Mobius Fermions

Improved DW Chiral Fermions

Sept 22, ILFT04 Shuzenji, Japan

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Outline

- Generalization of Shamir/Borici DW Fermions
- Modified EVEN/ODD Preconditioning
- M_{res} and the Ward-Takahashi identity at finite L_s
- Performance Test on Wilson 16^3 £ 32, $\beta = 6.0$
- Future Developments

Chiral Breaking Operator For Overlap

Approximate Ginsparg-Wilson relation:

$$\gamma_5 D_{ov}(0)+D_{ov}(0)\gamma_5-2D_{ov}(0)\gamma_5 D_{ov}(0) = 2\gamma_5\Delta_L$$

$$\Delta_L = \frac{1}{4}(1-\epsilon_L^2(H))$$

Breaking term in Neother theorem:

y

$$(1-m)^{-1}\delta(\bar{\psi}D_{ov}(m)\psi) = m_q\bar{\psi}(\gamma_5+\hat{\gamma}_5)\psi+2\bar{\psi}\gamma_5\Delta_L\psi$$

$m_q = m/(1-m)$ and $\hat{\gamma}_5 = \gamma_5(1-2D_0) = -\epsilon_L(H)$

y

Chiral breaking operator, $\Delta_L(x,y)$, is "local", not ultra local.

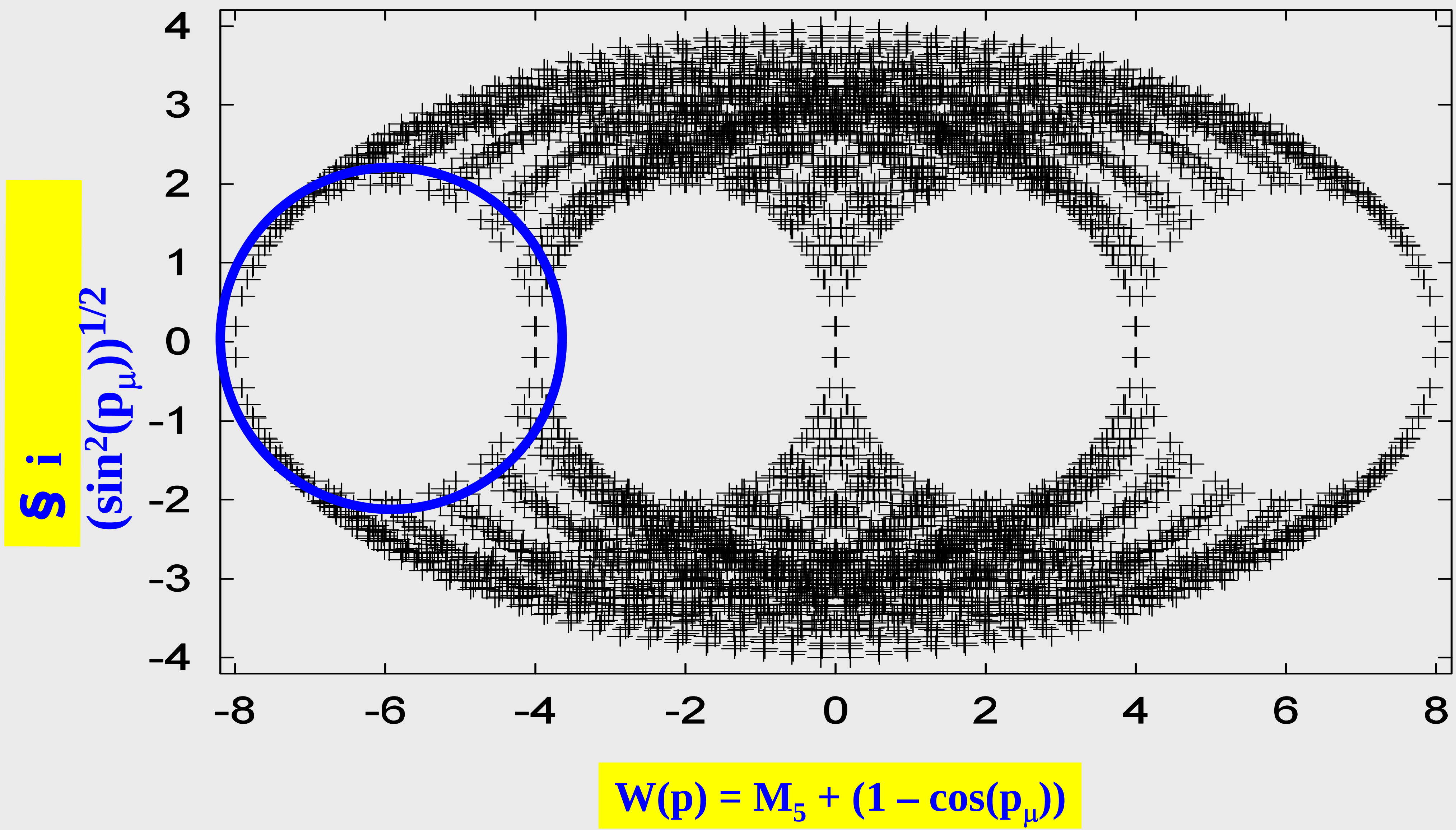
Overlap Solutions to GW Relation

$$D_{ov}(m) = \frac{1+m}{2} + \frac{1-m}{2}\,\gamma_5\,\epsilon_L[\gamma_5 D(M_5)]$$

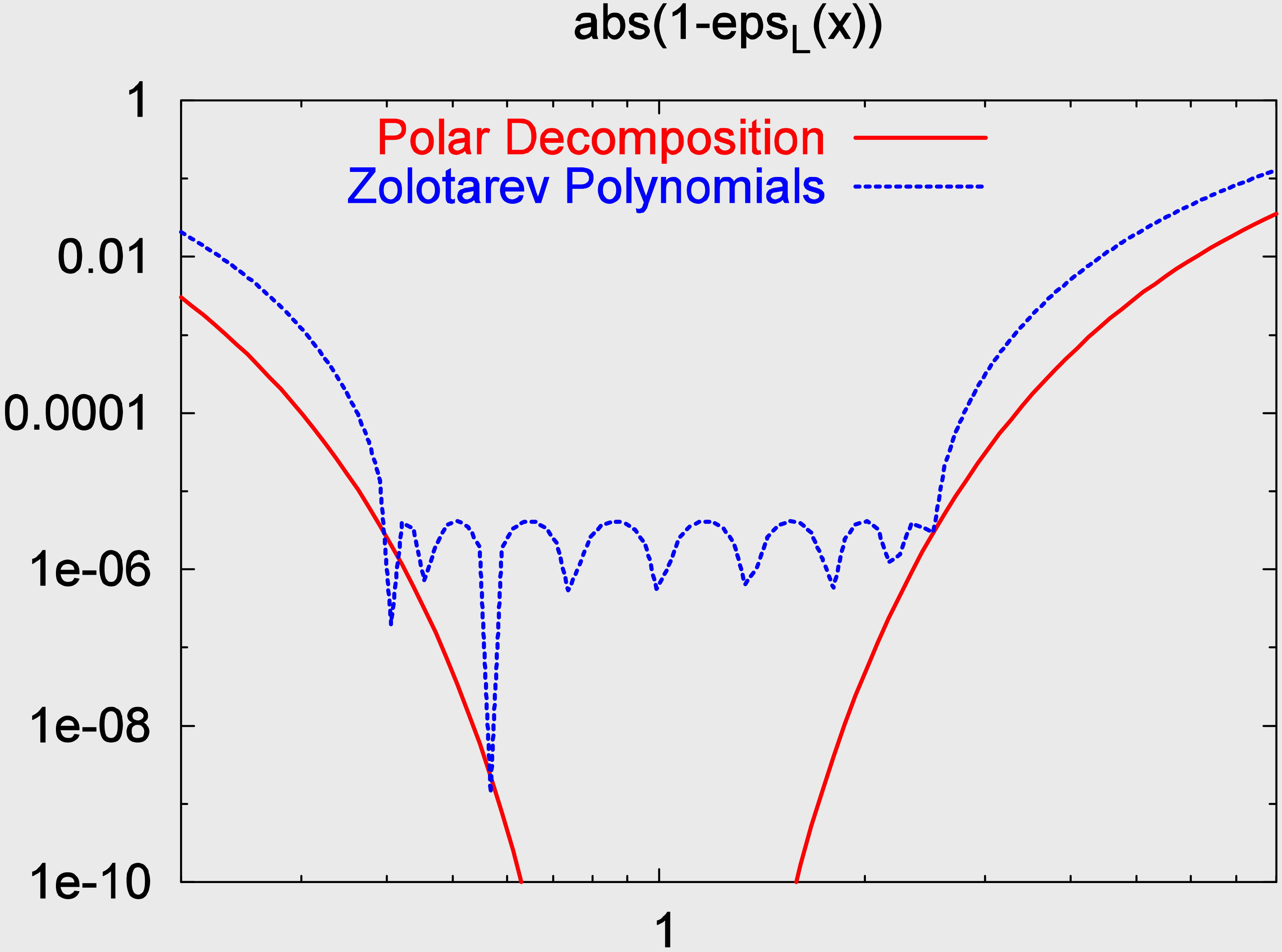
GOALS OF DOMAIN WALL IMPLEMENTATION

- Best choices for 4-d "kernel": H ´ $\gamma_5 D(M_5)$
- Best approx of sign function: $\varepsilon_L[x]$ ' $x/|x|$
- Define Chiral breaking : $\Delta_L[H] = (1 - \varepsilon^2_L[H])/4$

$D_{wilson}(-4)$ Free Eigenvalues



$1-\varepsilon_L[x]$ at $L_s = 16$ (polar) vs 8 (Zolotarev)



What is the best way to exploit the 5th time axis?

DW Construction: Shamir vs Borici

Shamir: $$D(M_5) = \frac{a_5 D_w(M_5)}{2 + a_5 D_w(M_5)}$$

or Borici: $$D(M_5) = a_5 D_W(M_5)$$

where the standard Wilson 4-d operator is

$$D_w(M) = (4+M_5)\delta_{x,y} - \frac{1}{2}\sum_{\mu=1}^{4}[(1-\gamma_\mu)U_\mu(x)\delta_{x+\mu,y} + (1+\gamma_\mu)U_\mu^\dagger(y)\delta_{x,y+\mu}]$$

implies

$$\epsilon_{L_s}[H] = \frac{T^{-L_s} - 1}{T^{-L_s} + 1} \text{ with } T = \frac{1-H}{1+H}, \quad H \equiv \gamma_5 D(M)$$

Mobius Generalization

$$D_{Mobius}(M_5) = \frac{a + b\, D_w(-1)}{c + d\, D_w(-1)}$$

$$\equiv \frac{(b_5 + c_5) D_w(M_5)}{2 + (b_5 - c_5) D_w(M_5)}$$

Parameters: M_5 , $a_5 = b_5 - c_5$ and scale: $\alpha = b_5 + c_5$

- **M_5 sets the zero for $H = \gamma_5\, D(M)$**
- **a_5 moves the doublers down**
- **NEW α slides e.v. in the window $1 - \varepsilon_L[H]$**

Domain Wall Implementation

L_s £ L_s DW Matrix:

$$D^{DW}_{s',s} = \begin{bmatrix} D_+^{(1)} & D_-^{(1)} P_- & 0 & \cdots & -m D_-^{(1)} P_+ \\ D_-^{(2)} P_+ & D_+^{(2)} & D_-^{(2)} P_- & \cdots & 0 \\ 0 & D_-^{(3)} P_+ & D_+^{(3)} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -m D_-^{(L_s)} P_- & 0 & 0 & \cdots & D_+^{(L_s)} \end{bmatrix}$$

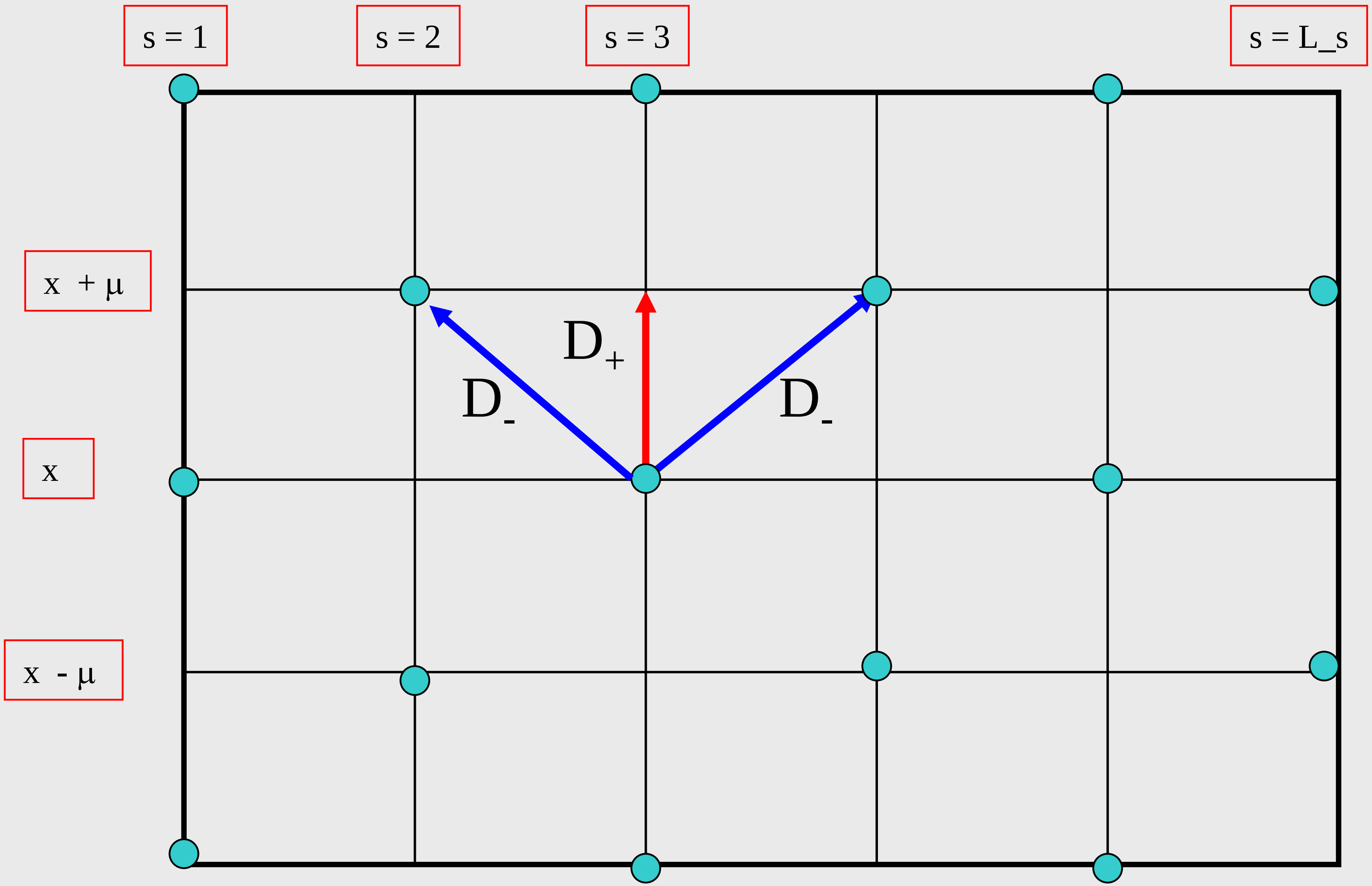
with L^4 Wilson Operators:

$$D_+^{(i)} = b_5\, \omega_i D_w(M) + 1 \,, \;\; D_-^{(i)} = c_5\, \omega_i D_w(M) - 1$$

$$P_\pm \equiv \tfrac{1}{2}(1 \pm \gamma_5)$$

(ω_i = 1, except for Chiu's Zolotarev form)

Mobius generalization of Shamir/Borici



Shamir: $b_5 = a_5$, $c_5 = 0$

Borici: $b_5 = c_5 = a_5$

$$D_+ = b_5 D_w(M) + 1\ ,\ D_- = c_5 D_w(M) - 1$$

Standard LDU ➔ 4-d Overlap Form

Define Boundary fields at s = 1, L_s

$$
\begin{aligned}
q_x &= P_-\Psi_{x,1} + P_+\Psi_{x,L_s} \\
\bar{q}_y &= -[\bar{\Psi}_1 D_-^{(1)}]_y P_+ - [\bar{\Psi}_{L_s} D_-^{(L_s)}]_y P_-
\end{aligned}
$$

$$\langle q_x \bar{q}_y \rangle = \frac{1}{1-m}[D_{ov}^{-1}(m) - 1]_{xy}$$

$$D_{ov} \equiv D_4^{-1}(1) D_4(m) = [\frac{1+m}{2} + \frac{1-m}{2}\gamma_5 \frac{T^{-L_s}-1}{T^{-L_s}+1}] \;,$$

where $T^{-L_s} \equiv T_1^{-1} T_2^{-1} \cdots T_{L_s}^{-1}$

$\varepsilon_L[H]$

Edwards & Heller use “Standard” UDL decomposition

$$' \; D^{DW}\,\mathcal{P} = U \; D \; L(m) \; '$$

$$D^{DW}\,\mathcal{P} = [\gamma_5 Q_- U Q_-^{-1}] \quad Q_- \begin{bmatrix} D_4(m) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad [L(m)]$$

Step #1: Prepare the Pivots by Permute Columns

$$\mathcal{P} = \begin{bmatrix} P_- & P_+ & 0 & 0 \\ 0 & P_- & P_+ & 0 \\ 0 & 0 & P_- & P_+ \\ P_+ & 0 & 0 & P_- \end{bmatrix}$$

Step #2: Do Gaussian Elimination to get U matrix

$$U = \begin{bmatrix} 1 & -T_1^{-1} & -T_1^{-1}T_2^{-1} & -T_1^{-1}T_2^{-1}T_3^{-1} \\ 0 & 1 & -T_2^{-1} & -T_2^{-1}T_3^{-1} \\ 0 & 0 & 1 & -T_3^{-1} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step #3 Back substitution to get L matrix

$$L(m) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -T_2^{-1}T_3^{-1}T_4^{-1}c_+ & 1 & 0 & 0 \\ -T_3^{-1}T_4^{-1}c_+ & 0 & 1 & 0 \\ -T_4^{-1}c_+ & 0 & 0 & 1 \end{bmatrix}$$

where

$$Q_-^{-1} = Diag[(Q_-^{(1)})^{-1}(Q_-^{(2)})^{-1}(Q_-^{(3)})^{-1}(Q_-^{(4)})^{-1}]$$

$$Q_-^{(s)} = \gamma_5[D_-^{(s)}P_+ + D_+^{(s)}P_-] \qquad c_- = P_- - mP_+$$

$$Q_+^{(s)} = \gamma_5[D_+^{(s)}P_+ + D_-^{(s)}P_-] \qquad c_+ = P_+ - mP_-$$

Generalized γ_5 Hermiticity and All That

- To get all the nice identities for Borici, Chiu and Mobius

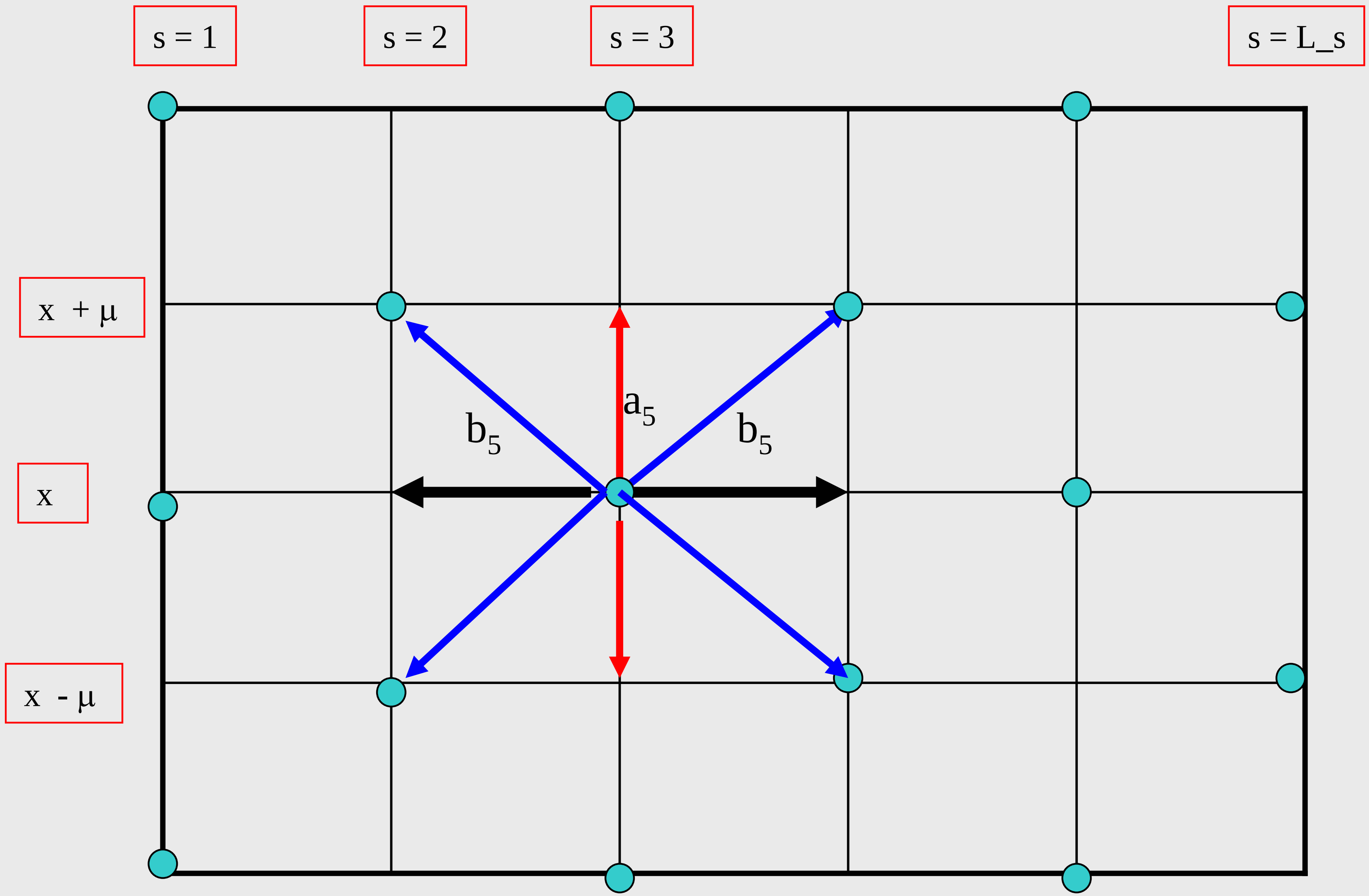
$$D_{DW} = D_- \times \widetilde{D}_{DW} \equiv$$

$$\begin{bmatrix} D_-^{(1)} & 0 & 0 & 0 \\ 0 & D_-^{(2)} & 0 & 0 \\ 0 & 0 & D_-^{(3)} & 0 \\ 0 & 0 & 0 & D_-^{(4)} \end{bmatrix} \times \begin{bmatrix} D_+^{(1)}/D_-^{(1)} & P_- & 0 & -mP_+ \\ P_+ & D_+^{(2)}/D_-^{(2)} & P_- & 0 \\ 0 & P_+ & D_+^{(3)}/D_-^{(3)} & P_- \\ -mP_- & 0 & P_+ & D_+^{(4)}/D_-^{(4)} \end{bmatrix}$$

So $\mathcal{R}D_-\gamma_5$ acts like the DW "gamma 5"

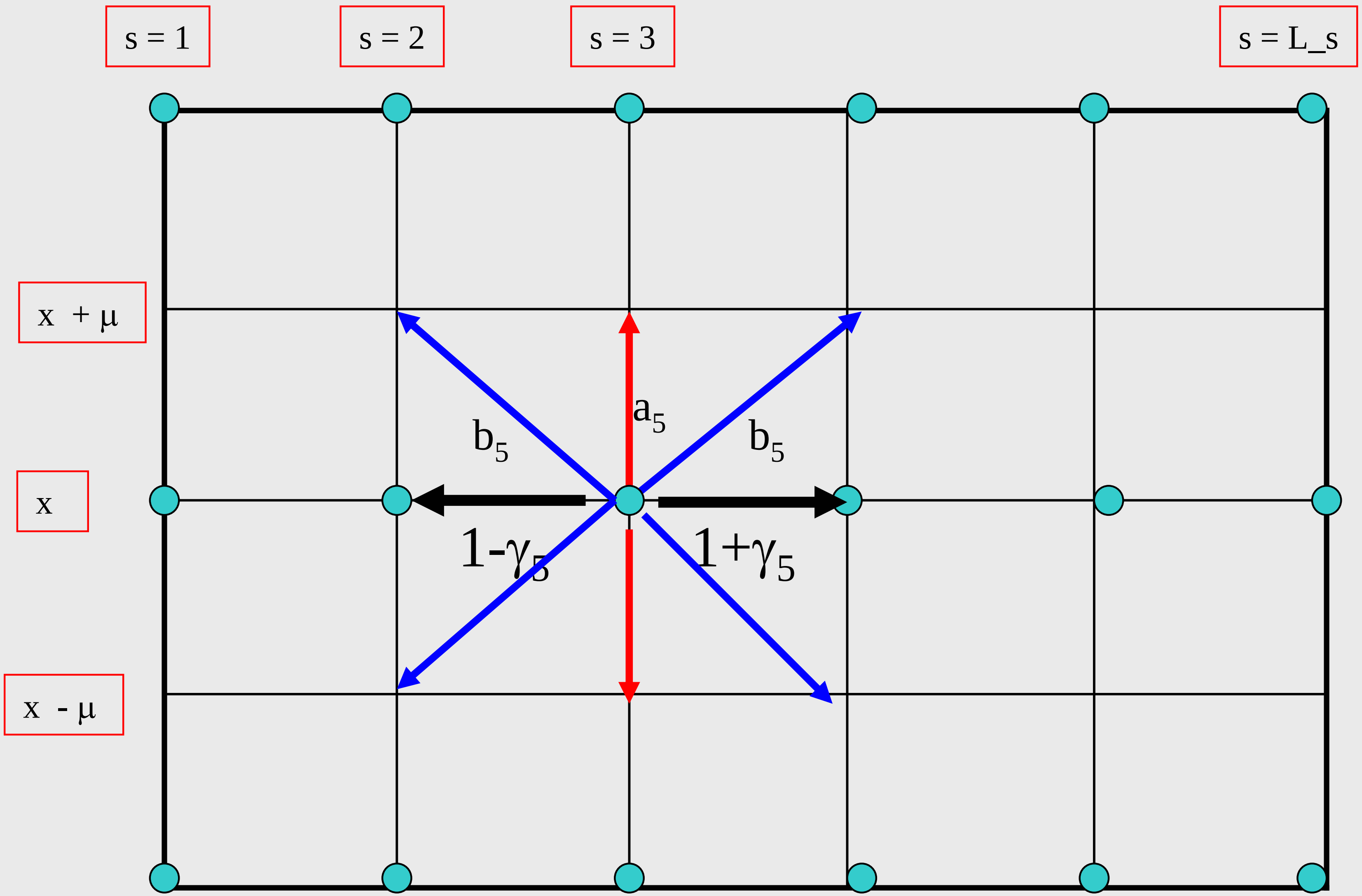
$$\gamma_5 \mathcal{R} \widetilde{D}_{DW} = \widetilde{D}^\dagger_{DW} \gamma_5 \mathcal{R}.$$

Standard Even/Odd ➔ 5d Checker board



Fails for Borici Form $b_5 > 0$

Modified Even/Odd ➔ 4-d Checkerboard



Even/Odd Partition of Matrix

$$D_w(M) = \begin{bmatrix} I_{ee} & D_{eo}^{DW'} \\ D_{oe}^{DW'} & I_{oo} \end{bmatrix}$$

I_{ee} and I_{oo} SHOULD be simple to invert

The Schur decomposition

$$D_w(M) = \begin{bmatrix} 1 & 0 \\ D_{oe}^{DW'} I_{ee}^{-1} & 1 \end{bmatrix} \begin{bmatrix} I_{ee} & 0 \\ 0 & I_{oo} - D_{oe}^{DW} I_{ee} D_{eo}^{DW} \end{bmatrix} \begin{bmatrix} 1 & I_{ee}^{-1} D_{eo}^{DW} \\ 0 & 1 \end{bmatrix}$$

$$D_{preconditioned}^{DW} = 1 - I_{oo}^{-1} D_{oe}^{DW} I_{ee}^{-1} D_{eo}^{DW}$$

On 16^3 x 32, β = 6.0 lattice

- For Shamir: Both 4-d & 5-d Even/Odd give » 2.7 speed up.
- For Borici: Even/Odd gives » 2.7 speed up

Ward Takashi Identity For Domain Wall

$$\Delta_\mu J^{a,DW}_\mu(x) = 2m\,\bar{q}_x\lambda^a\gamma_5 q_x + 2\bar{Q}_x\gamma_5\lambda^a Q_x$$

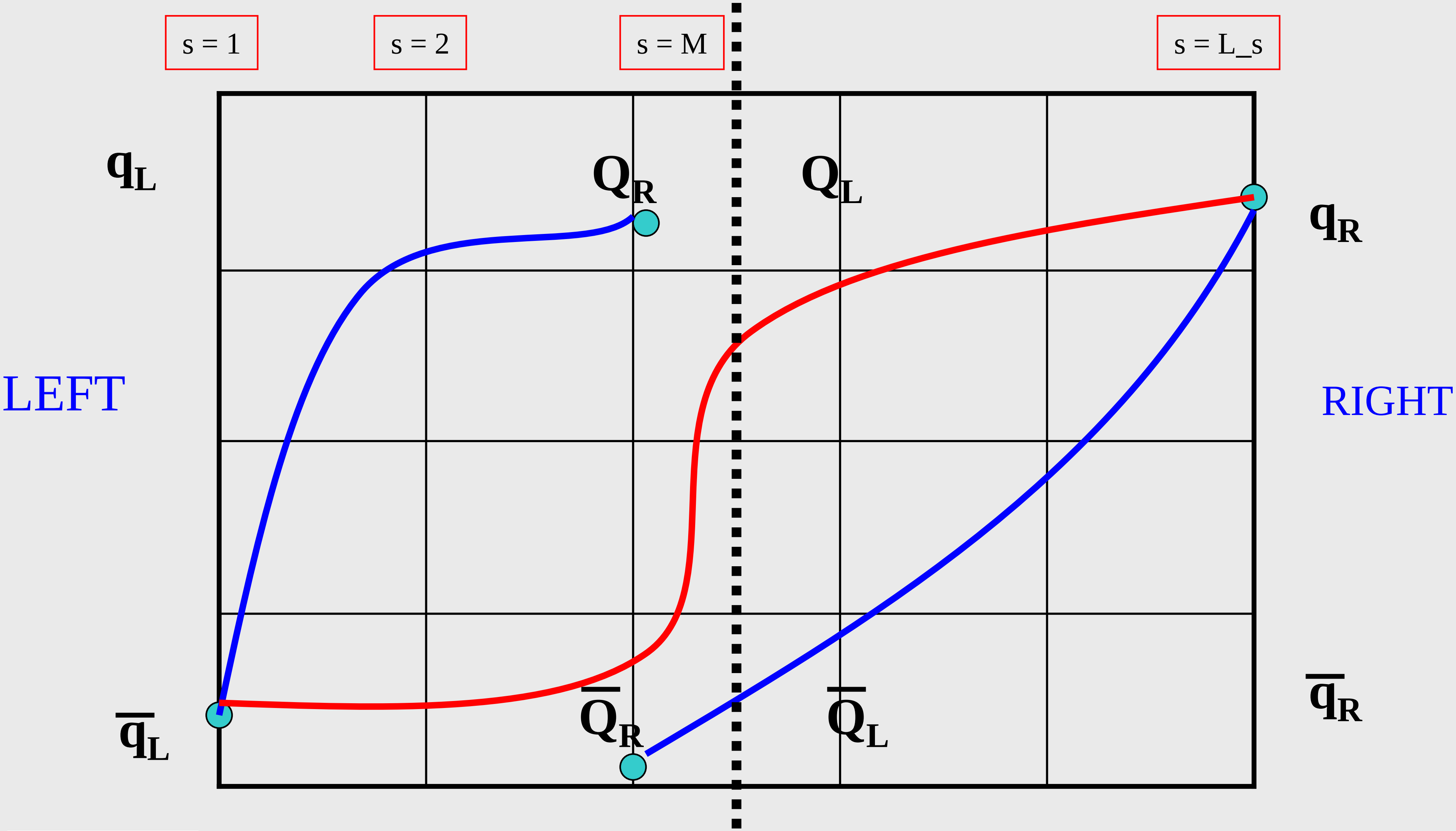
Noether's Theorem:
Rotate LEFT exp[i θ_x] & RIGHT exp[-i θ_x]

- **Is this breaking term UNIQUE?**
- **Is it the same as the overlap operators $\Delta_L(x)$?**

Nice Definition of Overlap Axial Current:

$$\langle J_\mu(x)\psi_y\bar{\psi}_z\rangle_c \equiv (1-m)\langle J^{DW}_\mu(x) q_y\bar{q}_z\rangle_c$$

Split Screen Correlators



$\langle Q_x \bar{q}_y \rangle$

$$\langle q_x \bar{q}_y \rangle = \frac{1}{1-m}[D_{ov}^{-1}(m) - 1]_{xy}$$

$\langle q_x \bar{Q}_y \rangle$

Split Screen Propagators

$$\langle Q_s \bar{q} \rangle = \frac{T_{s+1}^{-1} \cdots T_{L_s}^{-1}}{1 + T^{-L_s}} D_{ov}^{-1}(m)$$

y

$$\langle q \bar{Q}_s \rangle = D_{ov}^{-1}(m) \gamma_5 \frac{1}{1 + T^{-L_s}} [T_1^{-1} \cdots T_s^{-1}] \gamma_5$$

where s = M plane

$$Q_s = P_- \Psi_{s+1} + P_+ \Psi_s$$

$$\bar{Q}_s = -\bar{\Psi}_{s+1} D_-^{(L_s/2+1)} P_+ - \bar{\Psi}_{L/2} D_-^{(s)} P_-$$

y if $s = L_s/2 : \ \gamma_5 \langle Q_s \bar{q} \rangle^\dagger \gamma_5 = \langle q \bar{Q}_s \rangle$

(See Kikukawa and Noguchi, hep-lat/99902022)

Measuring the Operator Δ_{Ls}

(use Plateau region away from sources)

$$m_{res}(t) \equiv \frac{\sum_x \langle \bar{Q}_{t,x} \gamma_5 Q_{t,x} \; \bar{q}_0 \gamma_5 q_0 \rangle_c}{(1-m)^2 \sum_x \langle \bar{q}_{t,x} \gamma_5 q_{t,x} \; \bar{q}_0 \gamma_5 q_0 \rangle_c}$$

Sum over t ➔ Measure Matrix element of Δ_L operator

$$m_{res} \equiv \frac{Tr[\Delta_L(H) D_{ov}^{-1} D_{ov}^{\dagger -1}]}{Tr[D_{ov}^{-1} D_{ov}^{\dagger -1}]} = \sum_\lambda \rho(\lambda)\, \Delta_L(\lambda)$$

$|\lambda>$ in the Eigen basis of $H = \gamma_5 D(-M)$

Model for m_{res} dependence on L_s

$$m_{res} = \sum_{\lambda} \rho(\lambda)\ \Delta_L(\lambda)$$

$$\rho(\lambda) = \frac{\langle\lambda|G_{ov}G^{\dagger}_{ov}|\lambda\rangle}{\sum_{\lambda}\langle\lambda|G_{ov}G^{\dagger}_{ov}|\lambda\rangle} \geq 0$$

$$\begin{aligned}\Delta_L(\lambda) &= \langle\lambda|\Delta_L(H)|\lambda\rangle = \frac{4}{2+[\frac{1+\lambda}{1-\lambda}]^{-L}+[\frac{1+\lambda}{1-\lambda}]^{L}} \geq 0 \\ &\to e^{-L\,\log[(1+\lambda)/(1-\lambda)]} \quad \text{for } O(L^{-1}) < \lambda < O(L)\end{aligned}$$

Scaling Model: Assume that $\rho(\lambda)$ has negligible dependence on α and L_s

$$m_{res} \simeq \int dn(\lambda)\ \rho(\lambda)\ \Delta_L(\alpha\lambda)$$

Derivation:

$$m_{res} \equiv \frac{\sum_x \langle \bar{Q}_x \gamma_5 Q_x \; \bar{q}_0 \gamma_5 q_0 \rangle_c}{(1-m)^2 \sum_x \langle \bar{q}_x \gamma_5 q_x \; \bar{q}_0 \gamma_5 q_0 \rangle_c} \Rightarrow$$

$$\frac{\sum_x Tr[\langle Q_x \bar{q}_0 \rangle \; \gamma_5 \langle q_0 \bar{Q}_x \rangle \gamma_5]}{(1-m)^2 \sum_x Tr[\langle q_x \bar{q}_0 \rangle \; \gamma_5 \langle q_0 \bar{q}_x \rangle \gamma_5]} = \frac{\sum_x Tr[\Delta^R_{xy} \langle q_y \bar{q}_0 \rangle \; \langle q_z \bar{q}_0 \rangle^\dagger \Delta^L_{zx}]}{\sum_x Tr[\langle q_x \bar{q}_0 \rangle \; \langle q_x \bar{q}_0 \rangle^\dagger]}$$

$$= \frac{\sum_{zy} \langle \bar{q}_z \gamma_5 \Delta_{zy} q_y \; \bar{q}_0 \gamma_5 q_0 \rangle}{\sum_x \langle \bar{q}_x \gamma_5 q_x \; \bar{q}_0 \gamma_5 q_0 \rangle} = \frac{Tr[\Delta_L D_{ov}^{-1} D_{ov}^{\dagger -1}]}{Tr[D_{ov}^{-1} D_{ov}^{\dagger -1}]} \simeq \frac{\langle 0 | \bar{q} \gamma_5 \Delta q | \pi \rangle}{\langle 0 | \bar{q} \gamma_5 q | \pi \rangle}$$

where

$$\begin{aligned} \Delta^L \Delta^R &= \frac{(T_{Ls/2+1} .. T_{L_s})^{-1}}{1 + T^{-L_s}} \times \frac{(T_1 .. T_{L_s/2})^{-1}}{1 + T^{-L_s}} \\ &= \frac{T^{-L_s}}{(1 + T^{-L_s})^2} = \frac{1}{2 + T^{L_s} + T^{-L_s}} = \Delta_{L_s} \end{aligned}$$

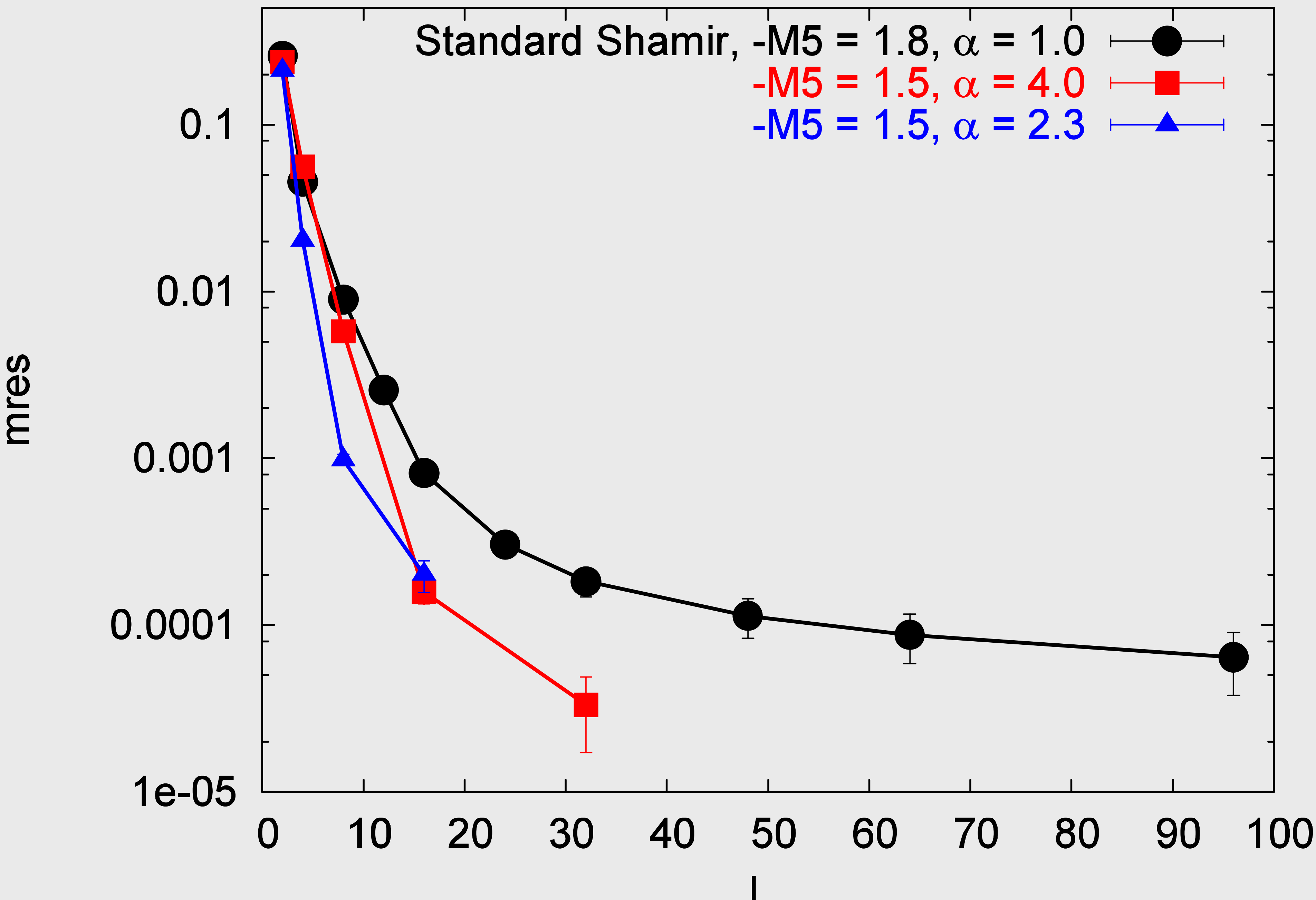
M_{res} is Good Measure of Chiral Symmetry

- M_{res} is independent of Placement of Split Screen
- Δ_L is local positive definite operator.
- $m_{res} = 0$ ➔ GW and Chiral Symmetry is EXACT
- Low energy matrix elements of Δ_L correct the effective Chiral Lagrangian. To leading O(a) this is a just a shift in the quark mass.

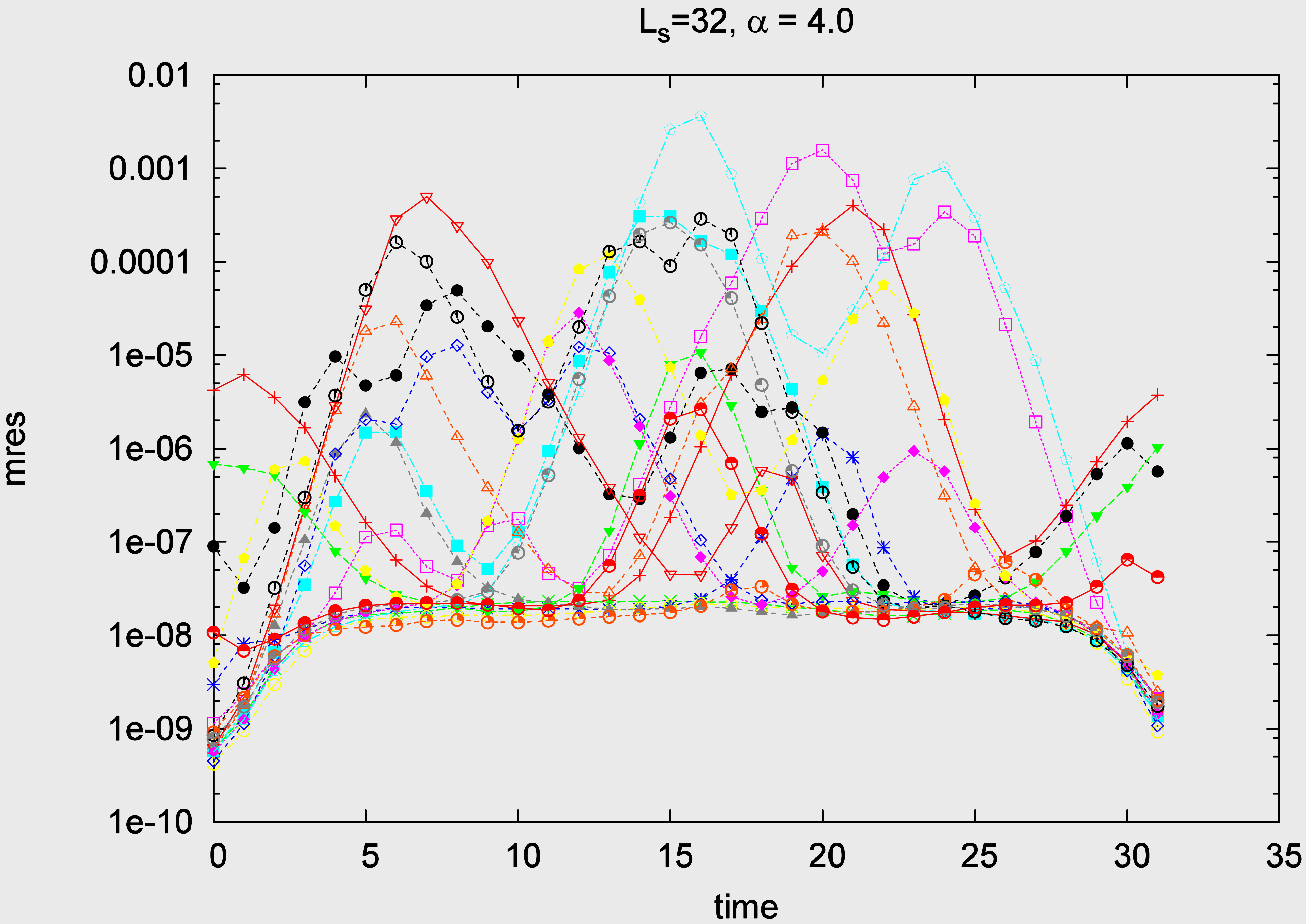
Preliminary Performance Tests

- 20 NERSC Gauge lattices: 16^3 x 32 at $\beta = 6.0$
- Set $m_q = 0.06$ for Shamir ($a_5 = 1$, $M_5 = 1.8$)
- As you vary M_5 and $\alpha = b_5 + c_5$ to fixed m_π

Shamir vs Mobius

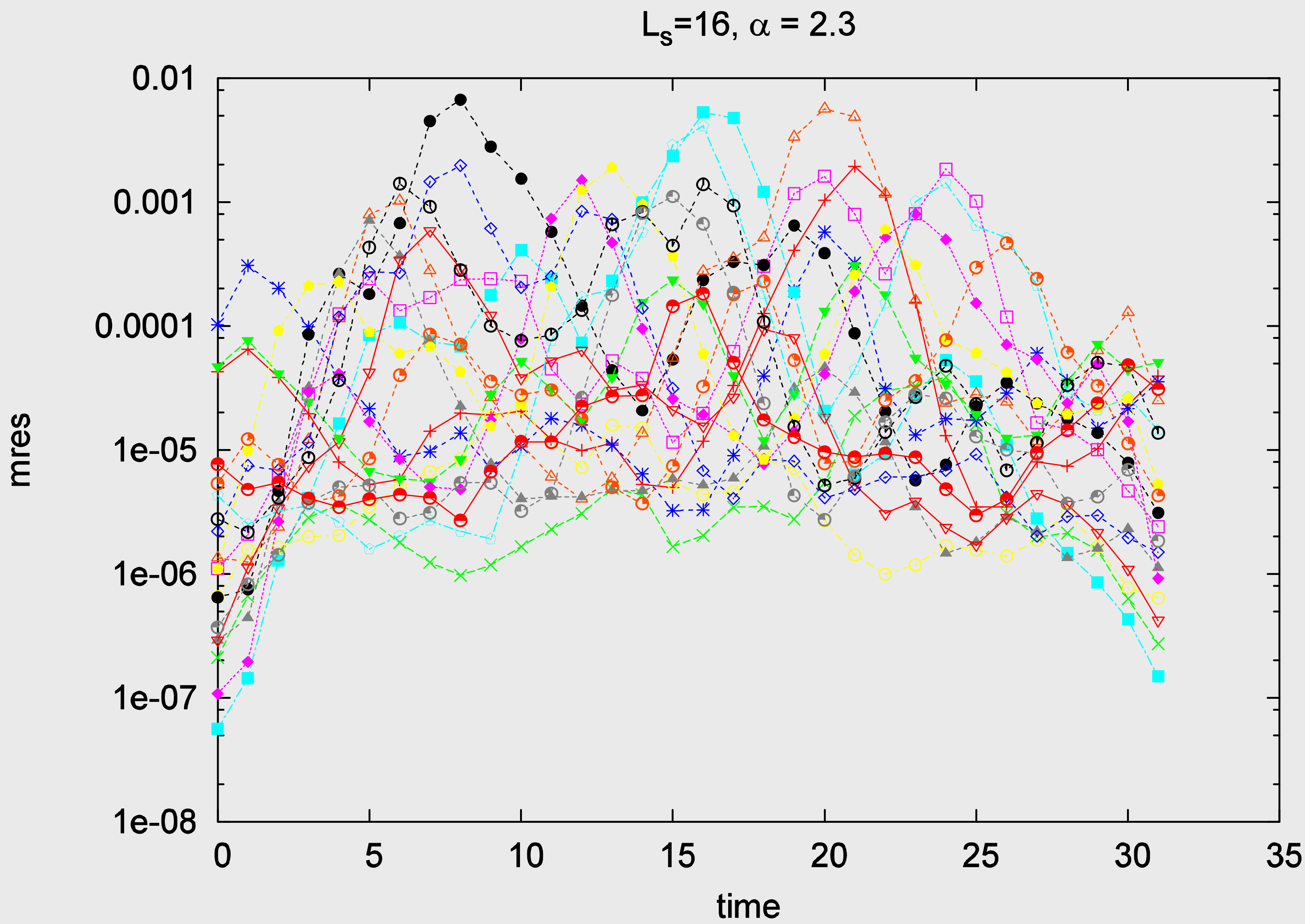


$m_{res}(t)$ for large α



Future Refinements and Projects

- Further tuning in M_5, a_5 and scale parameters
- Heuristic for m_{res}: Error $|1- \varepsilon_L(\lambda)|$ for lowest e. v. ?
- Test on HMC Lattices (RBC collaboration)
- Next to nearest neighbor ?
- Multi-grid in L_s ? (with C. Rebbi & David Keyes)



L_s=16, α = 2.3
mres
time
0.01
0.001
0.0001
1e-05
1e-06
1e-07
1e-08
0
5
10
15
20
25
30
35

Low Statistics Large L_s Behavior