Twisted Boundary Conditions

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Contents

- θ-BC: twisted boundary conditions
- TChPT: twisted chiral perturbation theory
- Partial twisting
- Applications?

Some recent work

N-N phase shifts	PF Bedaque	nucl-th/0402051
Pseudoscalar meson dispersion relation (quenched)	GM de Divitiis, R Petronzio and N Tantalo	hep-lat/0405002
θ-BC and two-particle states	GM de Divitiis and N Tantalo	hep-lat/0409154
ChPT analysis	CT Sachrajda and G Villadoro	hep-lat/0411033

Boundary Conditions

- PBC: lattice momenta quantised

$$p_i = \frac{2\pi}{L} n_i$$

- lowest non-zero momentum is quite large, big gaps
- non-periodic or *twisted* spatial boundary conditions: allow continuously variable offset in the comb of allowed three-momenta

Twisted BC in QCD

$$\mathcal{L}_q = \bar{q}(x)(\not{D} + M)q(x)$$

- observables should be single-valued: OK if action is single-valued on a torus
- $\Rightarrow$ field satisfies

$$\psi(x + e_i L) = U_i \psi(x)$$

for $i = 1, 2, 3$, where U_i is a symmetry of the action

- for general diagonal M, U_i should be diagonal (CSA of $U(3)$)

$$U_i = \exp(i\Theta_i)$$

- allowed momenta:

$$p_i = \frac{2\pi n_i}{L} + \frac{\theta_i}{L}$$

Twisted BC: 2

- change variable:

$$\tilde{q}(x) = e^{-i\Theta\cdot x} q(x) \qquad (\Theta_0 = 0)$$

- $\tilde{q}$ satisfies PBC
- Lagrangian:

$$\mathcal{L}_q = \bar{\tilde{q}}(x)(\tilde{\not{D}} + M)\tilde{q}(x)$$

with

$$\tilde{D}_\mu = D_\mu + iB_\mu, \qquad B_i = \Theta_i/L, \quad B_0 = 0$$

Twisted BC: 3

- propagator encodes shift: $S(x,y) \to \tilde{S}(x,y)$

$$\tilde{S}(x) = \langle \tilde{q}(x)\bar{\tilde{q}}(0)\rangle = \int \frac{dk_0}{2\pi}\frac{1}{L^3}\sum_{\mathbf{k}} \frac{e^{ik\cdot x}}{i(\not{k} + \not{B}) + M}$$

- sum over $\mathbf{k} = 2\pi\mathbf{n}/L$
- momentum in denominator is shifted by θ/L

Twisted BC on the Lattice

- change of variable modifies the lattice covariant derivatives:

$$
\begin{aligned}
\nabla_\mu^\Theta \psi(x) &= e^{i\Theta_\mu/L} U_\mu(x)\psi(x+\hat{\mu}) - \psi(x) \\
\nabla_\mu^{\Theta *} \psi(x) &= \psi(x) - e^{-i\Theta_\mu/L} U_\mu^\dagger(x-\hat{\mu})\psi(x-\hat{\mu})
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- hadron momentum shifted by sum of quark shifts
  - dDPT: quenched study of pseudoscalar meson dispersion relation
  - SV: ChPT analysis

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They construct effective Lagrangian in presence of θ-BC.

Twisted ChPT

Twisted BC:

$$\Sigma(x + e_i L) = U_i \Sigma(x) U_i^\dagger$$

Redefine fields:

$$\tilde{\Sigma}(x) = e^{-i\Theta\cdot x/L} \Sigma(x) e^{i\Theta\cdot x/L}$$

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to get

$$\mathcal{L}_{\mathrm{ChPT}} = \frac{f^2}{8}\langle \tilde{D}^\mu \tilde{\Sigma}^\dagger \tilde{D}_\mu \tilde{\Sigma}\rangle - \frac{f^2}{8}\langle \tilde{\Sigma}\chi^\dagger + \chi\tilde{\Sigma}^\dagger\rangle$$

where

$$\tilde{D}_\mu \tilde{\Sigma} = \partial_\mu + i[B_\mu, \tilde{\Sigma}]$$

Twisted ChPT: 2

Standard ChPT Lagrangian coupled to vector field B_μ.

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- allowed values of meson momenta shifted in external states and in propagators

dDPT Quenched Study

- fixed volume L^3T with $L = 3.2r_0 \approx 1.6\,\text{fm}$

$$16^3 \times 32$$
$$24^3 \times 48$$
$$32^3 \times 64$$

- $O(a)$ improved
- 4 quark masses
- invert for each of

$$|\theta| = 0,\ \sqrt{3}, 2\sqrt{3}, 3\sqrt{3}$$

- calculate pseudoscalar meson correlator with *one* quark twisted
- Expected meson momentum

$$|\mathbf{p}| = \frac{|\theta|}{L} = \begin{cases} 0.000\,\text{GeV} \\ 0.217\,\text{GeV} \\ 0.433\,\text{GeV} \\ 0.650\,\text{GeV} \end{cases}$$

cf. $2\pi/L \approx 0.785\,\text{GeV}$

dDPT: 2

- extract effective energies
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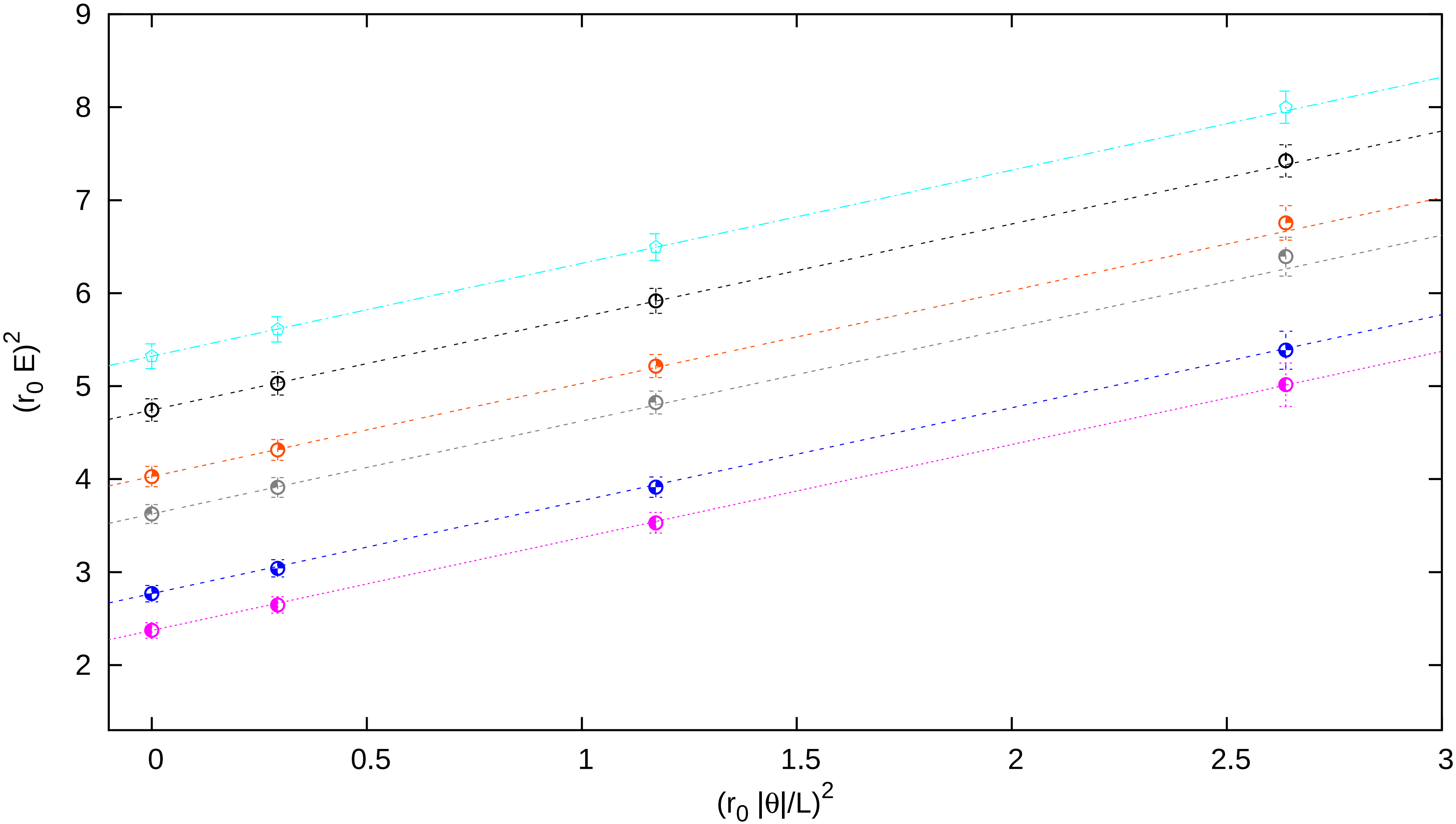
- relativistic dispersion relation

$$E_{ij}^2 = M_{ij}^2 + |\theta|^2/L^2$$

is well satisfied

dDPT: 3

Dispersion relation test



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- ChPT analysis above was in full QCD
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- Do you have to twist the sea quarks by the same amount as the valence quarks?
- SV say ‘Not always’
  - extend analysis to a *partially twisted* chiral Lagrangian corresponding to $N_v + N_s$ quarks with N_v ghost quarks
  - for processes with at most one hadron in external states and where shift does not introduce cuts in the correlator

Partial Twisting: 2

Example from SV: $f_{K^\pm}$ for untwisted d and s quarks:

$$\frac{f_K(L) - f_K(\infty)}{f_K(\infty)}$$

$$= \; -\frac{m_\pi^2}{f_\pi^2}\frac{e^{-m_\pi L}}{(2\pi m_\pi L)^{3/2}} \begin{cases} \frac{9}{4} & u \text{ untwisted} \\ (\frac{1}{2}\sum_i \cos\theta_i + \frac{3}{4}) & u \text{ fully twisted} \\ (\sum_i \cos\theta_i - \frac{3}{4}) & u \text{ partially twisted} \end{cases}$$

Applications?

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- Heavy-to-light semileptonic decays (eg: $D \to \pi l v$)
  - at fixed quark masses: map out full q^2 range
  - chiral extrapolation: generate points at fixed $E = v \cdot p_\pi$ for different light quarks and avoid fitting to a model FF