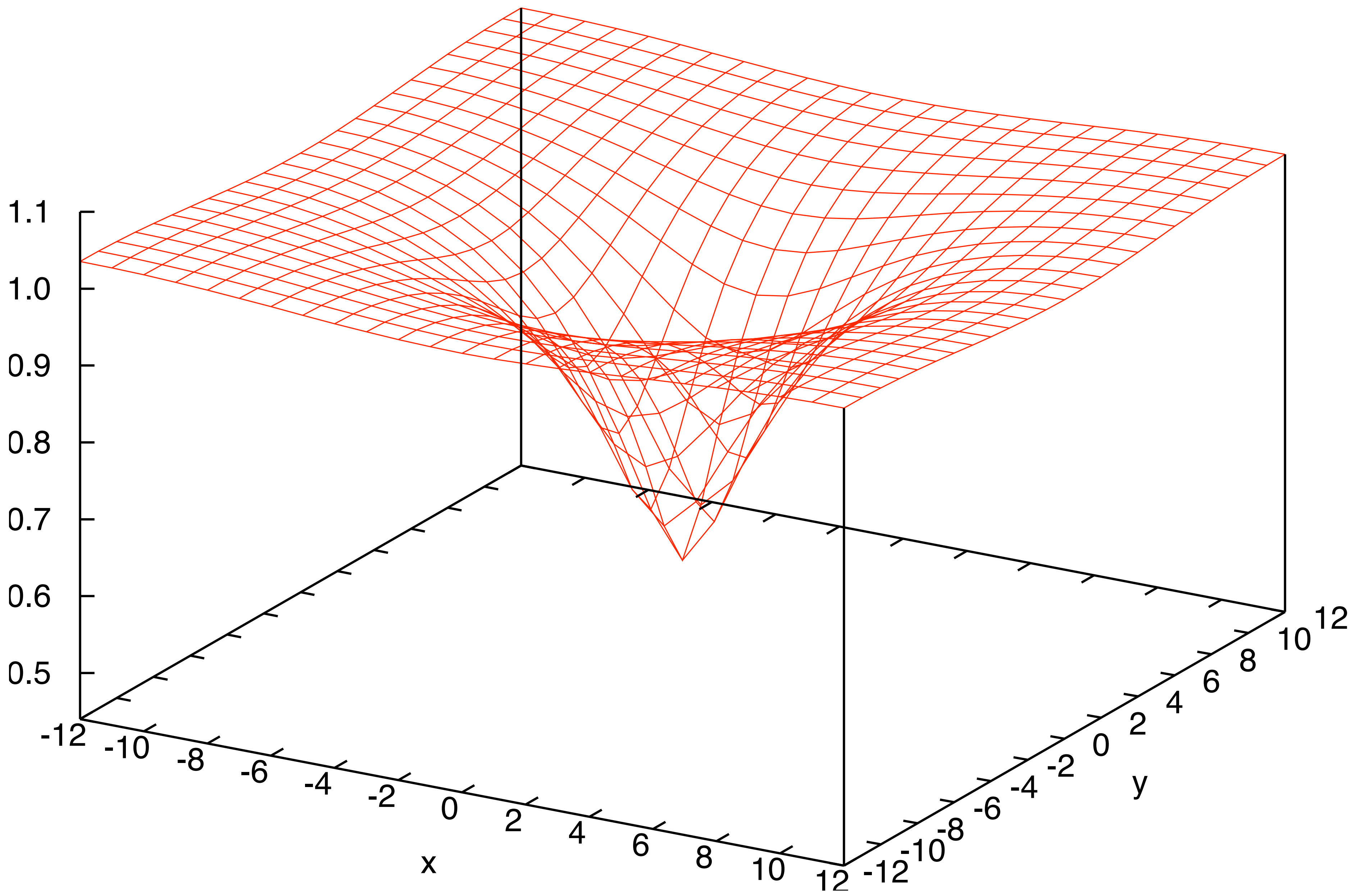
$$m_\pi^2 \ (\text{GeV}^2) = 0.27 \ , \ 0.35 \ , \ 0.44 \ , \ 0.59 \ , \ 0.74$$

- Wall source position : $t_0 = 12$

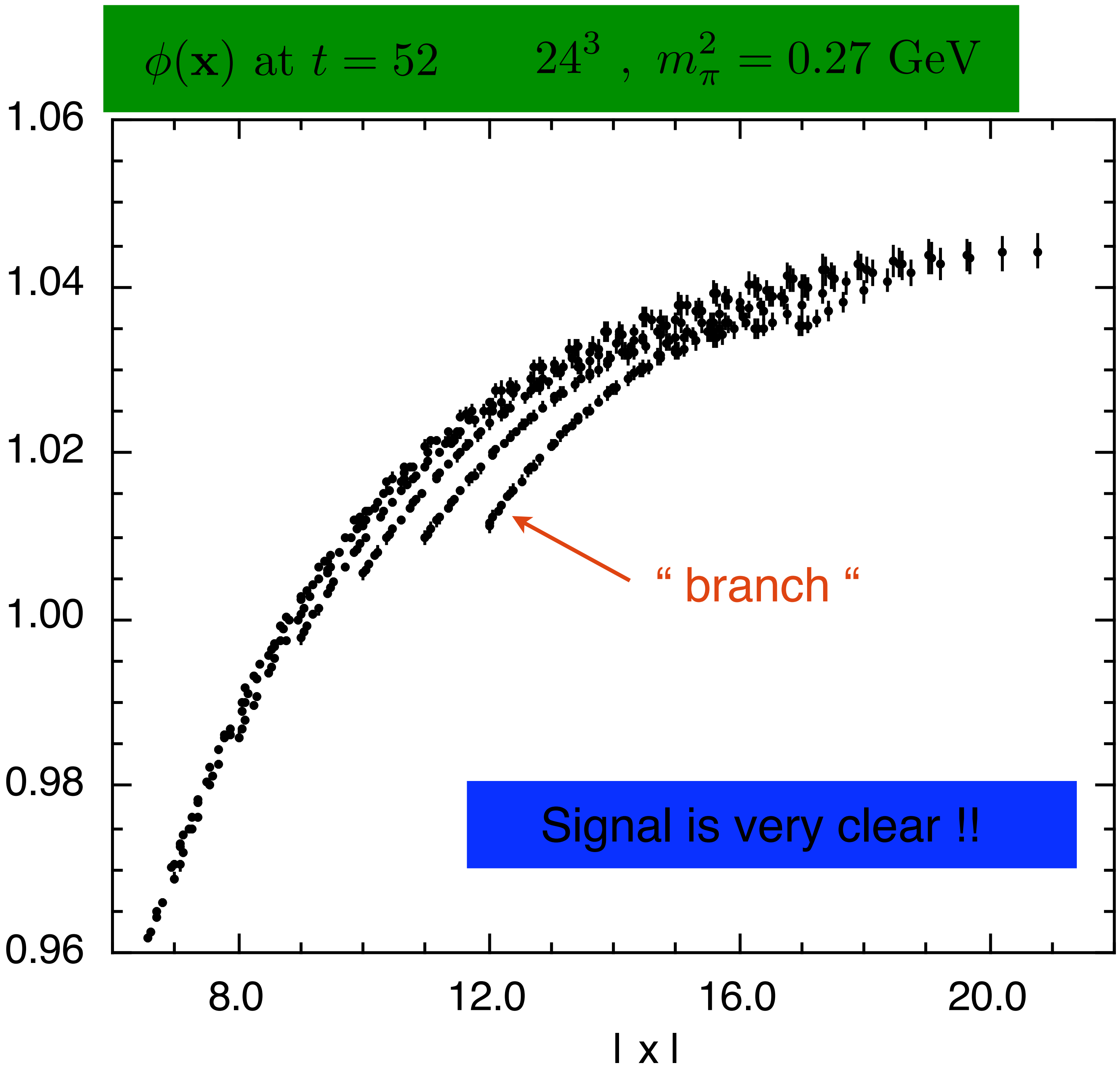
Coulomb gauge fix

4. Results of two-pion wave function

$\phi(\mathbf{x})$ at $t = 52$, $z = 0$ $\quad$ 24^3 , $m_\pi^2 = 0.27$ GeV



Signal is very clear !!



There are some "branches". Why ?

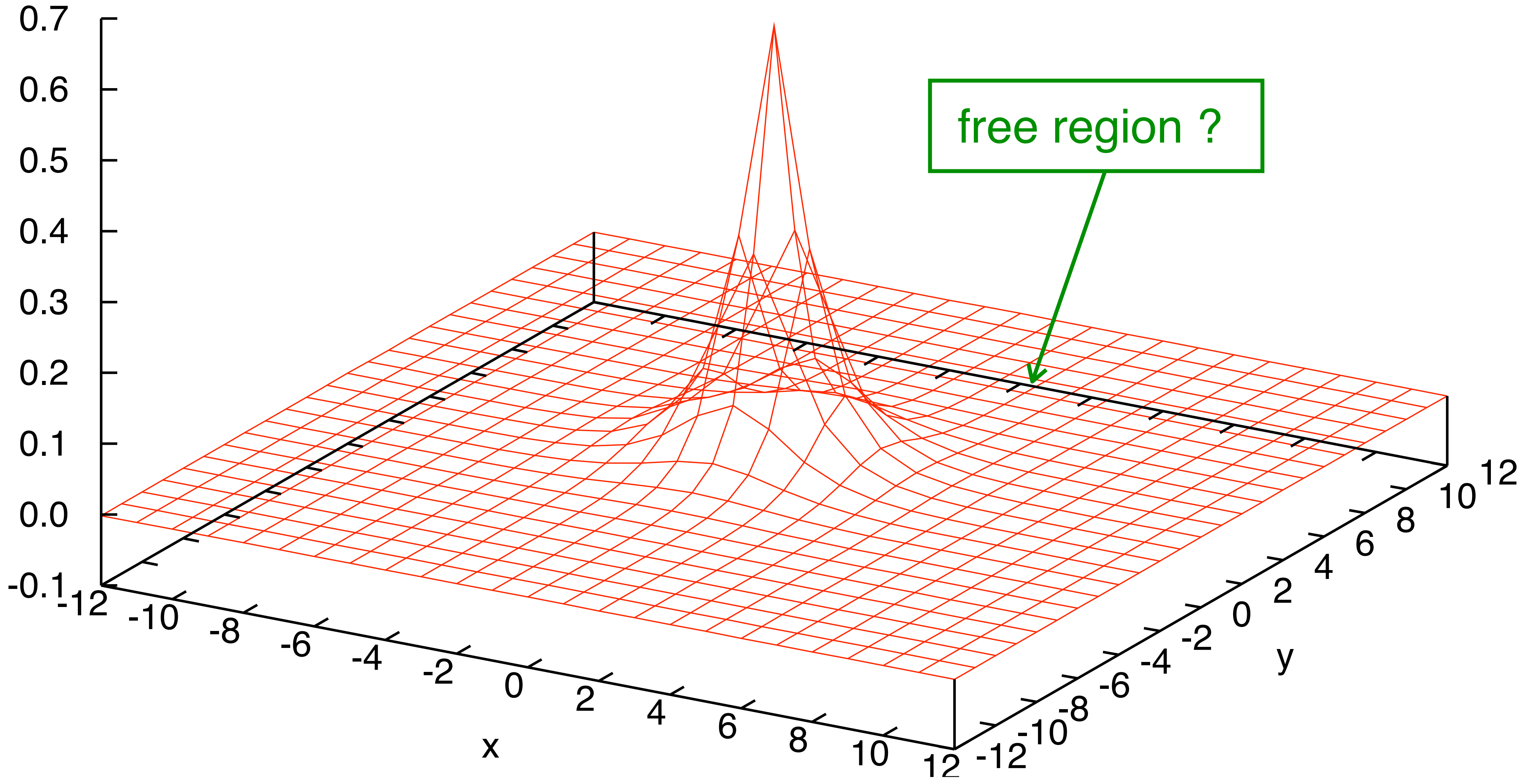
if $\phi(\mathbf{x}) = \phi(x = |\mathbf{x}|)$
then $\phi(\mathbf{x})$ does not satisfy periodic BC.,
because $\phi(\mathbf{x}) \neq 0$ at boundary.

$$\Rightarrow \quad \phi(\mathbf{x}) = f(x) + \sum_{lm} Y_{lm}(\Omega_x) \cdot f_l(x)$$

24^3 , $m_\pi^2 = 0.27$ GeV

$\nabla^2\phi(\mathbf{x})/\phi(\mathbf{x})$

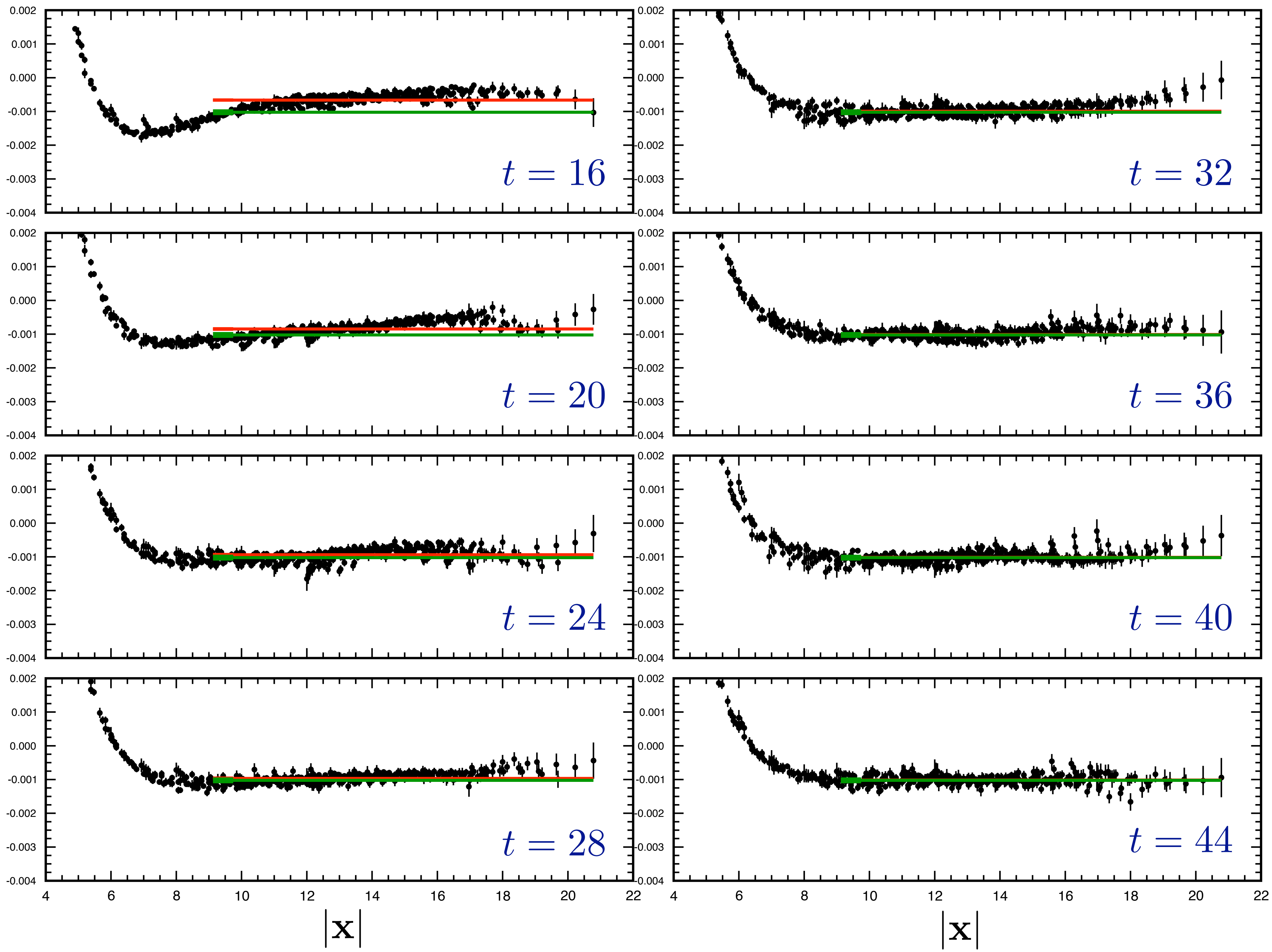
$\sim -k^2$



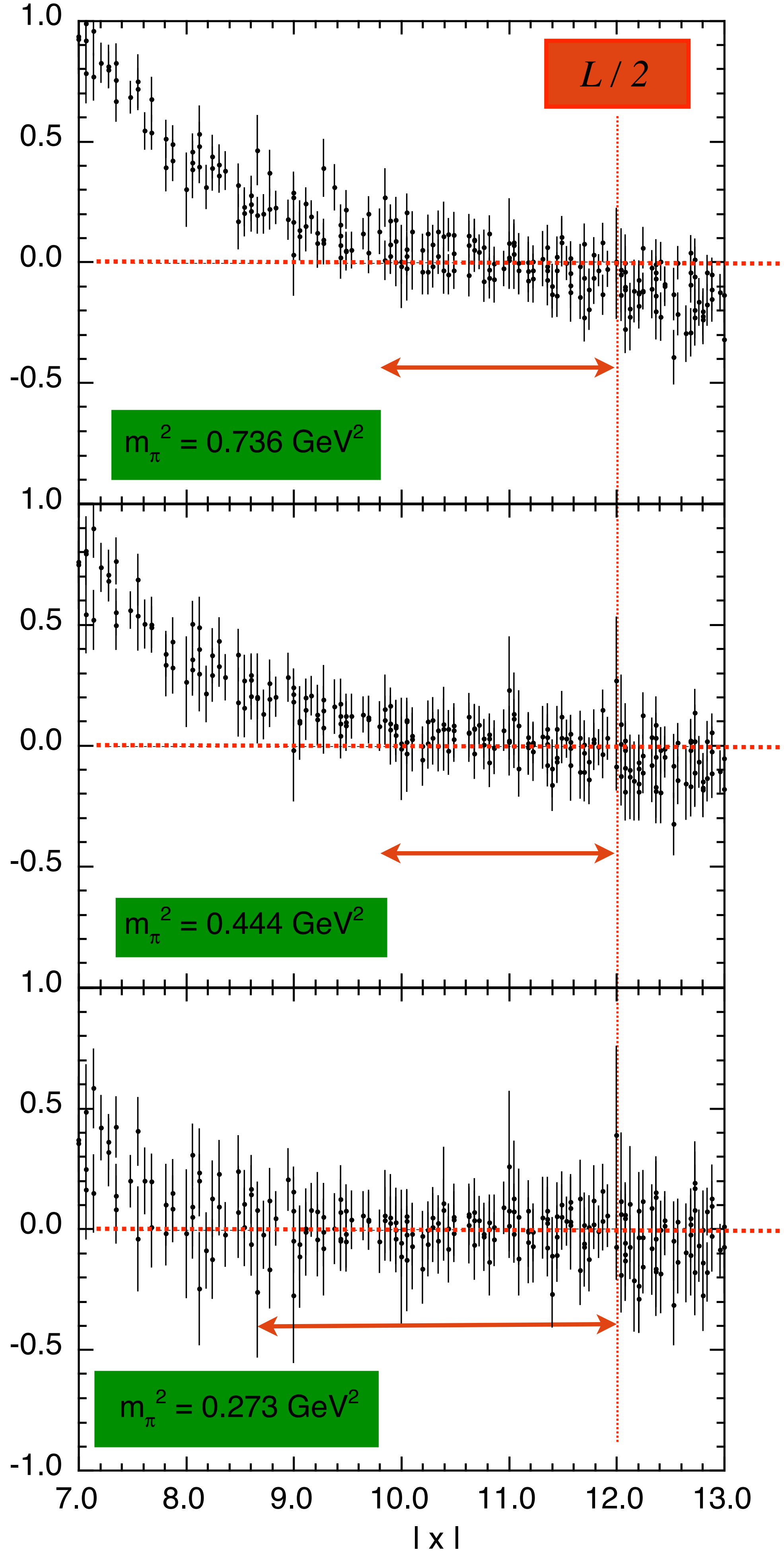
Signal is very clear !!

Very strong repulsive potential at origin

24^3 , $m_\pi^2 = 0.27$ GeV $\nabla^2\phi(\mathbf{x})/\phi(\mathbf{x}) \sim -k^2$



stable for $t \geq 44$



24^3 at $t = 52$

$$U(\mathbf{x}) \equiv \frac{(\nabla^2 + k^2)\phi(\mathbf{x})}{k^2\phi(\mathbf{x})} \sim 0$$

from two-pion time correlation function

Criterion of R :

$U(\mathbf{x}) = 0$ within Stat. Err. for $R < |\mathbf{x}| < L/2$

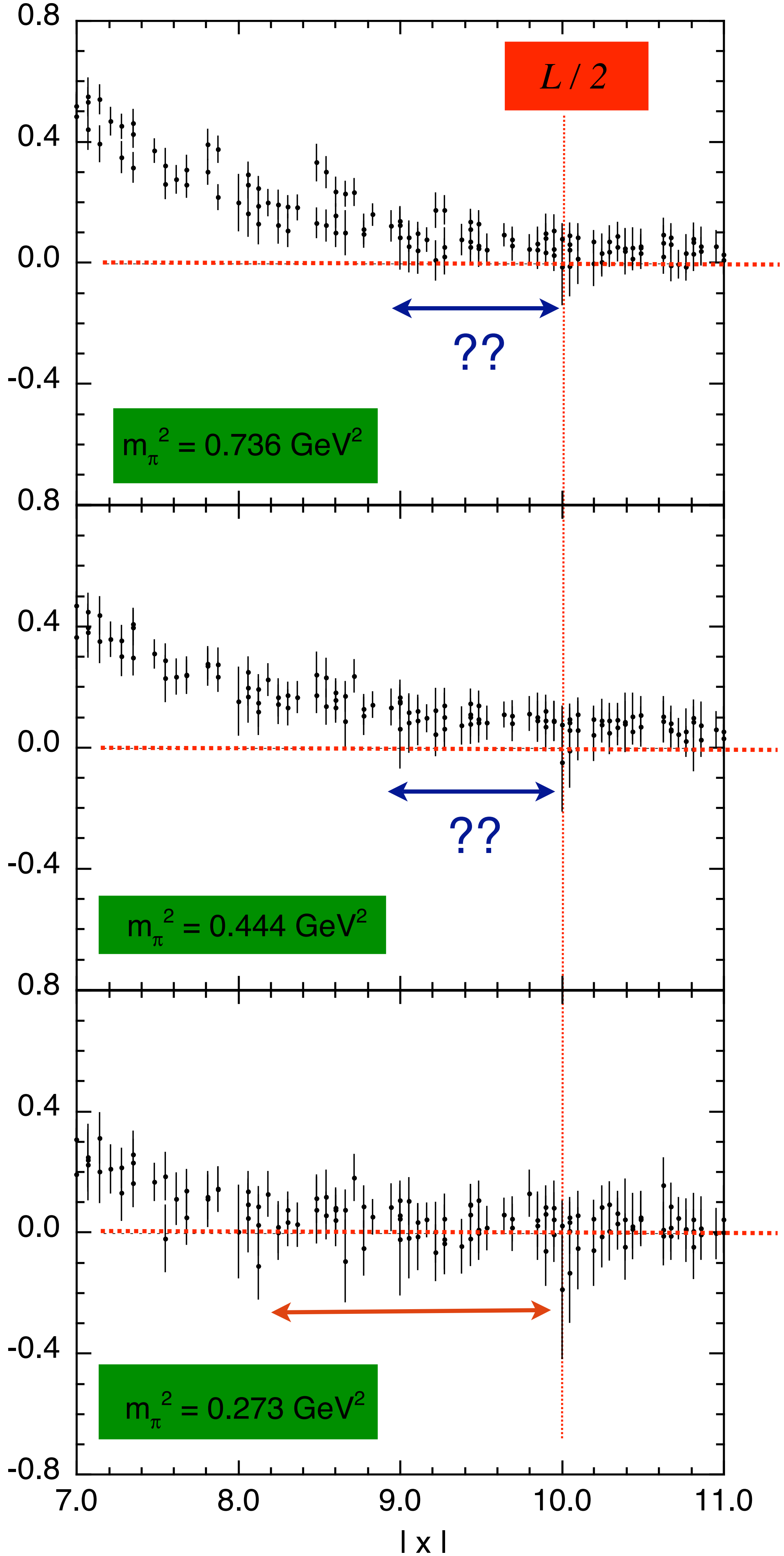
without any assumptions
for two-pion interaction

Interaction range R is larger
for larger quark mass.

Worst case (largest quark mass)

$R \sim 9.8$ (1.6 fm) $< (L/2 = 12)$

→ We can apply the formula
for all quark masses.



20^3 at $t = 52$

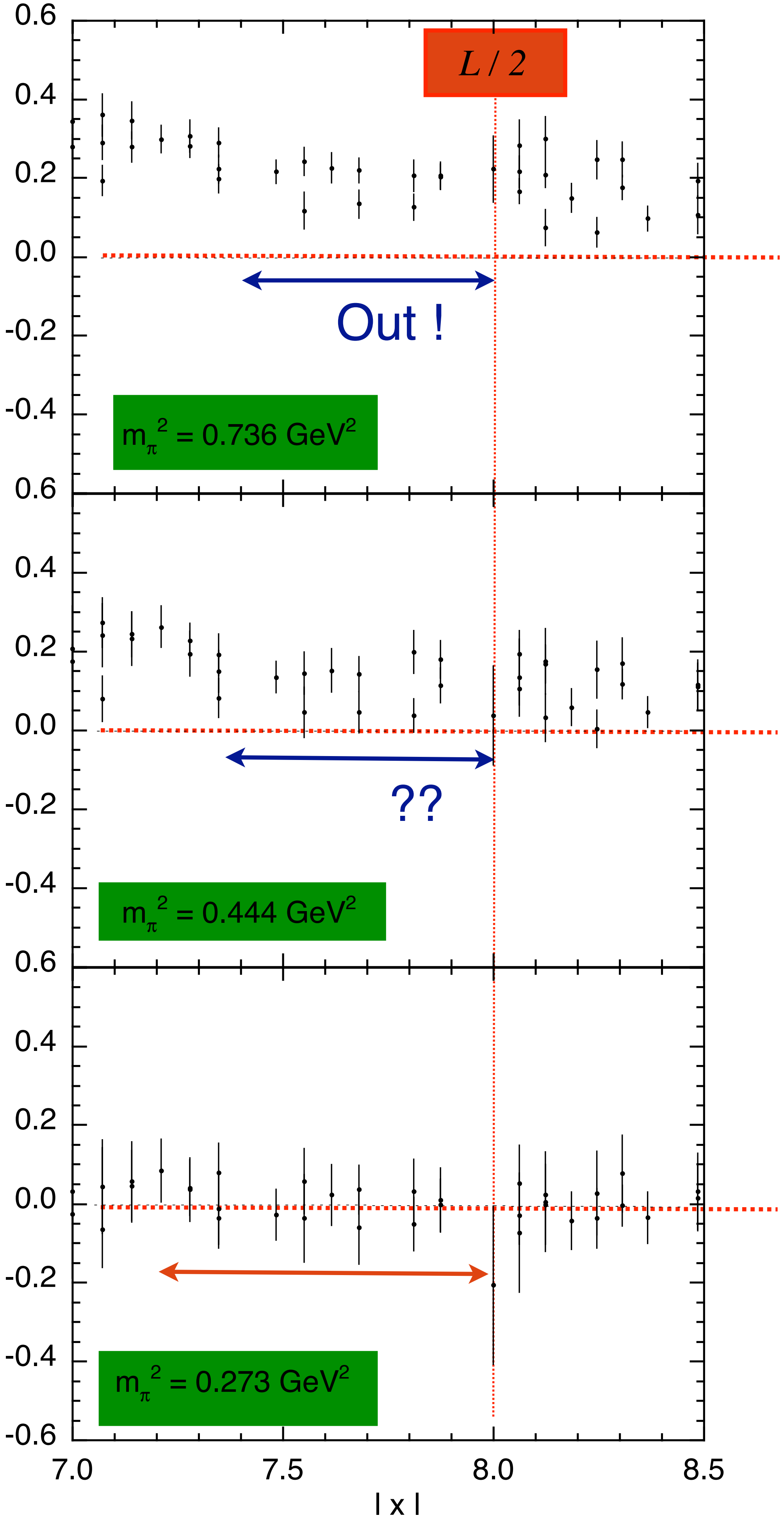
$$U(\mathbf{x}) \equiv \frac{(\nabla^2 + k^2)\phi(\mathbf{x})}{k^2\phi(\mathbf{x})} \sim 0$$

$$m_\pi^2 \ (\mathrm{GeV}^2) = 0.27 \ , \ 0.35 \ , \ 0.44 \ , \ 0.59 \ , \ 0.74$$

??

We can not clearly show zero region in $|\mathbf{x}| < L/2$ for larger quark masses.

Validity of necessary condition of the formula is not supported for these masses.



16^3 at $t = 52$

$$U(\mathbf{x}) \equiv \frac{(\nabla^2 + k^2)\phi(\mathbf{x})}{k^2\phi(\mathbf{x})} \sim 0$$

m_π^2 (GeV^2) = 0.27 , 0.35 , 0.44 , 0.59 , 0.74

??

We can not clearly show zero region in $|\mathbf{x}| < L/2$ for larger quark masses.

Validity of necessary condition of the formula is not supported for these masses.

4. Scattering length from wave function

Procedure :

(1) Extracting k^2 by

(a) from

$$-k^2 = \nabla^2\phi(\mathbf{x})/\phi(\mathbf{x})$$

(b) from fitting wave function with periodic Green's function

$$\phi(\mathbf{x}) = A \cdot G(\mathbf{x}; k)$$

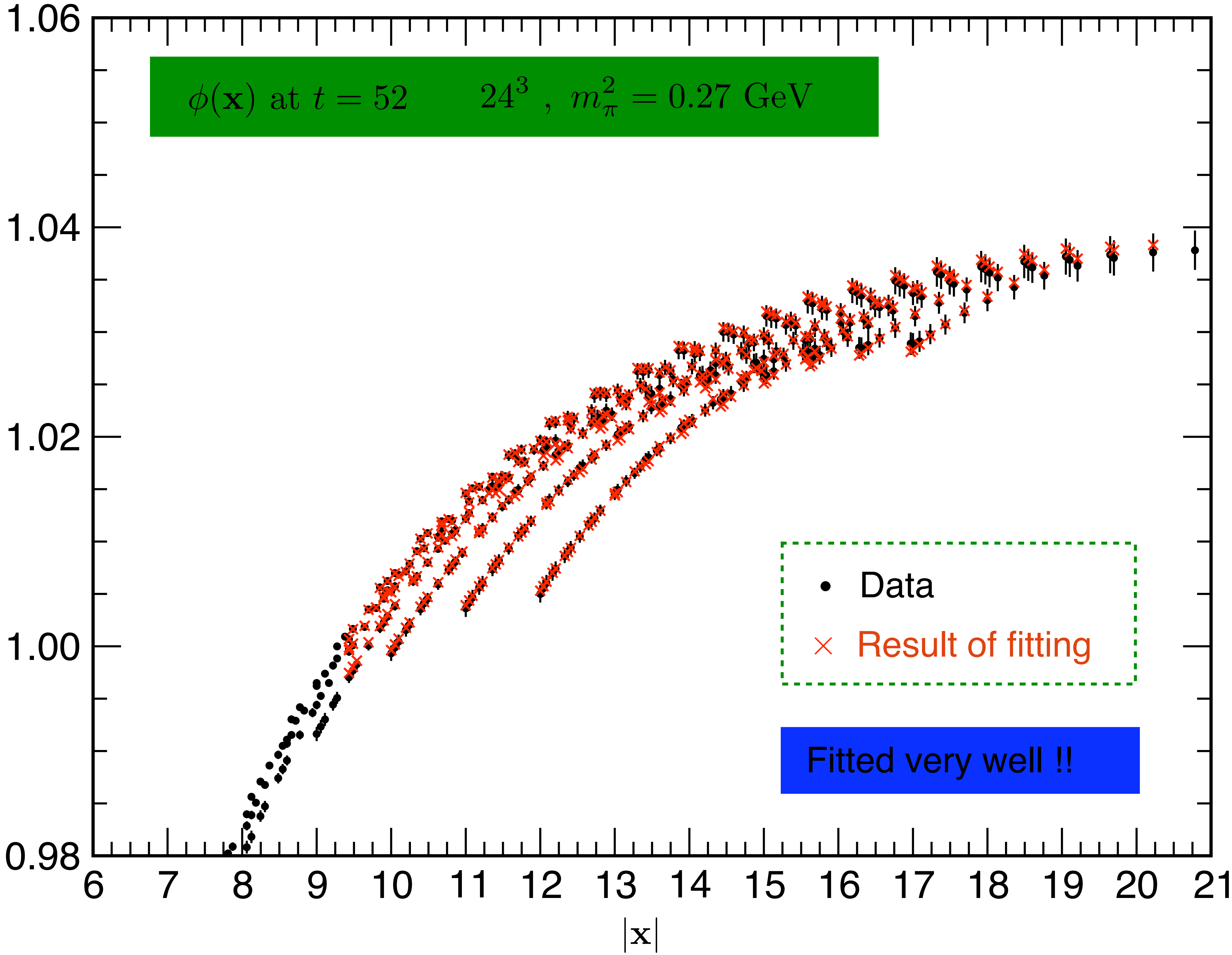
- fitting parameters : A , k^2

fitting range : $R \leq x \leq \{\max x = \sqrt{3}L/2\}$

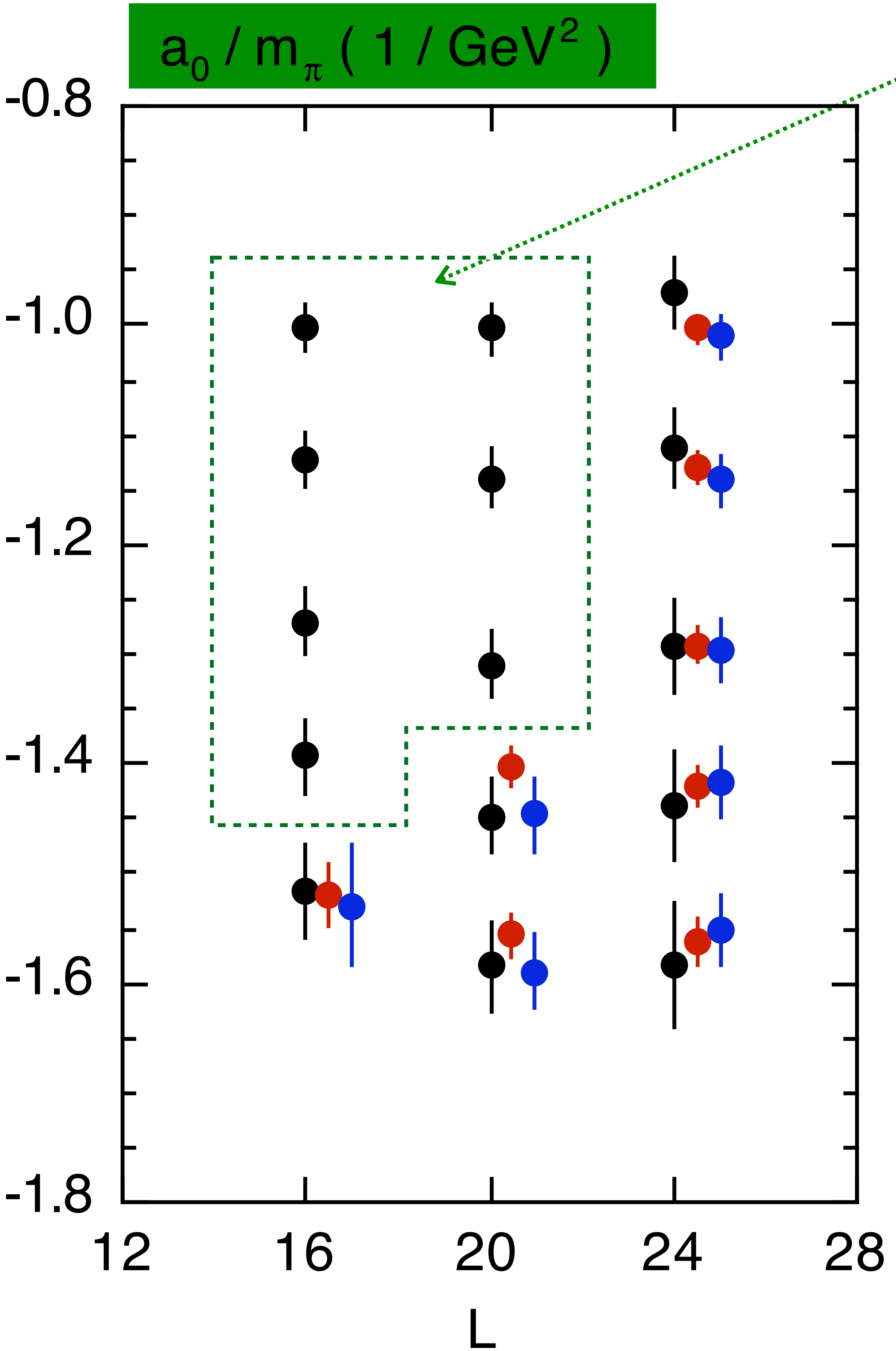
(2) Substituting k^2 into Lüscher's formula

$$\rightarrow \quad \frac{\tan\delta_0(k)}{k} = a_0 + O(k^2)$$

Results of fitting wave function



Results of SC. length



We can not clearly show zero region of $U(\mathbf{x})$ for these masses

● from time correlation function (by traditional method)

● from $-k^2 = \nabla^2\phi(\mathbf{x})/\phi(\mathbf{x})$ ($t = 52$)

● from $\phi(\mathbf{x}) = A \cdot G(\mathbf{x}; k)$ ($t = 52$)

Consistent with results given by traditional method.

Statistical error is smaller !!

5. Summary

We calculate two-pion wave function
of ground state of $I=2$ two-pion.

We find :

(1) Calculation is possible and not difficult.

(2) Scattering length can be obtained from wave function.

Statistical error is small.

(3) Two-pion interaction range can be estimated from wave function.

Interaction range is larger for larger quark mass.

For $m_\pi^2 \leq 0.74 \text{ GeV}^2$

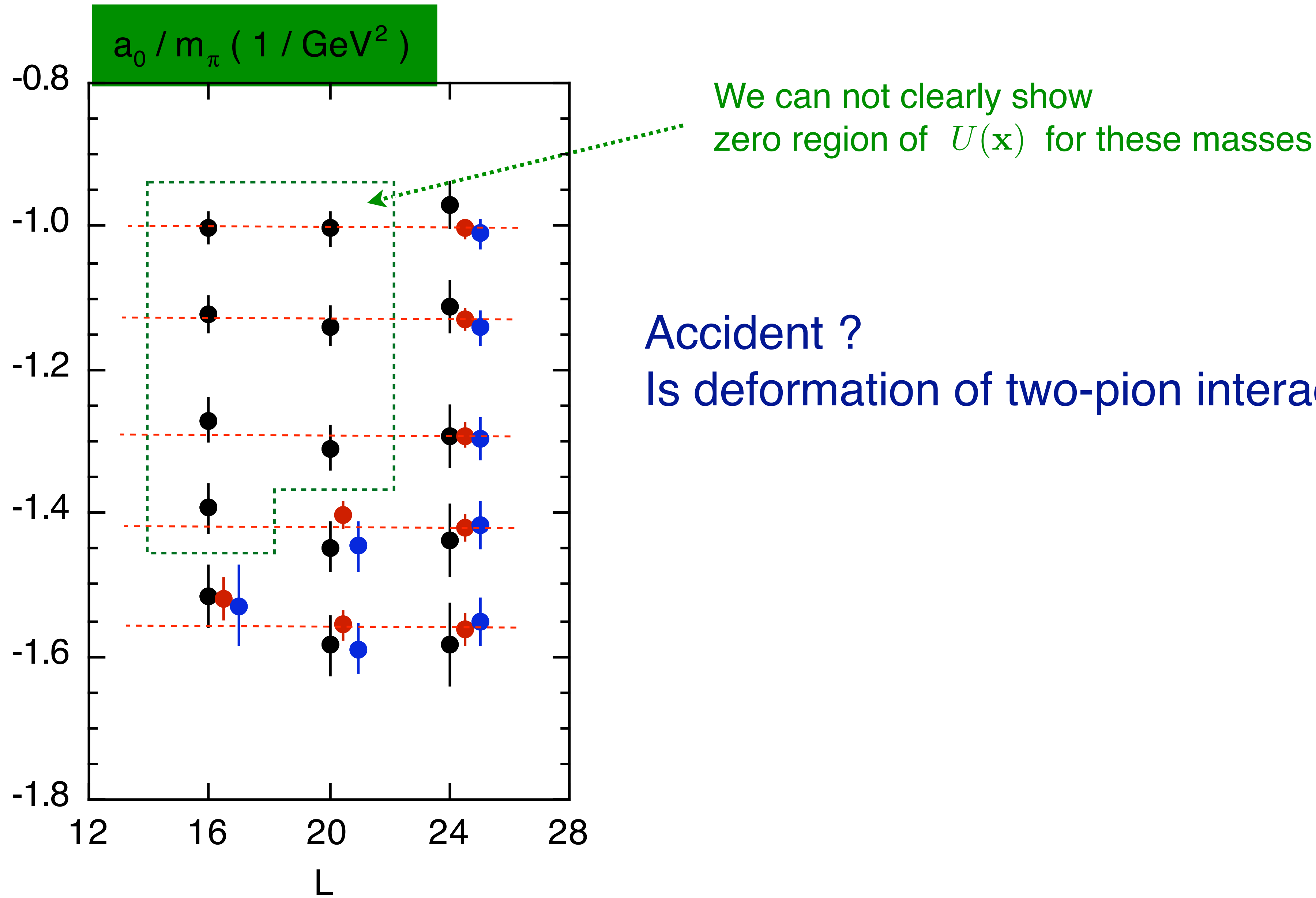
Necessary volume : $L > 24 \ (\ 4\text{fm} \ )$

As shown, necessary volume is smaller for lighter quark mass

The two-pion interaction does not exactly equals to zero
in the quantum field theory.
There is exponential tail in potential at large distance.
But we do not consider it.

But ...... , How can we estimate it ?

(4) We find no significant volume dependence
for all quark masses on $L > 16$ (2.6 fm),
including data
for which are not satisfied with necessary condition of Lüscher's formula.



Accident ?
Is deformation of two-pion interaction negligible ?

Extensions of this works :

(1) Scattering phase shift for $I=2$ two-pion system.

Wave function has “node” in this case.

(2) $I=0$ and $I=1$ two-pion system.

What happens at resonance point ?

(3) Estimation of contribution from in-elastic scattering.

If it is not small, Lüscher’s formula can not be applied.

$$\phi(\mathbf{x}) = \langle 0|\pi(\mathbf{0})|\pi\rangle \underbrace{\langle\pi|\pi(\mathbf{x})|\pi\pi\rangle}_{\pi\pi\to\pi\pi} + \langle 0|\pi(\mathbf{0})|\pi\pi\pi\rangle \underbrace{\langle\pi\pi\pi|\pi(\mathbf{x})|\pi\pi\rangle}_{3\pi\to 3\pi\ ,\ 4\pi\to 2\pi} + \cdots$$

We can access this question by estimating each terms.

(4) Two Nucleon system.

$$a(^1S_0) \ = \ -23.714 \text{ fm}$$

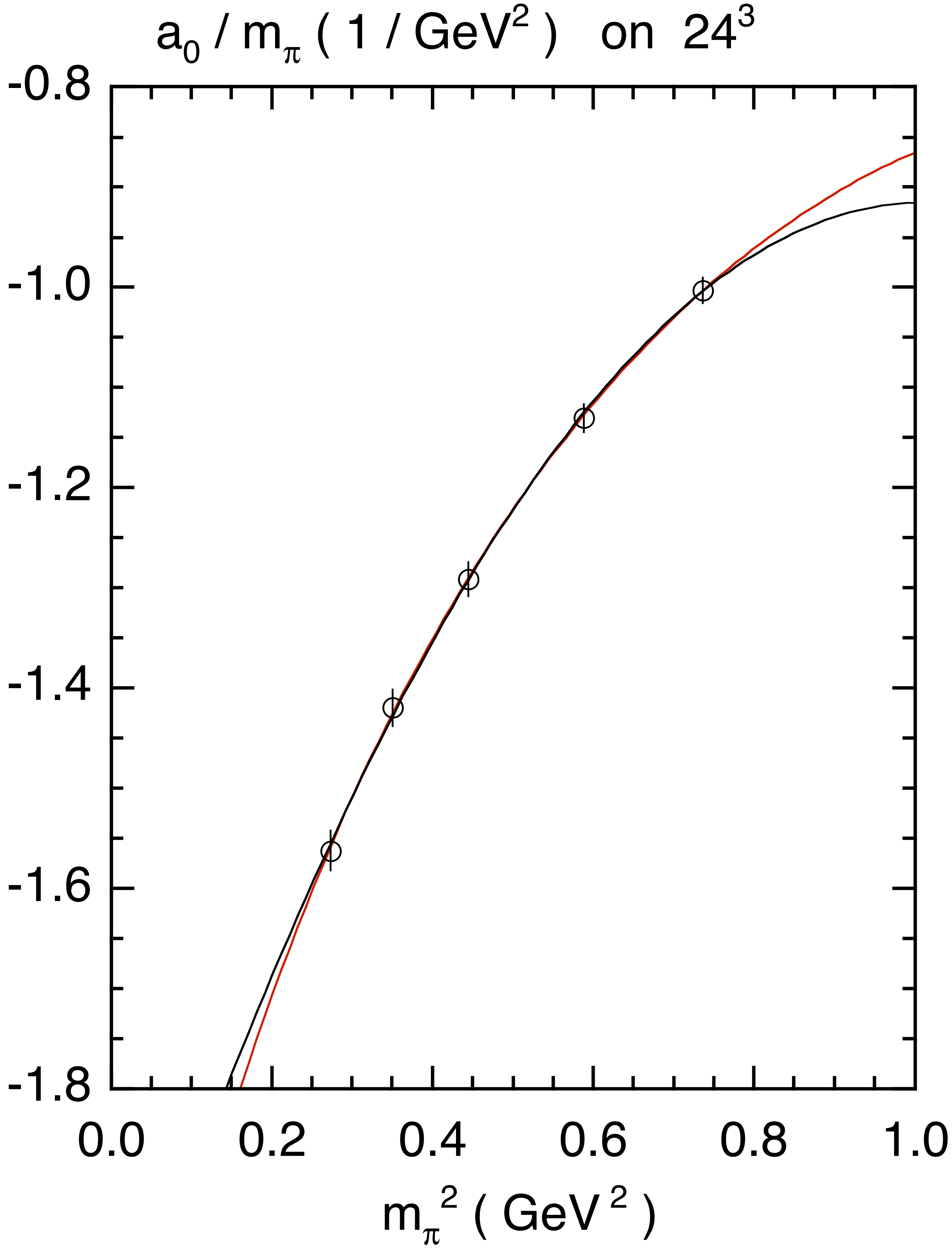
Is lattice calculation possible ?

We can access this question from wave function.

(5) Pentaquark ?

Using wave function to separate

Pentaquak and K+N scattering states.



$(a_0/m_\pi) = A + B\, m_\pi^2 \log(\, m_\pi^2\,) + C\, m_\pi^2$

$A = -\ 2.39(16)$

$(a_0/m_\pi) = A + B\, m_\pi^2 + C\, m_\pi^4$

$A = -\ 2.117(83)$

CHPT = - 2.265(51)