Twisted Boundary Conditions

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Boundary Conditions

- PBC: lattice momenta quantised
  \[ p_i = \frac{2\pi}{L} n_i \]

- lowest non-zero momentum is quite large, big gaps

- non-periodic or twisted spatial boundary conditions: allow continuously variable offset in the comb of allowed three-momenta
Twisted BC in QCD

\[ \mathcal{L}_q = \bar{q}(x)(\bar{\Psi} + M)q(x) \]

- observables should be single-valued: OK if action is single-valued on a torus

⇒ field satisfies

\[ \psi(x + e_iL) = U_i\psi(x) \]

for \( i = 1, 2, 3 \), where \( U_i \) is a symmetry of the action

- for general diagonal \( M \), \( U_i \) should be diagonal (CSA of \( U(3) \))

\[ U_i = \exp(i\Theta_i) \]

- allowed momenta:

\[ p_i = \frac{2\pi n_i}{L} + \frac{\theta_i}{L} \]
Twisted BC: 2

- change variable:

\[ \tilde{q}(x) = e^{-i\Theta \cdot x} q(x) \quad (\Theta_0 = 0) \]

- \( \tilde{q} \) satisfies PBC

- Lagrangian:

\[ \mathcal{L}_q = \tilde{q}(x)(\bar{\mathcal{D}} + M)\tilde{q}(x) \]

with

\[ \tilde{D}_\mu = D_\mu + iB_\mu, \quad B_i = \Theta_i/L, \quad B_0 = 0 \]
propagator encodes shift: $S(x, y) \rightarrow \tilde{S}(x, y)$

$$\tilde{S}(x) = \langle \tilde{q}(x)\tilde{q}(0) \rangle = \int \frac{dk_0}{2\pi} \frac{1}{L^3} \sum_k \frac{e^{ik \cdot x}}{i(k + \beta) + M}$$

- sum over $k = 2\pi n / L$
- momentum in denominator is shifted by $\theta / L$
Twisted BC on the Lattice

A change of variable modifies the lattice covariant derivatives:

\[ \nabla^{\Theta}_\mu \psi(x) = e^{i\Theta_\mu/L} U_\mu(x) \psi(x + \hat{\mu}) - \psi(x) \]

\[ \nabla^{\Theta*}_\mu \psi(x) = \psi(x) - e^{-i\Theta_\mu/L} U^*_\mu(x - \hat{\mu}) \psi(x - \hat{\mu}) \]
Twisted BC on the Lattice

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- inverting the modified operator encodes the momentum shift \( \Theta/L \) in the calculated propagator
Twisted BC on the Lattice

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- hadron momentum shifted by sum of quark shifts
  - dDPT: quenched study of pseudoscalar meson dispersion relation
  - SV: ChPT analysis
Exponential suppression of finite-volume corrections from $\theta$-BC for quantities without FSI (masses, decay constants, semileptonic FF’s)
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Not possible in general to extract matrix elements using $\theta$-BC for amplitudes involving FSI (eg $K \rightarrow \pi\pi$)
SV Chiral PT Analysis

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- The above remain true for ‘partial twisting’: $\theta$-BC for valence, PBC for sea
Exponential suppression of finite-volume corrections from $\theta$-BC for quantities without FSI (masses, decay constants, semileptonic FF’s)

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The above remain true for ‘partial twisting’: $\theta$-BC for valence, PBC for sea

They construct effective Lagrangian in presence of $\theta$-BC.
Twisted BC:
\[ \Sigma(x + e_i L) = U_i \Sigma(x) U_i^\dagger \]

Redefine fields:
\[ \tilde{\Sigma}(x) = e^{-i\Theta \cdot x/L} \Sigma(x) e^{i\Theta \cdot x/L} \]
Twisted ChPT

Twisted BC:

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Redefine fields:

\[ \tilde{\Sigma}(x) = e^{-i\Theta \cdot x / L} \Sigma(x) e^{i\Theta \cdot x / L} \]

to get

\[ \mathcal{L}_{\text{ChPT}} = \frac{f^2}{8} \langle \tilde{D}_\mu \tilde{\Sigma}^\dagger \tilde{D}_\mu \tilde{\Sigma} \rangle - \frac{f^2}{8} \langle \tilde{\Sigma} \chi^\dagger + \chi \tilde{\Sigma}^\dagger \rangle \]

where

\[ \tilde{D}_\mu \tilde{\Sigma} = \partial_\mu + i [B_\mu, \tilde{\Sigma}] \]
Twisted ChPT: 2

Standard ChPT Lagrangian coupled to vector field $B_\mu$.
Effect of twist on mesons found from $[B_i, \tilde{\Sigma}]$: 
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Standard ChPT Lagrangian coupled to vector field $B_\mu$. Effect of twist on mesons found from $[B_i, \tilde{\Sigma}]$:

- neutral mesons:
  
  $$ [B_i, \pi^0] = 0 \quad \rightarrow \text{no shift} $$
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- charged mesons shifted by difference of the twists of the two valence quarks

  $[B_i, \pi^\pm] = \pm \frac{(\theta_{ui} - \theta_{di})}{L} \pi^\pm$
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- allowed values of meson momenta shifted in external states and in propagators
• fixed volume $L^3T$ with $L = 3.2r_0 \approx 1.6$ fm
  
  $16^3 \times 32$
  $24^3 \times 48$
  $32^3 \times 64$

• $O(a)$ improved

• 4 quark masses

• invert for each of

  $|\theta| = 0, \sqrt{3}, 2\sqrt{3}, 3\sqrt{3}$

• calculate pseudoscalar meson correlator with one quark twisted

• Expected meson momentum

\[ |p| = \frac{|\theta|}{L} = \begin{cases} 0.000 \text{ GeV} \\ 0.217 \text{ GeV} \\ 0.433 \text{ GeV} \\ 0.650 \text{ GeV} \end{cases} \]

\[ \text{cf. } 2\pi/L \approx 0.785 \text{ GeV} \]
• extract effective energies
• interpolate to fixed physical quark masses
• extrapolate $a \to 0$ at fixed quark mass, fixed $L$
extract effective energies
interpolate to fixed physical quark masses
extrapolate $a \rightarrow 0$ at fixed quark mass, fixed $L$

relativistic dispersion relation

$$E_{ij}^2 = M_{ij}^2 + |\theta|^2 / L^2$$

is well satisfied
Dispersion relation test

$$(r_0 E)^2$$

$$(r_0 |\theta|/L)^2$$

Tsukuba LQCD&PP 15 Dec 2004
Partial Twisting

- ChPT analysis above was in full QCD
- Do you have to twist the sea quarks by the same amount as the valence quarks?
Partial Twisting

- ChPT analysis above was in full QCD
- Do you have to twist the sea quarks by the same amount as the valence quarks?
- SV say ‘Not always’
  - extend analysis to a *partially twisted* chiral Lagrangian corresponding to $N_v + N_s$ quarks with $N_v$ ghost quarks
  - for processes with at most one hadron in external states and where shift does not introduce cuts in the correlator
Example from SV: \( f_{K^\pm} \) for untwisted \( d \) and \( s \) quarks:

\[
\frac{f_K(L) - f_K(\infty)}{f_K(\infty)} = -\frac{m_\pi^2}{f_\pi^2} \frac{e^{-m_\pi L}}{(2\pi m_\pi L)^{3/2}} \left\{ \begin{array}{l}
\frac{9}{4} \\
\frac{1}{2} \sum_i \cos \theta_i + \frac{3}{4} \\
\sum_i \cos \theta_i - \frac{3}{4}
\end{array} \right\} \]

\( u \) untwisted

\( u \) fully twisted

\( u \) partially twisted
Applications?

- Numerical tests with twisted valence quarks on untwisted sea quarks: meson dispersion relations, meson decay constants with different meson momenta
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- Numerical tests with twisted valence quarks on untwisted sea quarks: meson dispersion relations, meson decay constants with different meson momenta

- Heavy-to-light semileptonic decays (eg: $D \rightarrow \pi l\nu$)
  - at fixed quark masses: map out full $q^2$ range
  - chiral extrapolation: generate points at fixed $E = v \cdot p_\pi$ for different light quarks and avoid fitting to a model FF