$\bar{\psi}\gamma_5\psi$ condensates and topology fixing action

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Hidenori Fukaya

YITP, Kyoto Univ.

Collaboration with T.Onogi (YITP)

1. Introduction

- Why topology fixing action ?

An action proposed by Luscher provide us to simulate with a fixed topological charge by suppressing the field strength.

$$S_G = \begin{cases} \dfrac{1}{g^2} \displaystyle\sum_{x,\mu,\nu} \dfrac{(1 - \mathrm{Re} P_{\mu\nu}(x))}{1 - |1 - P_{\mu\nu}(x)|^2/\epsilon^2} & \text{if admissible} \\ \infty & \text{otherwise} \end{cases}$$

- Better statistics of higher topological sectors.
- Locality of Overlap D is improved.
- Chiral symmetry is also improved.
- Theta vacuum if one has a reweighting way.

M.Lüscher,Nucl.Phys.B538,515(1999), Nucl.Phys.B549,295(1999)

1. Introduction

- Overview

In QCD or 2-d QED ($\theta = 0$),

$$\langle \bar{\psi}\psi \rangle \neq 0 \quad \text{and} \quad \langle \bar{\psi}\gamma_5\psi \rangle = 0,$$

but in $\theta \neq 0$ case, we expect

$$\langle \bar{\psi}\psi \rangle \neq 0 \quad \text{and} \quad \langle \bar{\psi}\gamma_5\psi \rangle \neq 0.$$

Then the η has a long-range correlation as

$$\langle \eta(x)^\dagger \eta(0) \rangle \to_{|x|\to\infty} \langle \bar{\psi}\gamma_5\psi \rangle \langle \bar{\psi}\gamma_5\psi \rangle \neq 0.$$

1. Introduction

- The 2-flavor massive Schwinger model

$$S=\int d^2x\left(\frac{1}{4g^2}F_{\mu\nu}^2+\sum_{f=1}^{2}\bar{\psi}_f(\not{D}+m)\psi_f+\frac{i\theta}{4\pi}\epsilon_{\mu\nu}F_{\mu\nu}\right)$$

  - Confinement
  - Chiral condensation
  - U(1) problem ($m_\eta > m_\pi$)
  - Bosonization picture ⇒ strong coupling limit

We define

$$\begin{aligned}\pi(x) &= \bar{\psi}_1(x)\gamma_5\psi_1(x)-\bar{\psi}_2(x)\gamma_5\psi_2(x),\\ \eta(x) &= \bar{\psi}_1(x)\gamma_5\psi_1(x)+\bar{\psi}_2(x)\gamma_5\psi_2(x).\end{aligned}$$

1. Introduction

- How to evaluate **θ** vacuum

$$\langle O(x)\rangle^{\theta} = \frac{\int dA d\psi O(x) e^{-S_g - S_f - S_\theta}}{\int dA d\psi e^{-S_g - S_f - S_\theta}}$$

$\Downarrow$ using $\int dA = \sum_Q \int dA^Q$

$$= \frac{\sum_Q e^{iQ\theta} \overbrace{\left(\int dA^Q d\psi O(x) e^{-S_g - S_f} / Z_Q \right)}^{\langle O(x)\rangle^Q} Z_Q}{\sum_Q e^{iQ\theta} \underbrace{\int dA^Q d\psi e^{-S_g - S_f}}_{=Z_Q}} = \frac{\sum_Q e^{iQ\theta} \langle O(x)\rangle^Q Z_Q}{\sum_Q e^{iQ\theta} Z_Q}$$

$\Rightarrow$ We need $\langle O(x)\rangle^Q$ and Z_Q .

1. Introduction

- How to evaluate **θ** vacuum

We can calculate $\langle O(x) \rangle^Q$ by generating link variables which satisfy Lüscher's bound (which are called "admissible".);

$|1 - P_{\mu\nu}(x)| < \epsilon$ for all $x, \mu, \nu (0 < \epsilon < 2)$,

realized by the following gauge action;

$$S_G = \begin{cases} \dfrac{1}{g^2} \displaystyle\sum_{x,\mu,\nu} \dfrac{(1 - \mathrm{Re}P_{\mu\nu}(x))}{1 - |1 - P_{\mu\nu}(x)|^2/\epsilon^2} & \text{if admissible} \\ \infty & \text{otherwise} \end{cases}$$

⇒ topologcal charge is conserved !!!

※ $0 < \boldsymbol{\varepsilon} < \boldsymbol{\pi} / 5$ → GW Dirac operator is local.

M.Lüscher,Nucl.Phys.B538,515(1999), Nucl.Phys.B549,295(1999)

1. Introduction

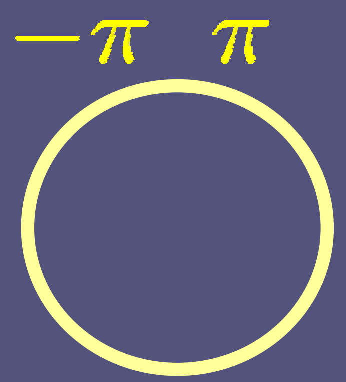
- How to evaluate θ vacuum

Topological charge is defined as

$$Q = \frac{1}{4\pi}\sum_x \epsilon_{\mu\nu} F^{\text{lat}}_{\mu\nu}(x), \quad (F^{\text{lat}}_{\mu\nu} \equiv -i \ln P_{\mu\nu}).$$

- Without Lüscher's bound

$$-\frac{\pi}{a^2} < F^{\text{lat}}_{\mu\nu}(x) \leq \frac{\pi}{a^2}.$$



⇒ topological charge can jump； Q → Q±1.

1. Introduction

- How to evaluate **θ** vacuum

Note that the continuum limit is the same as the standard plaquette action at any **ε**,

$$S_G = \begin{cases} \dfrac{1}{g^2} \displaystyle\sum_{x,\mu,\nu} \dfrac{(1 - \mathrm{Re} P_{\mu\nu}(x))}{1 - |1 - P_{\mu\nu}(x)|^2/\epsilon^2} & \text{if admissible} \\ \infty & \text{otherwise} \end{cases}$$

$$\rightarrow_{a\rightarrow 0} \frac{1}{g^2} \int d^2x \frac{F^2_{\mu\nu}(x) + O(a^2)}{1 - \frac{a^4}{\epsilon^2}(F^2_{\mu\nu}(x) + O(a^2))}.$$

1. Introduction

- How to evaluate θ vacuum

Using the following expression,

$$Z^Q(\beta, m) = Z^Q(\infty, m) e^{\int_\beta^\infty d\beta' \langle S_g \rangle_{\beta'}^Q},$$

$$= \lim_{\beta_{max} \to \infty} \int DA_{cl}^Q e^{-\beta_{max} S_{min}^Q} \det(D+m)^2 e^{\int_\beta^\infty d\beta' \langle S_g \rangle_{\beta'}^Q},$$

$$= \underbrace{e^{-\beta S_{min}^Q}}_{\text{classical solution}} \underbrace{\left[\int DA_{cl}^Q \det(D+m)^2\right]}_{\text{moduli integral}} \underbrace{e^{\int_\beta^\infty d\beta' \langle S_g - S_{min}^Q \rangle_{\beta'}^Q}}_{\text{fluctuation}},$$

we can evaluate Z_Q normalized by $Z_{Q=0}$!

1. Introduction

- How to evaluate **θ** vacuum

$$R^Q \equiv \frac{Z^Q(\beta,m)}{Z^0(\beta,m)}$$

$$= \underbrace{e^{-\beta S^Q_{min}}}_{\text{classical solution}} \underbrace{\left[\frac{\int DA^Q_{cl}\,\mathrm{det}(D+m)^2}{\int DA^0_{cl}\,\mathrm{det}(D+m)^2}\right]}_{\text{moduli integral}} \underbrace{e^{\int_\beta^\infty d\beta' \Delta S^Q_g}}_{\text{fluctuation}}$$

  - Classical solution ⇒ Constant field strength.
  - Moduli (constant potential which affects Polyakov loops) integral of the determinants

    ⇒ Householder and QL method.
  - Integral of $\Delta S^Q_g(\beta') \equiv \langle S_g - S^Q_{min}\rangle^Q_{\beta'} - \langle S_g\rangle^0_{\beta'} = O(\frac{1}{\beta'^2})$

    ⇒ fitting with polynomiyals.

1. Introduction

- How to evaluate θ vacuum

Now we can evaluate

$$\langle O(x)\rangle^{\theta} = \frac{\sum_Q e^{iQ\theta}\langle O(x)\rangle^{Q} Z_Q/Z_0}{\sum_Q e^{iQ\theta} Z_Q/Z_0},$$

by Lüscher's gauge action and our reweighting method.

Details are shown in

HF,T.Onogi,Phys.Rev.D68,074503(2003).

1. Introduction

The simulations were done on the Alpha work station at YITP and SX-5 at RCNP.

- Numerical simulation
  - Action : Lüscher's action + DW fermion with PV's.
  - Algorithm : The hybrid Monte Carlo method.
  - Gauge coupling : $g = 1.0$. ($\beta = 1/g^2 = 1.0$.)
  - Fermion mass : $m_1 = m_2 = 0.1, 0.15, 0.2, 0.25, 0.3$.
  - Lattice size : 16×16 ($\times 6$).
  - Admissibility condition : $\epsilon = \sqrt{2}$
  - Topological charge : $Q = -4 \sim +4$
  - 50 molecular dynamics steps with the step size $\Delta\tau = 0.035$ in one trajectory.
  - 300 configurations are generated in each sector.
  - Admissibility is checked at the Metropolis test.

2. Fermion Condensates

- Condensations in each sector

Integrating the anomaly equation;

$$\partial_\mu J^5_\mu = 2im\bar{\psi}\gamma_5\psi + \frac{i}{2\pi}\epsilon_{\mu\nu}F_{\mu\nu}$$

$$\Downarrow$$

$$0 = \int_{T^2} d^2x \langle \partial_\mu J^5_\mu \rangle^Q = 2imV\langle\bar{\psi}\gamma_5\psi\rangle^Q + \langle \int_{T^2} d^2x \frac{i}{2\pi}\epsilon_{\mu\nu}F_{\mu\nu}\rangle^Q$$

one obtains the following relation

$$-\langle\bar{\psi}\gamma_5\psi\rangle^Q = \frac{Q}{mV}$$

2. Fermion Condensations

- Condensations in θ vacuum

We evaluate

$$\langle \bar{\psi}\psi \rangle^{\theta} = \frac{\sum_{Q=-4}^{4} e^{iQ\theta} \langle \bar{\psi}\psi \rangle^{Q} R^{Q}}{\sum_{Q=-4}^{4} e^{iQ\theta} R^{Q}},$$

and

$$\langle \bar{\psi}\gamma_5\psi \rangle^{\theta} = \frac{\sum_{Q=-4}^{4} e^{iQ\theta} \langle \bar{\psi}\gamma_5\psi \rangle^{Q} R^{Q}}{\sum_{Q=-4}^{4} e^{iQ\theta} R^{Q}}.$$

3. The η meson in θ vacuum

- The η correlations in each sector
- Connected part (pion results)

Our numerical data show a good agreement with the continuum theory at small θ ;

$$m_\pi \propto m^{2/3} g^{1/3} \cos^{2/3} \frac{\theta}{2}$$

3. The η meson in θ vacuum

- The η correlations in each sector

$$\langle \bar{\psi}\gamma_5\psi\rangle^Q \neq 0$$

⇒ We expect the η meson has a long-range correlation in each topological sector.

We measure

$$\langle \eta^\dagger(x)\eta(y)\rangle^Q = 2\left\langle \mathrm{tr}\left(\gamma_5\frac{1}{D}(x,y)\gamma_5\frac{1}{D}(y,x)\right)\right\rangle^Q$$
$$-4\left\langle \mathrm{tr}\left(\gamma_5\frac{1}{D}(x,x)\right)\mathrm{tr}\left(\gamma_5\frac{1}{D}(y,y)\right)\right\rangle^Q,$$

where 2nd part is calculated exactly.

3. The η meson in θ vacuum

- The η correlations in θ vacuum

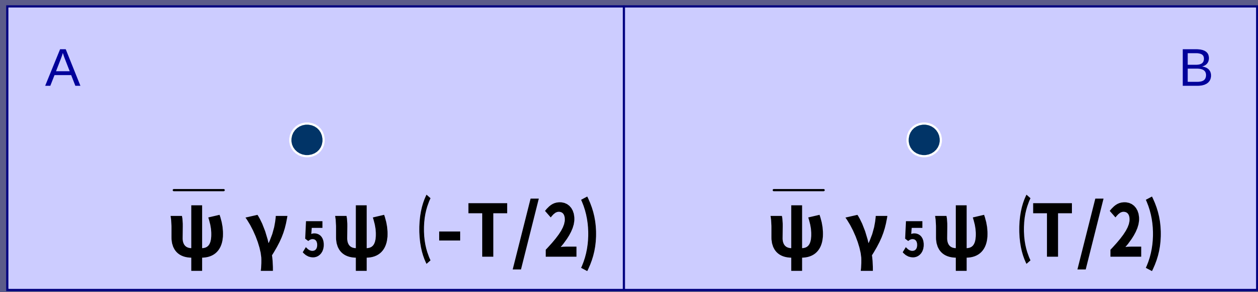
We evaluate

$$\langle \eta^\dagger(x)\eta(y)\rangle^\theta = \frac{\sum_{Q=-4}^{4} e^{iQ\theta}\langle \eta^\dagger(x)\eta(y)\rangle^Q R^Q}{\sum_{Q=-4}^{4} e^{iQ\theta} R^Q}.$$

3. The η meson in θ vacuum

- Clustering decomposition

Consider the operators put on t = -T/2 and t=T/2 of a large box divided into two parts;

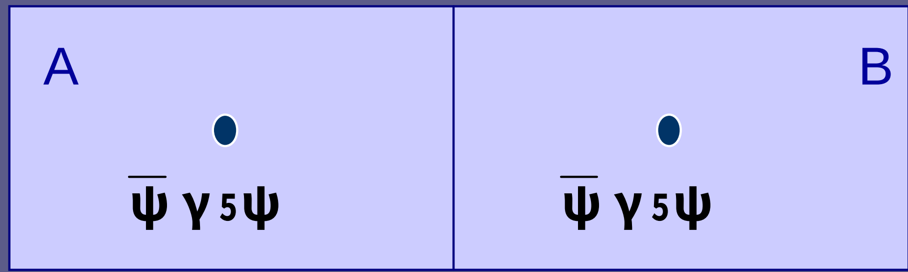


We expect the η correlation in each sector is expressed as

$$\langle \eta^\dagger(T/2)\eta(-T/2)\rangle^Q \to_{T\to\infty} -4\sum_{Q'} P_{Q,Q'} \langle\bar{\psi}\gamma_5\psi\rangle_A^{Q'} \langle\bar{\psi}\gamma_5\psi\rangle_B^{Q-Q'}.$$

3. The η meson in θ vacuum

- Clustering decomposition
  - Q=0 case



$$\langle \eta^\dagger(T/2)\eta(-T/2)\rangle^{Q=0}$$

$$\rightarrow_{T\rightarrow \text{large}} -4\sum_{Q'} P_{0,Q'}\langle\bar{\psi}\gamma_5\psi\rangle_A^{Q'}\langle\bar{\psi}\gamma_5\psi\rangle_B^{-Q'}$$

$$= +4\sum_{Q'} P_{0,Q'}\left(\langle\bar{\psi}\gamma_5\psi\rangle^{Q'}\right)^2 > 0 \ \ !!!$$

3. The η meson in θ vacuum

- Clustering decomposition
  - Q>>0 case

We assume Q' distribution is expressed as Gaussian around Q/2 ; $P_{Q,Q'} \sim \sqrt{\alpha/\pi} e^{-\alpha(Q'-\frac{Q}{2})^2}$. $(\sqrt{1/\alpha} << Q/2)$

$$\langle \eta^\dagger(T/2)\eta(-T/2)\rangle^{Q>>0}$$

$$\sim -4\sqrt{\frac{\alpha}{\pi}}\int dQ' \underbrace{\langle\bar{\psi}\gamma_5\psi\rangle^{Q'}}_{=Q'/mV}\langle\bar{\psi}\gamma_5\psi\rangle^{Q-Q'} e^{-\alpha(Q'-Q/2)^2}$$

$$= -4\sqrt{\frac{\alpha}{\pi}}\int dq \frac{(Q/2-q)(Q/2+q)e^{-\alpha q^2}}{(mV/2)^2}$$

$$= -\frac{4}{m^2}\left(\frac{Q^2}{V^2}-\frac{1}{\alpha V^2}\right) \sim -\frac{4Q^2}{m^2V^2} < 0 \;\;!!!$$

4. Summary and Discussion

- The results of 2-d QED

We obtain

$\langle \bar{\psi}\gamma_5\psi\rangle^{\theta} \neq 0,$ and

$$\langle \eta^{\dagger}(x)\eta(y)\rangle^{\theta} = A(e^{-m_\eta|x-y|} + e^{-m_\eta(L-|x-y|)}) + C_\theta.$$

It is important that vanishing of C_θ is non-trivial even in $\theta = 0$ case and

$\Delta\langle \bar{\psi}\gamma_5\psi\rangle^{Q}$ ⇒ the fluctuation of disconnected diagrams !!!

4. Summary and Discussion

- How about 4-d QCD ?

We expect

$$\langle \bar{\psi}\gamma_5\psi \rangle^{\theta} \neq 0,$$

and

$$\langle \eta^{\dagger}(x)\eta(y) \rangle^{\theta} = A(e^{-m_\eta|x-y|} + e^{-m_\eta(L-|x-y|)}) + C_\theta.$$

It is important that vanishing of C_θ is non-trivial even in $\theta = 0$ case and

$\Delta \langle \bar{\psi}\gamma_5\psi \rangle^{Q}$ ⇒ the fluctuation of disconnected diagrams !!!

4. Summary and Discussion

- How about 4-d QCD ?

In QCD, one obtains the same anomaly equation;

$$\langle \bar{\psi}\gamma_5\psi\rangle^Q = -Q/mV.$$

From the clustering decomposition,

$$\lim_{|x|\to\text{large}} \langle \eta^\dagger(x)\eta(0)\rangle^Q = -\frac{4Q^2}{m^2V^2},$$

is also expected, which is consistent with

ChPT results in the **ε** -regime !!!!

(P.H.Damgaard et al., Nucl.Phys B629(2002)445),

$$\langle P^0(x)P^0(0)\rangle = -\frac{N_f^2Q^2}{m^2V^2} + \frac{N_f\Sigma_Q^{\text{1-loop}}(m\Sigma V)}{mV} + (x \text{ dependent terms}).$$

4. Summary and Discussion

- Application to 4-d QCD ?

Are $R^Q = Z^Q / Z^0$'s calculable in 4-d QCD?

⇒ Yes (in principle.).

Difficulties

- Fermion determinants

  ⇒ eigenvalue truncation ?

- Moduli integrals

  ⇒ we need all the solutions on 4-d torus.

- VEV's of action (ΔS^Q)

  ⇒ perturbative analysis ?