Twisted-mass QCD, O(a) improvement and Wilson chiral perturbation theory

Sinya Aoki
Tsukuba University

in collaboration with
Oliver Baer
Tsukuba University

Startup workshop "Lattice QCD simulation via International Research Network"
21-24 September 2004 at Shuzenji, Izu, Japan

Twisted-mass QCD(tmQCD)

Twisted mass term: $m' e^{i\omega\gamma_5\tau_3} = m + i\mu\gamma_5\tau_3$ ω: twist angle

Advantages of tmQCD with Wilson fermion:

- No zero mode at $\mu \neq 0$ ⇨ No exceptional configurations
- Simplified renormalization of weak matrix elements
- Automatic O(a) improvement at maximal twist $\omega = \frac{\pi}{2}$ Frezzotti, Rossi '03

But: Definition of maximal twist on the lattice is non-trivial !

Caveat: Definition of Frezzotti/Rossi works only if $m' \gg a^2$

Outline of the talk

- Automatic O(a) improvement at maximal twist
  - Argument a la Frezzotti and Rossi
  - Caveat for small quark masses
  - Alternative proposal for maximal twist
- Pion mass in ChPT at non-zero lattice spacing
  - Brief introduction into ChPT at non-zero lattice spacing
  - Example: O(a) improvement of the pion mass at maximal twist
- Alternative scenario ($c_2 < 0$)
- Summary

Twisted mass term on the lattice

Mass term + Wilson term on the lattice :

$$\bar{\psi}(x)\left[\left(-a\frac{r}{2}\sum_{\mu}\nabla_{\mu}^{\star}\nabla_{\mu}+M_{\rm cr}(r)\right)+m_q\exp(iw\gamma_5\tau_3)\right]\psi(x)$$

$$m_q = m_0 - M_{\rm cr}(r)$$

m_0 ↑ bare quark mass; $M_{\rm cr}(r)$ ↑ critical quark mass

Field redefinition:

$$\psi_{\rm ph} = \exp(i\frac{\omega}{2}\gamma_5\tau_3)\psi,$$
$$\bar{\psi}_{\rm ph} = \bar{\psi}\exp(i\frac{\omega}{2}\gamma_5\tau_3)$$

$$\bar{\psi}_{\rm ph}(x)\left[\left(-a\frac{r}{2}\sum_{\mu}\nabla_{\mu}^{\star}\nabla_{\mu}+M_{\rm cr}(r)\right)\exp(-iw\gamma_5\tau_3)+m_q\right]\psi_{\rm ph}(x)$$

Wilson average and O(a) improvement

The Wilson average $$\langle O \rangle^{WA}(r, m_q, \omega) \equiv \frac{1}{2}\Big[\langle O \rangle(r, m_q, \omega) + \langle O \rangle(-r, m_q, \omega)\Big]$$

can be shown to be O(a) improved: $$= \langle O \rangle^{\mathrm{cont}}(m_q) + O(a^2)$$

Crucial assumption: $$M_{\mathrm{cr}}(-r) = -M_{\mathrm{cr}}(r)$$

Automatic O(a) improvement at maximal twist

Consider the twist average:

$$\langle O\rangle^{TA}(r, m_q, \omega = \frac{\pi}{2}) \equiv \frac{1}{2}\Big[\langle O\rangle(r, m_q, \omega = \frac{\pi}{2}) + \langle O\rangle(r, m_q, \omega = -\frac{\pi}{2})\Big]$$

$$\exp(-i\frac{\pi}{2}\gamma_5\tau_3) = -\exp(i\frac{\pi}{2}\gamma_5\tau_3)$$

$$= \frac{1}{2}\Big[\langle O\rangle(r, m_q, \omega = \frac{\pi}{2}) + \langle O\rangle(-r, m_q, \omega = \frac{\pi}{2})\Big]$$

For observables even in ω (e.g. masses):

$$\langle O\rangle(r, m_q, \omega = \frac{\pi}{2}) = \langle O\rangle^{TA}(r, m_q, \omega = \frac{\pi}{2}) = \langle O\rangle^{\mathrm{cont}}(m_q) + O(a^2)$$

O(a) improvement without taking an average !

Crucial assumption: $M_{\rm cr}(-r) = -M_{\rm cr}(r)$

"Proof":

1. Defining equation: $m_\pi(r, M_{\rm cr}(r)) = 0,$

2. Symmetries of the lattice theory: $m_\pi(r, m_0) = m_\pi(-r, -m_0)$

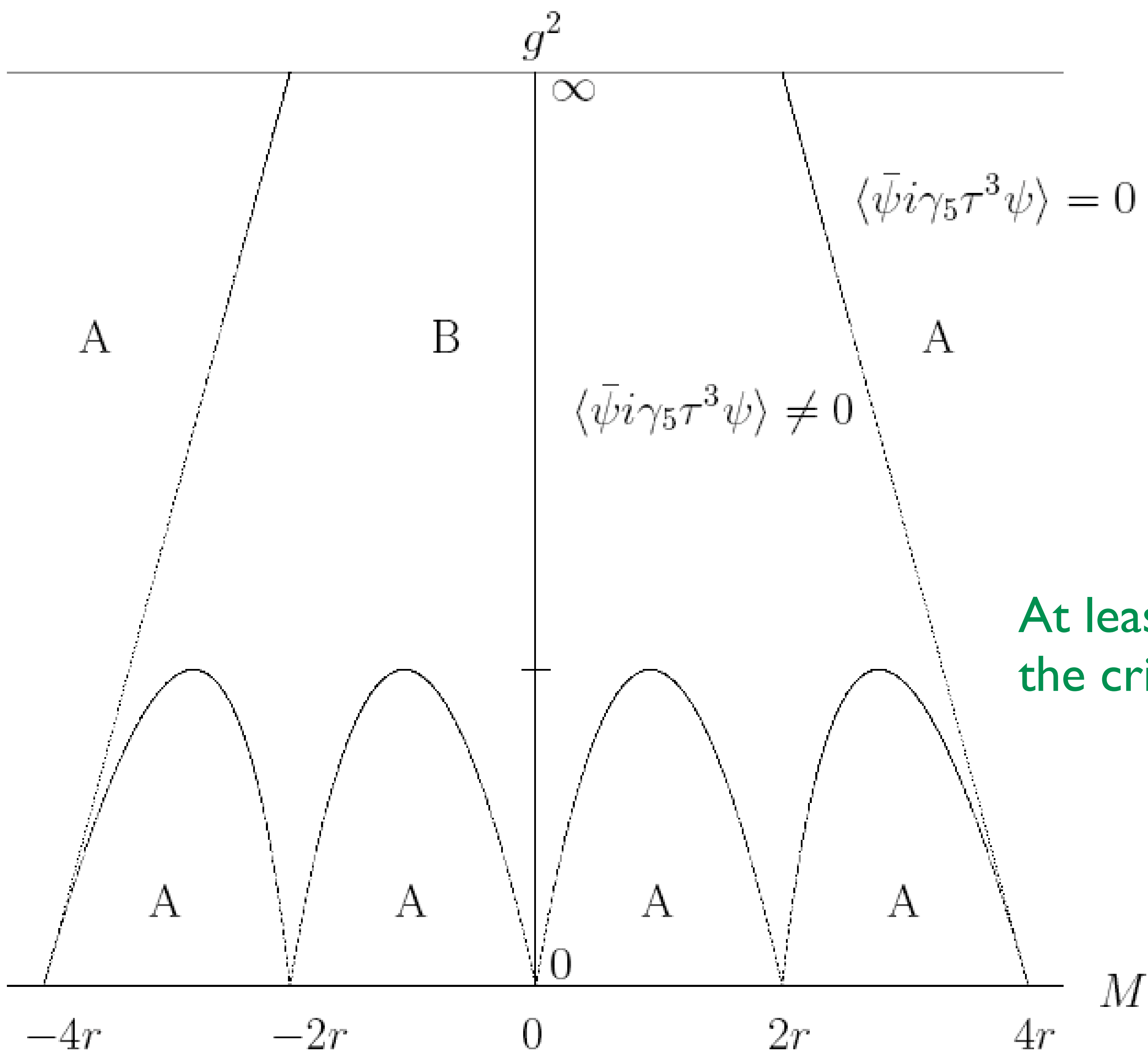
1 & 2: $m_\pi(r, M_{\rm cr}(r)) = m_\pi(-r, M_{\rm cr}(-r)) = m_\pi(r, -M_{\rm cr}(-r)) = 0$

3. "Conclusion": $M_{\rm cr}(r) = -M_{\rm cr}(-r)$

Only true if the defining equation has a unique solution !

Critical Mass: Phase diagram with Wilson fermion

Aoki '85

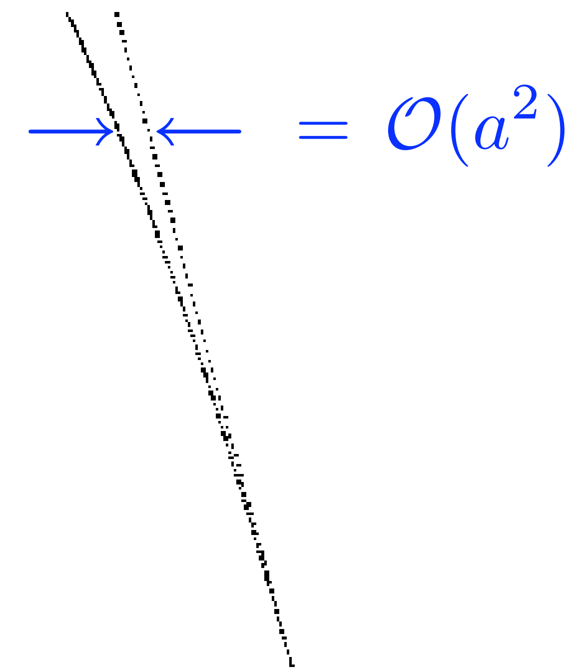


At least 2 values for the critical mass !

Critical Mass: ChPT analysis

Sharpe, Singleton '98

Two possible scenarios:



Scenario 1: No massless pion at non-zero a

Scenario 2: Spontaneous breaking of flavor and parity

⇨ Massless pions

Quantitative result for critical mass:

Width of "fingers" is $\mathcal{O}(a^2)$

$$\Longrightarrow \qquad M_{\rm cr}(r) = M_{\rm odd}(r) + a^2 c\, M_{\rm even}(r) \;\equiv\; M_{\rm cr}^{(1)}(r)$$

$$-M_{\rm cr}(-r) = M_{\rm odd}(r) - a^2 c\, M_{\rm even}(r) \;\equiv\; M_{\rm cr}^{(2)}(r)$$

$$\Longrightarrow \qquad M_{\rm cr}(r) \neq -M_{\rm cr}(-r)$$

What happens to O(a) improvement ?

Ansatz: $$M_{\rm cr}(r) = M_{\rm odd}(r) + a^2 c\, M_{\rm even}(r)$$

$$\Longrightarrow \quad \langle O\rangle(r, m_q, \omega = \frac{\pi}{2})^{TA} = \frac{1}{2}\Big[\langle O\rangle(r, m_q, \omega = \frac{\pi}{2}) + \langle O\rangle(-r, m_q', \omega = \frac{\pi}{2} + \omega')\Big]$$

$$m_q' = \sqrt{m_q^2 + (2a^2 c M_{\rm even}(r))^2} \qquad \tan\omega' = \frac{2a^2 c\, M_{\rm even}(r)}{m_q}$$

$\Longrightarrow$ Twist average = Wilson average only if $m_q \gg a^2$

$\Longrightarrow$ Automatic O(a) improvement only if $m_q \gg a^2$

Alternative definition for the twist angle

Define:

$$\overline{M}_{\rm cr}(r) = \frac{M_{\rm cr}(r) - M_{\rm cr}(-r)}{2} = -\overline{M}_{\rm cr}(-r)$$

$$\Delta M_{\rm cr}(r) = \frac{M_{\rm cr}(r) + M_{\rm cr}(-r)}{2} = \Delta M_{\rm cr}(-r)$$

$\Longrightarrow$ New definition for the twist angle:

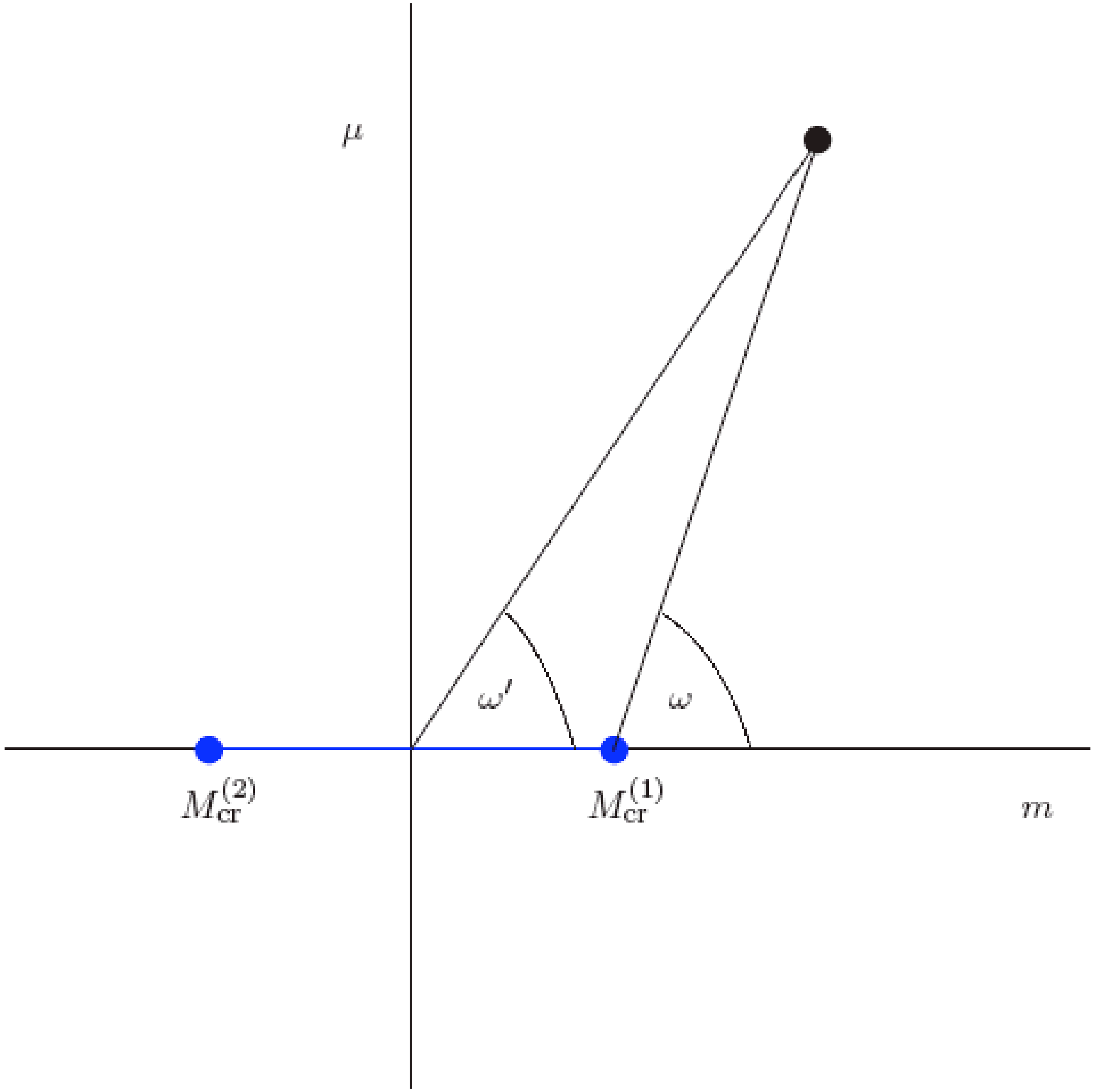
$$\bar{\psi}_{\rm ph}(x)\left[-\left(-a\frac{r}{2}\sum_{\mu}\nabla^{\star}_{\mu}\nabla_{\mu} + \overline{M}_{\rm cr}(r)\right)\exp(-iw\gamma_5\tau_3) + m_q + \Delta M_{\rm cr}(r)\right]\psi_{\rm ph}(x)$$

You can show:

Twist average = Wilson average irrespective of m_q

Automatic O(a) improvement holds for all m_q

Sketch of the different definitions



ω : Frezzotti / Rossi

ω' : Alternative definition

Part 2: ChPT analysis of the pion mass

- Brief introduction into ChPT at non-zero a (Wilson ChPT)
- Example: Pion mass at maximal twist: Are the linear a-effects absent ?

ChPT at nonzero a: Strategy

Two-step matching to effective theories:

Lee, Sharpe '98
Sharpe, Singleton '98

1. Lattice theory $\longrightarrow$ Symanzik's effective theory

   continuum theory making
   the a-dependence explicit

2. Symanzik's effective theory $\longrightarrow$ ChPT

   including the a-dependence

$\Longrightarrow$ Chiral expressions for m_π, f_π ... with explicit a-dependence

Symanzik's action for Lattice QCD with Wilson fermions

Locality and symmetries of the lattice theory

$\Rightarrow$ $$S_{eff} = S_{QCD} + a\,c \int \overline{\psi}\, i\sigma_{\mu\nu} G_{\mu\nu} \psi + \mathcal{O}(a^2)$$

- At $O(a)$ only one additional operator (making use of EOM)
- c : unknown coefficient (”low-energy constant”)
- $O(a^2)$: dim-6 operators: - fermion bilinears
  - 4-fermion operators

  Sheikholeslami, Wohlert
- $\frac{1}{a}$ divergence in quark mass must be subtracted

Reminder: Chiral Lagrangian

Fields: $$\Sigma(x) = \exp\left(\frac{2\,i}{F}\,\pi^a(x)T^a\right)$$ T^a: Group generators

Lagrangian: $$\mathcal{L}_{eff}\left[\Sigma, M\right] = \mathcal{L}_{eff}\left[\Sigma', M'\right]$$ M: Quark mass matrix

$$\Sigma' = L\Sigma R^\dagger \qquad M' = LMR^\dagger$$ L, R : Left, Right transformations

Expand in powers of derivatives and masses: $\mathcal{L}_{eff} = \mathcal{L}_2 + \mathcal{L}_4 + \ldots$

$$\mathcal{L}_2 = \frac{f^2}{4}\mathrm{tr}\left[\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\right] - \frac{f^2 B}{2}\,\mathrm{tr}\left[\Sigma^\dagger M + M^\dagger\Sigma\right]$$

f, B : undetermined low-energy constants

Chiral Lagrangian including a

$$S_{eff} = S_{QCD} + a\, c \int \overline{\psi}\, i\sigma_{\mu\nu} G_{\mu\nu} \psi + \mathcal{O}(a^2)$$

Pauli term breaks the chiral symmetry exactly like the mass term in S_{QCD}

$\Longrightarrow$ a enters chiral Lagrangian exactly like the mass term

$$\Longrightarrow \qquad \mathcal{L}_2 = \frac{f^2}{4} \mathrm{tr}\left[\partial_\mu \Sigma \partial_\mu \Sigma^\dagger\right] - \frac{f^2 B}{2} \mathrm{tr}\left[\Sigma^\dagger M + M^\dagger \Sigma\right] - \frac{f^2 W_0}{2}\, a\, \mathrm{tr}\left[\Sigma + \Sigma^\dagger\right]$$

Sharpe, Singleton '98
Rupak, Shoresh '02

W_0 : new undetermined low-energy constant

includes $c = c(g_0^2)$ not really a constant (weak a dependence)

$\mathcal{L}_4$ -Lagrangian:

$$\mathcal{L}_4 = \mathcal{L}_4(p^4, p^2 m, m^2) \;+\; \mathcal{L}_4(p^2 a, ma) \;+\; \mathcal{L}_4(a^2)$$

Gasser, Leutwyler '85

Rupak, Shoresh '02

Rupak, Shoresh, Baer '03
Aoki '03

- No O(4) symmetry breaking terms in $\mathcal{L}_4(a^2)$ (start at $\mathcal{O}(a^2 p^4)$)
- Total number of low-energy constants: 10 L_i + (5 + 3)W_i = 18

Power counting

The power counting is non-trivial because of

1. the additive mass renormalization $\propto \frac{1}{a}$

2. two symmetry breaking parameters $, m_{\text{quark}}$

   $\Longrightarrow$ their relative size matters

Leading order pion mass (degenerate case)

$$M_\pi^2 = 2Bm + 2aW_0 \qquad m = m_u = m_d$$

- Leading a-effect: Shift in the pion mass
- M_π^2 does not vanish for $m = 0$

Common practice on the lattice:

$$M_\pi^2 = 0 \quad \text{for} \quad m' = Z_m(m_0 - m_{\rm cr}) = 0$$

In ChPT this corresponds to $m' = m\left(1 + \frac{W_0}{B}a\right)$

$$\Longrightarrow \quad M_\pi^2 = 2Bm'$$

$$\Downarrow$$

$M_\pi^2 < 0 \text{ for } m' < 0$ Tachyon !

a^2 effect must be included for small m'

Different power countings have been discussed:

If $m' \gg a^2 \rightarrow$ continuum like ChPT + small $\mathcal{O}(a^n)$ corrections

Rupak, Shoresh, Baer ‘03

If $m' \approx a^2 \rightarrow$ qualitatively different :

Non-trivial phase diagram — Sharpe, Singleton ‘98

Modification of chiral logs — Aoki ‘03

Spontaneous flavor and parity breaking

Potential energy:

$(\mathrm{N_f} = 2)$

Sharpe, Singleton '98

$$V = -c_1 m' \mathrm{tr}\left[\Sigma + \Sigma^\dagger\right] + c_2 a^2 \left(\mathrm{tr}\left[\Sigma + \Sigma^\dagger\right]\right)^2$$

$c_1(f, B)$

$c_2(f, B, W_i)$

A: $\mathrm{sign}\, c_2 = +1 \quad \Longrightarrow \quad \Sigma_{\mathrm{vacuum}} \neq \pm 1$

flavor and parity are broken
massless pions at $a \neq 0$

B: $\mathrm{sign}\, c_2 = -1 \quad \Longrightarrow \quad \Sigma_{\mathrm{vacuum}} = \pm 1$

no flavor/parity breaking
no massless pions

The realized scenario depends on the details of the underlying lattice theory

(i.e. the particular Lattice action)

ChPT for tmQCD

Symanzik action: $S_{eff} = S_{\mathrm{tmQCD}} + a\,\mathrm{Pauli\,term} + \mathcal{O}(a^2)$

$\Longrightarrow$ $$\mathcal{L}_{\mathrm{chiral}}\left[m, \omega, a, a^2\right]$$

a: Muenster, Schmidt

a^2: Sharpe, Wu

$\Longrightarrow$ m_π^2, f_π as a function of m, ω, a, a^2

Again: Proper parameter matching required !

Here m and ω

Check for O(a) improvement of the pion mass

1. Lagrangian $\mathcal{L}_{\rm chiral}$ $\longrightarrow$ potential Energy $\mathcal{V}_{\rm chiral}$
2. Find ground state $\Sigma_0 = e^{i\phi\tau_3}$ by $\frac{\mathrm{d}\mathcal{V}_{\rm chiral}}{\mathrm{d}\phi} = 0$
3. Expand around Σ_0 and find M_π^2 (to LO)
4. Express M_π^2 in terms of the twist angle ω corresponding to the lattice theory
5. Go to $\omega = \frac{\pi}{2}$ and check for $\mathcal{O}(a)$

$$m_{\pi_a}^2 = \frac{2Bm + 2W_0 a}{\cos\phi} - 2c_2 a^2$$

Aoki, Baer
hep-lat/0409006

$$m = m' \cos w$$

Definition of Frezotti / Rossi

Definition of ω :

Lattice theory $$(m_0 - M_{\rm cr}(r))e^{i\omega\tau_3\gamma_5}$$

Effective theory $$\tan\omega = \frac{2Bm_L\sin\omega_L}{2Bm_L\cos\omega_L + 2W_0a - 2c_2a^2}$$

For $\omega = \pi/2 \quad (\mu := m_L\sin\omega_L)$

1. $2B\mu \geq \mathcal{O}(a) \quad \Longrightarrow$
$$m_{\pi_a}^2 = 2B\mu$$
$$m_{\pi_3}^2 - m_{\pi_a}^2 = 2c_2a^2$$

2. $2B\mu \ll 2c_2a^2 \quad \Longrightarrow$
$$m_{\pi_a}^2 = (c_2a^2)^{1/3}(2B\mu)^{2/3}$$
$$m_{\pi_3}^2 - m_{\pi_a}^2 = 2(c_2a^2)^{1/3}(2B\mu)^{2/3}$$

O(a) improvement only in case 1

Alternative definition for the twist

Definition of ω : Lattice theory

$$\left(m_0 - \frac{M_{\rm cr}(r) - M_{\rm cr}(-r)}{2} \right) e^{i\omega\tau_3}$$

Effective theory

$$\tan\omega = \frac{2Bm_L \sin\omega_L}{2Bm_L\cos\omega_L + 2W_0 a}$$

For $\omega = \pi/2 \quad (\mu := m_L \sin\omega_L)$

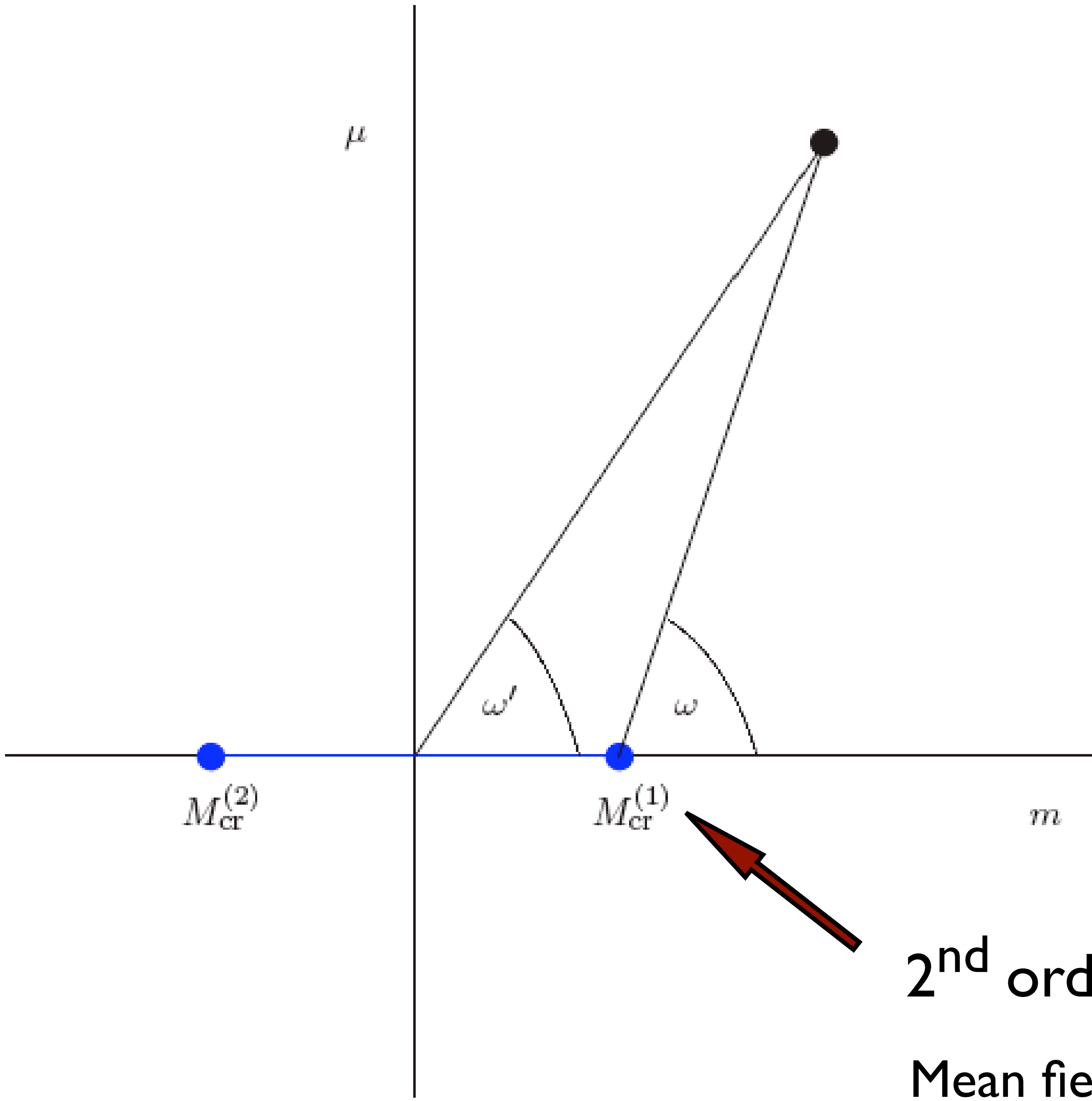
Without restrictions on $2B\mu \quad \Longrightarrow$

$$m_{\pi_a}^2 = 2B\mu$$

$$m_{\pi_3}^2 - m_{\pi_a}^2 = 2c_2 a^2$$

Automatic O(a) improvement irrespective of the size of μ !

Note: $m_{\pi_3} = \sqrt{2c_2}\, a = \mathcal{O}(a) \text{ at } \mu = 0$



ω : Frezzotti / Rossi

ω' : Alternative definition

2nd order phase transition point

Mean field critical exponent = 2/3

Twist angle from Ward identities

In continuum tmQCD:

$$\tan\omega_{\rm WT} = \frac{\langle \partial_\mu V_\mu^2 \; P^1 \rangle}{\langle \partial_\mu A_\mu^1 \; P^1 \rangle}$$

Vector and Axial vector WT identities:

$$\partial_\mu V_\mu^a = -2\mu\epsilon^{3ab} P^b$$

$$\partial_\mu A_\mu^a = 2mP^a + 2i\mu S^0 \delta_{a3}$$

$\Longrightarrow$

$$\tan\omega_{\rm WT} = \frac{\mu}{m}$$

ω_{WT} in the effective theory

1. Maximal twist of Frezzotti / Rossi :

For $2B\mu \ll 2c_2 a^2$ $\Longrightarrow$ $\tan\omega_{\mathrm{WT}} \simeq \left(\frac{2B\mu}{2c_2 a^2}\right)^{1/3}$

$\Longrightarrow$ $\omega_{\mathrm{WT}} \neq \pi/2$ $\quad (\omega_{\mathrm{WT}} = 0 \ \text{for}\ \mu = 0)$

2. Alternative definition :

$$\tan\omega_{\mathrm{WT}} = \infty$$

$$\omega_{\mathrm{WT}} = \pi/2 = \omega$$

Part 3: Alternative Scenario($c_2 < 0$)

$$\mu = 0$$

- Vacuum

$$\cos\phi = \begin{cases} 1 & c_1 > 0 \\ -1 & c_1 < 0 \end{cases} \qquad c_1 := 2Bm + 2W_0 a$$

- Pion mass

$$m_{\pi_3}^2 = m_{\pi_a}^2 = |c_1| - 2c_2 a^2$$

- AWI quark mass

$$m_{\mathrm{AWI}} \propto \begin{cases} m_\pi^2 & c_1 > 0 \\ -m_\pi^2 & c_1 < 0 \end{cases}$$

⇨ No massless point (Minimum: $m_\pi^2 = -2c_2 a^2$) ⇨ No M_{cr}

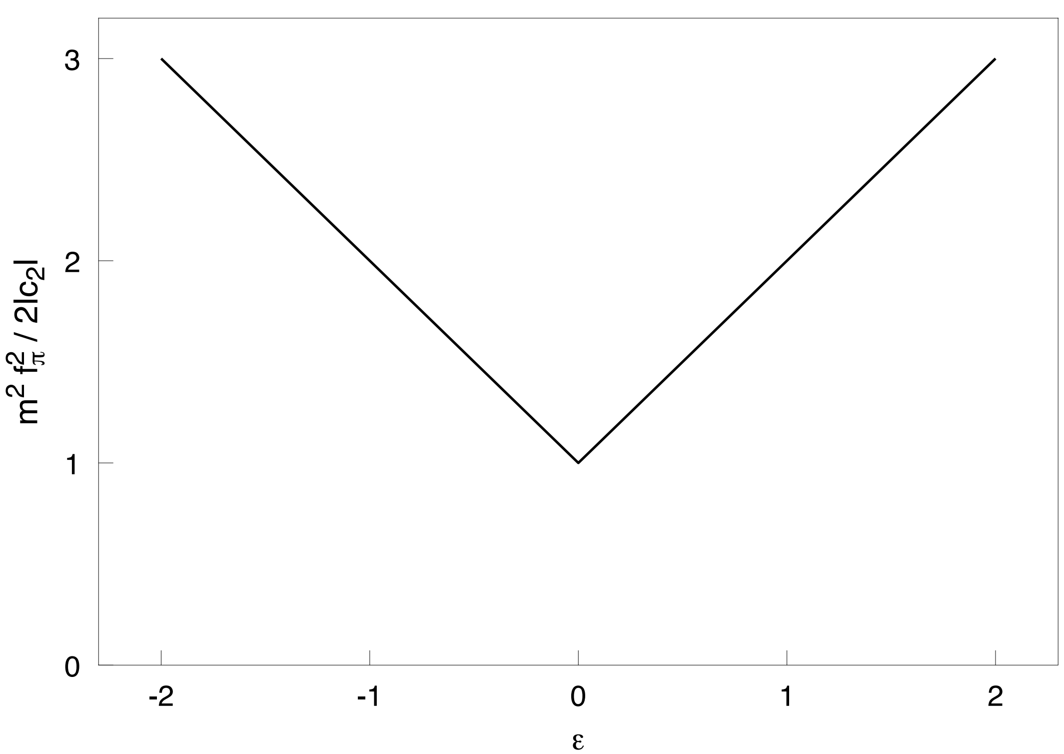
⇩ ⇧

No Flavor-Parity breaking phase

⇩ ⇧

No zero modes for Wilson-Dirac operator

Sharpe, Singleton '98



$\mu \neq 0$

Define $\quad M_{\rm cr} \Leftrightarrow c_1 = 2Bm + 2W_0 a = 0$

Maximal twist $\quad c_1 = 0, \mu \neq 0$

- Vacuum

$$\cos\phi = \begin{cases} \pm\sqrt{1 - \frac{(2B\mu)^2}{(2c_2 a)^2}} & |2B\mu| < -2c_2 a^2 \quad c_1 \to 0^{\pm} \\ 0 & |2B\mu| \geq -2c_2 a^2 \quad \forall c_1 \end{cases}$$

- Twist angle from WTI

$$\tan w_{\rm WT} = \begin{cases} \pm\frac{2B\mu}{-2C_2 a^2} & |2B\mu| < -2c_2 a^2 \quad c_1 \to 0^{\pm} \\ \pm\infty & |2B\mu| \geq -2c_2 a^2 \quad \pm|\mu| \end{cases}$$

Twist angle becomes maximal ($w_{\rm WT} = \pm\pi/2$) only for $|2B\mu| \geq -2c_2 a^2$

- Pion mass

$$m_{\pi_a}^2 = \begin{cases} -2c_2a^2 & |2B\mu| < -2c_2a^2 \\ |2B\mu| & |2B\mu| \geq -2c_2a^2 \end{cases}$$

$$m_{\pi_3}^2 = \begin{cases} -2c_2a^2\left(1 - \frac{(2B\mu)^2}{(2c_2a^2)^2}\right) & |2B\mu| < -2c_2a^2 \\ 0 & |2B\mu| = -2c_2a^2 \\ |2B\mu| + 2c_2a^2 & |2B\mu| > -2c_2a^2 \end{cases}$$

$\Downarrow$

$$m_{\pi_3}^2 < m_{\pi_a}^2$$

Summary

- Twisted mass QCD can be automatically O(a) improved at max. twist
- Tricky: Definition of the proper twist angle
- Definition of Frezzotti / Rossi works only for $m \gg a^2$
- Alternative definition can be given that ensures O(a) improvement without restrictions on the quark mass
- O(a) improvement can be explicitly demonstrated for the pion mass in ChPT at non-zero lattice spacing
- Physics at $c_2 < 0$ is different from the one at $c_2 > 0$
  - No massless pion
  - No critical quark mass
  - No zero mode