O(a) improvement in tmLQCD for arbitrarily small quark masses

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tmQCD

Twisted mass term: $m' e^{i\omega\gamma_5\tau_3} = m + i\mu\gamma_5\tau_3$ ω: twist angle

Advantages of tmQCD on the lattice:

- Quenched: No exceptional configurations
- Unquenched: Numerically cheaper than untwisted mass A. Kennedy, Lattice 2004
- Simplified renormalization of weak matrix elements
- Automatic O(a) improvement at maximal twist $\omega = \frac{\pi}{2}$ Frezzotti, Rossi '03

No improvement factors need to be determined !

Quark bounds for O(a) improvement

Frezzotti/Rossi: O(a) improvement works only for

Frezzotti, Rossi '03

1. $m \gg a\Lambda_{\rm QCD}^2$
2. $m \gg a^2\Lambda_{\rm QCD}^3$ for O(a) improved Lattice QCD (a la Symanzik)

Example: $a^{-1} = 2\text{GeV}$ $\Lambda_{\rm QCD} = 300\text{MeV}$

1. $m \gg 45\text{GeV}$ too restrictive to be useful
2. $m \gg 7\text{GeV}$ accessible (but...)

Quark bounds for O(a) improvement

Our claim: O(a) improvement works

Aoki, OB '04

1. either for arbitrary $m > 0$
2. or for $m \geq a^2 \Lambda_{\rm QCD}^3$

provided maximal twist is defined appropriatly

Note: Our bounds hold independently of O(a) improvement

Outline of the talk

- Automatic O(a) improvement at maximal twist
  - Argument a la Frezzotti and Rossi
  - Caveat for small quark masses
  - Alternative proposal for maximal twist
- Pion mass in ChPT at non-zero lattice spacing
  - Brief introduction: ChPT at non-zero lattice spacing
  - Example: O(a) improvement of the pion mass at maximal twist
- Conclusion

Twisted mass term on the lattice

Mass term + Wilson term on the lattice :

$$\bar\psi(x)\left[\left(-a\frac{r}{2}\sum_\mu \nabla^\star_\mu\nabla_\mu + M_{\rm cr}(r)\right) + m_q \exp(iw\gamma_5\tau_3)\right]\psi(x)$$

$$m_q = m_0 - M_{\rm cr}(r)$$

↑ bare quark mass ↑ critical quark mass

Field redefinition:

$$\psi_{\rm ph} = \exp(i\frac{\omega}{2}\gamma_5\tau_3)\psi,$$
$$\bar\psi_{\rm ph} = \bar\psi\exp(i\frac{\omega}{2}\gamma_5\tau_3)$$

$$\bar\psi_{\rm ph}(x)\left[\left(-a\frac{r}{2}\sum_\mu \nabla^\star_\mu\nabla_\mu + M_{\rm cr}(r)\right)\exp(-iw\gamma_5\tau_3) + m_q\right]\psi_{\rm ph}(x)$$

Spurionic Symmetry

The Lattice action is invariant under

$$\mathcal{R}_5^{SP} = [r \to -r] \times [m_0 \to -m_0] \times \mathcal{R}_5$$

$$\mathcal{R}_5 : \begin{cases} \psi & \to & \psi' = \gamma_5 \psi \\ \overline{\psi} & \to & \overline{\psi}' = -\overline{\psi}\gamma_5 \end{cases}$$

Remember: $m_q = m_0 - M_{\rm cr}(r)$

$\Longrightarrow$ Symmetry holds if $M_{\rm cr}(-r) = -M_{\rm cr}(r)$

Wilson average and O(a) improvement

The Wilson average $\langle O \rangle^{WA}(r, m_q, \omega) \equiv \frac{1}{2}\Big[\langle O \rangle(r, m_q, \omega) + \langle O \rangle(-r, m_q, \omega)\Big]$

is O(a) improved: $= \langle O \rangle^{\mathrm{cont}}(m_q) + O(a^2)$

Symanzik's effective action:

$$S_{\mathrm{Symanzik}} = \int \overline{\psi}(x)\Big[\gamma_\mu D_\mu + m_q + r\, a\, C\, \sigma_{\mu\nu} G_{\mu\nu}\Big]\psi(x)$$

some low-energy constant

lattice spacing

Pauli term is odd in r $\Longrightarrow$ contribution cancels by taking the Wilson average

Automatic O(a) improvement at maximal twist

Consider the twist average:

$$\langle O\rangle^{TA}(r,m_q,\omega=\frac{\pi}{2}) \equiv \frac{1}{2}\Big[\langle O\rangle(r,m_q,\omega=\frac{\pi}{2})+\langle O\rangle(r,m_q,\omega=-\frac{\pi}{2})\Big]$$

$$\exp(-i\frac{\pi}{2}\gamma_5\tau_3) = -\exp(i\frac{\pi}{2}\gamma_5\tau_3)$$

$$=\frac{1}{2}\Big[\langle O\rangle(r,m_q,\omega=\frac{\pi}{2})+\langle O\rangle(-r,m_q,\omega=\frac{\pi}{2})\Big]$$

For observables even in ω (e.g. masses):

$$\langle O\rangle(r,m_q,\omega=\frac{\pi}{2}) = \langle O\rangle^{TA}(r,m_q,\omega=\frac{\pi}{2}) = \langle O\rangle^{\mathrm{cont}}(m_q)+O(a^2)$$

O(a) improvement without taking an average !

(My) puzzle

The argument is based entirely on symmetries

Where does the size of the quark mass enter ?

The critical mass under $r \longrightarrow -r$

Crucial assumption for O(a) improvement: $M_{\rm cr}(-r) = -M_{\rm cr}(r)$

"Proof": Consider $m_\pi(r, m_0)$

1. Defining equation: $m_\pi(r, M_{\rm cr}(r)) = 0,$

2. Spurionic symmetry: $m_\pi(r, m_0) = m_\pi(-r, -m_0)$

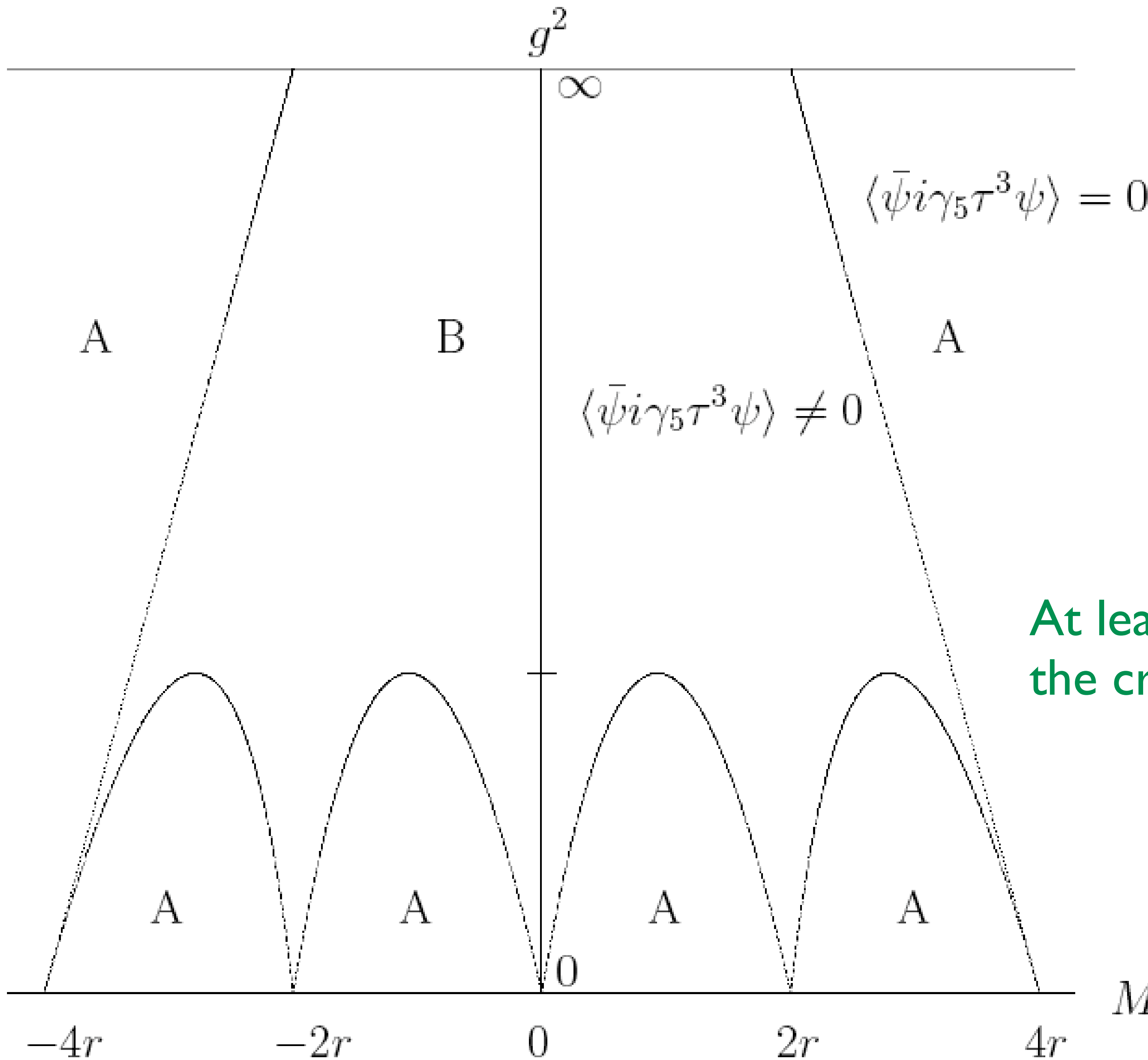
1. & 2: $m_\pi(r, M_{\rm cr}(r)) \; = \; m_\pi(r, -M_{\rm cr}(-r)) \; = \; 0$

3. "Conclusion": $M_{\rm cr}(r) \; = \; -M_{\rm cr}(-r))$

Only true if the defining equation has a unique solution !

Critical Mass: Aoki's phase diagram

Aoki '85

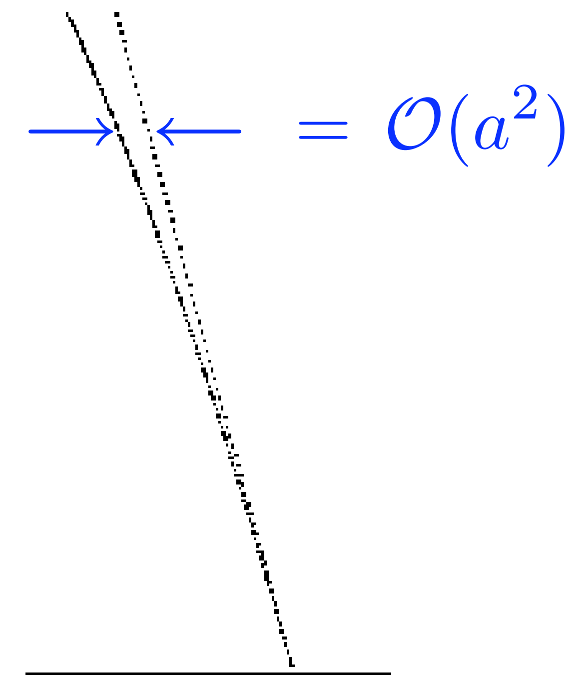


At least 2 values for the critical mass !

Critical Mass: ChPT analysis

Sharpe, Singleton '98

Two possible scenarios:



Scenario 1: No massless pion at non-zero a

Scenario 2: Spontaneous breaking of flavor and parity
Massless pions a la Aoki
Quantitative result for critical mass:
Width of Aoki fingers is $\mathcal{O}(a^2)$

$$\Longrightarrow \quad M_{\rm cr}(r) = M_{\rm odd}(r) + a^2 c\, M_{\rm even}(r) \;\equiv\; M_{\rm cr}^{(1)}(r)$$

$$-M_{\rm cr}(-r) = M_{\rm odd}(r) - a^2 c\, M_{\rm even}(r) \;\equiv\; M_{\rm cr}^{(2)}(r)$$

$$\Longrightarrow \quad M_{\rm cr}(r) \neq -M_{\rm cr}(-r)$$

What happens to O(a) improvement ?

Ansatz: $$M_{\rm cr}(r) = M_{\rm odd}(r) + a^2 c\, M_{\rm even}(r)$$

$$\Longrightarrow \quad \langle O\rangle(r, m_q, \omega = \frac{\pi}{2})^{TA} = \frac{1}{2}\Big[\langle O\rangle(r, m_q, \omega = \frac{\pi}{2}) + \langle O\rangle(-r, m_q', \omega = \frac{\pi}{2} + \omega')\Big]$$

$$m_q' = \sqrt{m_q^2 + (2a^2 c M_{\rm even}(r))^2} \qquad \tan\omega' = \frac{2a^2 c\, M_{\rm even}(r)}{m_q}$$

$\Longrightarrow$ Twist average = Wilson average only if $m_q \gg a^2$

$\Longrightarrow$ Automatic O(a) improvement only if $m_q \gg a^2$

Alternative definition for the twist angle

Define: $$\overline{M}_{\rm cr}(r) = \frac{M_{\rm cr}(r) - M_{\rm cr}(-r)}{2} = -\overline{M}_{\rm cr}(-r)$$ odd part

$$\Delta M_{\rm cr}(r) = \frac{M_{\rm cr}(r) + M_{\rm cr}(-r)}{2} = \Delta M_{\rm cr}(-r)$$ even part

$\Longrightarrow$ New definition for the twist angle:

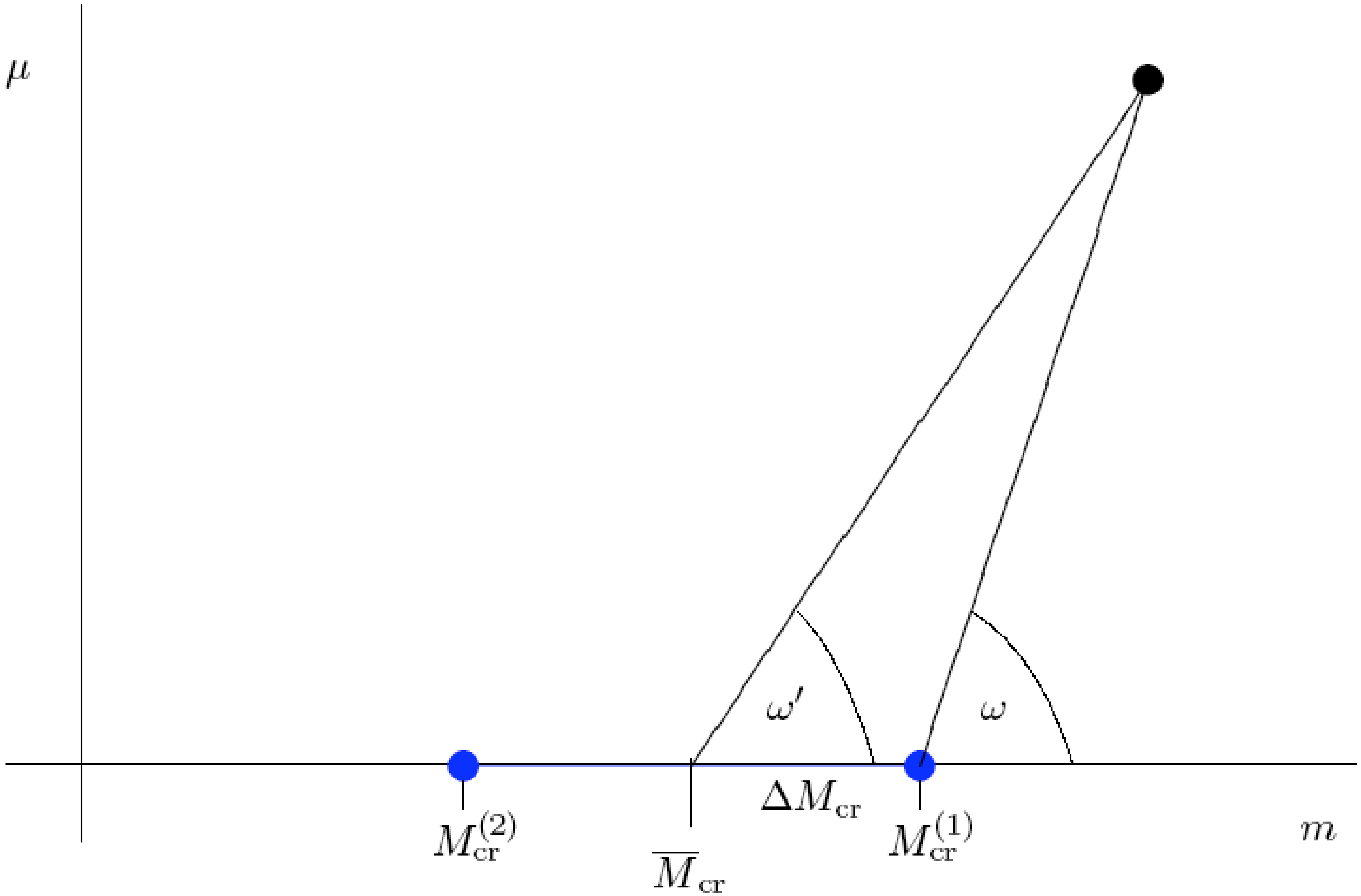
$$\bar{\psi}_{\rm ph}(x)\left[-\left(-a\frac{r}{2}\sum_{\mu}\nabla_{\mu}^{\star}\nabla_{\mu} + \overline{M}_{\rm cr}(r)\right)\exp(-iw\gamma_5\tau_3) + m_q + \Delta M_{\rm cr}(r)\right]\psi_{\rm ph}(x)$$

You can show:

Twist average = Wilson average irrespective of m_q

Automatic O(a) improvement holds for all m_q

Sketch of the different definitions



ω : Frezzotti / Rossi

ω' : New definition

Part 2: ChPT analysis of the pion mass

- Brief reminder: ChPT at non-zero a
- Example: Pion mass at maximal twist: Are the linear a-effects absent ?

ChPT at nonzero a: Strategy

Two-step matching to effective theories:

Lee, Sharpe '98
Sharpe, Singleton '98

1. Lattice theory $\longrightarrow$ Symanzik's effective theory

   continuum theory making the a-dependence explicit

2. Symanzik's effective theory $\longrightarrow$ ChPT

   including the a-dependence

$\Longrightarrow$ Chiral expressions for m_π, f_π ... with explicit a-dependence

Symanzik's action for Lattice QCD with Wilson fermions

Locality and symmetries of the lattice theory

$$\Longrightarrow \qquad S_{eff} = S_{QCD} + a\, c \int \overline{\psi}\, i\sigma_{\mu\nu} G_{\mu\nu} \psi + \mathcal{O}(a^2)$$

- At $O(a)$ only one additional operator (making use of EOM)
- c : unknown coefficient (”low-energy constant”)
- $O(a^2)$: dim-6 operators: - fermion bilinears
  - 4-fermion operators

Sheikholeslami, Wohlert

Chiral Lagrangian including a

$$S_{eff} = S_{QCD} + a\, c \int \overline{\psi}\, i\sigma_{\mu\nu} G_{\mu\nu} \psi + \mathcal{O}(a^2)$$

Pauli term breaks the chiral symmetry exactly like the mass term in S_{QCD}

$$\Rightarrow \qquad \mathcal{L}_2 = \frac{f^2}{4} \mathrm{tr}\left[\partial_\mu \Sigma \partial_\mu \Sigma^\dagger\right] - \frac{f^2 B}{2}\, \mathrm{tr}\left[\Sigma^\dagger M + M^\dagger \Sigma\right]$$

$$- \frac{f^2 W_0}{2}\, a\, \mathrm{tr}\left[\Sigma + \Sigma^\dagger\right]$$

Sharpe, Singleton '98
Rupak, Shoresh '02

W_0 : new undetermined low-energy constant

$\mathcal{L}_4$ - Lagrangian

$$\mathcal{L}_4 = \mathcal{L}_4(p^4, p^2 m, m^2) \;+\; \mathcal{L}_4(p^2 a, ma) \;+\; \mathcal{L}_4(a^2)$$

Gasser, Leutwyler '85

Rupak, Shoresh '02

Rupak, Shoresh, OB '03
Aoki '03

One example: $r^2 W' a^2 \left(\mathrm{tr}\left[\Sigma + \Sigma^\dagger\right]\right)^2$

W': undetermined low-energy constant

Note: This contribution is even in r !

Pion mass including O(a^2)

$$m_\pi^2 = 2Bm + 2W_0 a - 2c_2 a^2 \qquad c_2 = -64 r^2 W' \frac{W_0^2}{f^2}$$

- a-effect: Shift in the pion mass
- m_π^2 does not vanish for $m = 0$

If we define $m_\pi^2 = 0$ for $m' = Z_m(m_0 - m_{\mathrm{cr}}) = 0$

In ChPT this corresponds to: $m' = m + \frac{W_0}{B} a - \frac{c_2}{B} a^2 \quad \Rightarrow \quad m_\pi^2 = 2Bm'$

Matching the effective theory to the lattice theory
requires proper parameter matching!

Spontaneous flavor and parity breaking

Potential energy: $(\mathrm{N_f} = 2)$

Sharpe, Singleton '98

$$V = -c_1 m \,\mathrm{tr}\left[\Sigma + \Sigma^\dagger\right] + c_2 a^2 \left(\mathrm{tr}\left[\Sigma + \Sigma^\dagger\right]\right)^2$$

$c_1(f, B)$

A: $c_2 > 0$ $\Longrightarrow$ $\Sigma_{\mathrm{vacuum}} \neq \pm 1$

Aoki phase
flavor and parity are broken
massless pions at $a \neq 0$

B: $c_2 < 0$ $\Longrightarrow$ $\Sigma_{\mathrm{vacuum}} = \pm 1$

no flavor/parity breaking
no massless pions

The realized scenario depends on the details of the underlying lattice theory

(i.e. the particular Lattice action)

ChPT for tmLQCD

Symanzik action: $S_{eff} = S_{\rm tmQCD} + a\,{\rm Pauli\,term} + \mathcal{O}(a^2)$

$\Longrightarrow$ $$\mathcal{L}_{\rm chiral}\left[m, \omega, a, a^2\right]$$

a: Muenster, Schmidt

a^2: Sharpe, Wu

$\Longrightarrow$ m_π^2, f_π as a function of m, ω, a, a^2

Again: Proper parameter matching required !

Here m and ω

Check for O(a) improvement of the pion mass

1. Lagrangian $\mathcal{L}_{\text{chiral}} \longrightarrow$ potential Energy $\mathcal{V}_{\text{chiral}}$

2. Find ground state $\Sigma_0 = e^{i\phi\tau_3}$ by $\frac{\mathrm{d}\mathcal{V}_{\text{chiral}}}{\mathrm{d}\phi} = 0$

3. Expand around Σ_0 and find m_π^2 (to LO)

4. Express m_π^2 in terms of the twist angle ω corresponding to the lattice theory

5. Go to $\omega = \frac{\pi}{2}$ and check for $\mathcal{O}(a)$

I assume: $c_2 > 0$ Existence of an Aoki phase and massless pions

Definition of Frezotti / Rossi

Definition of ω : Lattice theory $(m_0 - M_{\text{cr}}(r))e^{i\omega\tau_3\gamma_5}$

Effective theory $\tan\omega = \dfrac{2Bm_L \sin\omega_L}{2Bm_L\cos\omega_L + 2W_0 a - 2c_2 a^2}$

For $\omega = \pi/2 \quad (\mu := m_L \sin\omega_L)$

1. $2B\mu \geq \mathcal{O}(a) \quad \Longrightarrow$

$$m_{\pi_a}^2 = 2B\mu + 2c_2 a^2$$

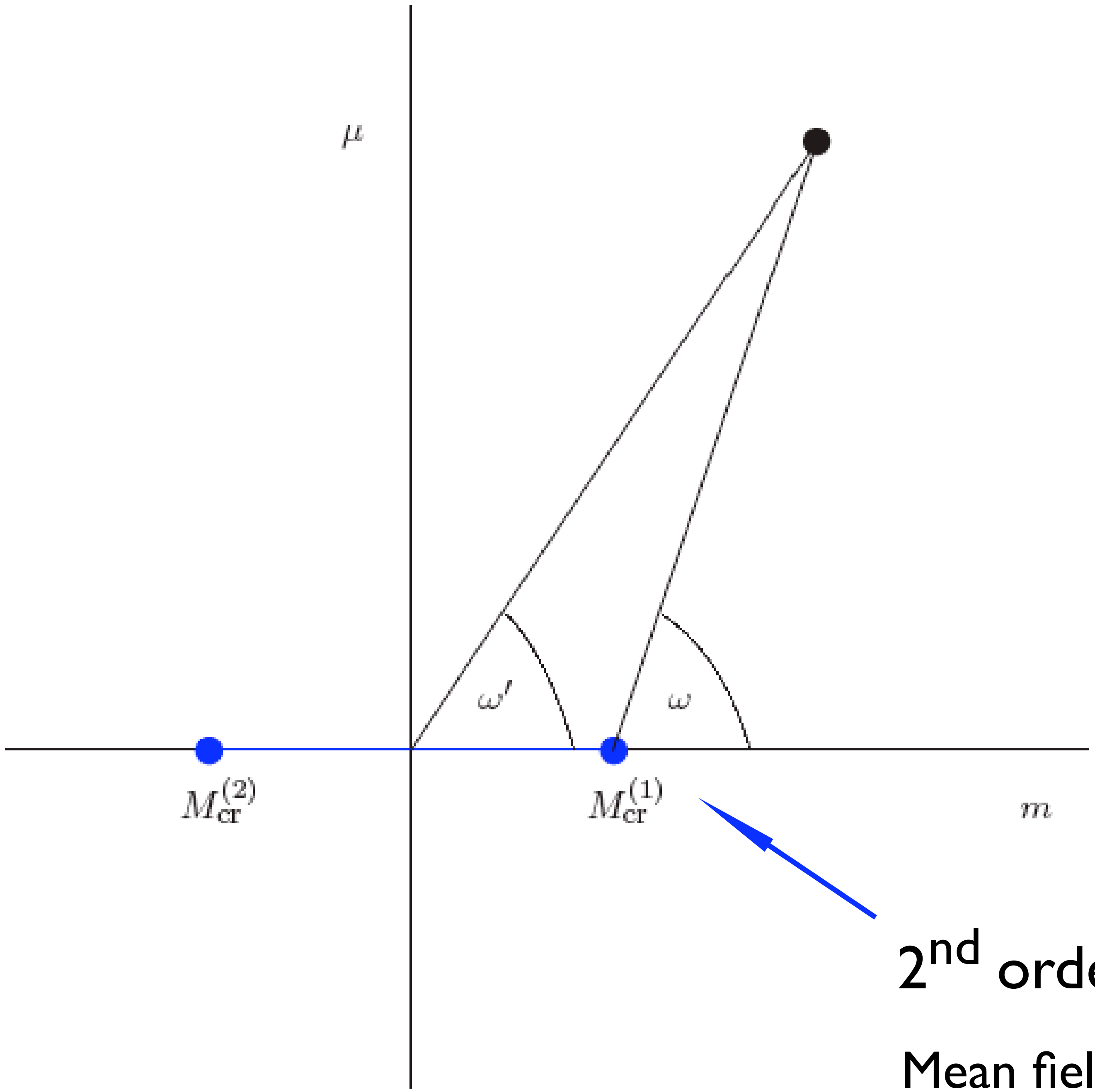
$$m_{\pi_a}^2 - m_{\pi_3}^2 = 2c_2 a^2$$

2. $2B\mu \ll 2c_2 a^2 \quad \Longrightarrow$

$$m_{\pi_a}^2 = (c_2 a^2)^{1/3}(2B\mu)^{2/3}$$

$$m_{\pi_a}^2 - m_{\pi_3}^2 = 2(c_2 a^2)^{1/3}(2B\mu)^{2/3}$$

O(a) improvement only in case 1



ω : Frezzotti / Rossi

ω' : Alternative definition

2nd order phase transition point

Mean field critical exponent = 2/3

Alternative definition for the twist

Definition of ω : Lattice theory $\left(m_0 - \frac{M_{\rm cr}(r) - M_{\rm cr}(-r)}{2}\right) e^{i\omega\tau_3}$

Effective theory $\tan\omega = \frac{2Bm_L \sin\omega_L}{2Bm_L\cos\omega_L + 2W_0 a}$

For $\omega = \pi/2 \quad (\mu := m_L \sin\omega_L)$

Without restrictions on $2B\mu$ $\Longrightarrow$

$$m_{\pi_a}^2 = 2B\mu$$

$$m_{\pi_a}^2 - m_{\pi_3}^2 = 2c_2 a^2$$

Automatic O(a) improvement irrespective of the size of μ !

Note: Shown only for the purely linear term in a , not for $am, am^2, \ldots$

Twist angle from Ward identities

In continuum tmQCD: $$\tan\omega_{\rm WT} = \frac{\langle \partial_\mu V_\mu^2 \ P^1 \rangle}{\langle \partial_\mu A_\mu^1 \ P^1 \rangle}$$

Vector and Axial vector WT identities: $$\partial_\mu V_\mu^a = -2\mu\epsilon^{3ab} P^b$$

$$\partial_\mu A_\mu^a = 2mP^a + 2i\mu S^0 \delta_{a3}$$

$\Longrightarrow$ $$\tan\omega_{\rm WT} = \frac{\mu}{m}$$

ω_{WT} in the effective theory

1. Maximal twist of Frezzotti / Rossi :

For $2B\mu \ll 2c_2a^2$ $\Longrightarrow$ $\tan\omega_{\mathrm{WT}} \simeq (\frac{2B\mu}{2c_2a^2})^{1/3}$

$\Longrightarrow$ $\omega_{\mathrm{WT}} \neq \pi/2$ $(\omega_{\mathrm{WT}} = 0 \text{ for } \mu = 0)$

2. Alternative definition :

$$\tan\omega_{\mathrm{WT}} = \infty$$

$$\omega_{\mathrm{WT}} = \pi/2 = \omega$$

Second scenario

$c_2 < 0 \quad \Longrightarrow$

- 1st order phase transition
- No massless pions at $a \neq 0$

Consider $m_\pi(r, m_0)$

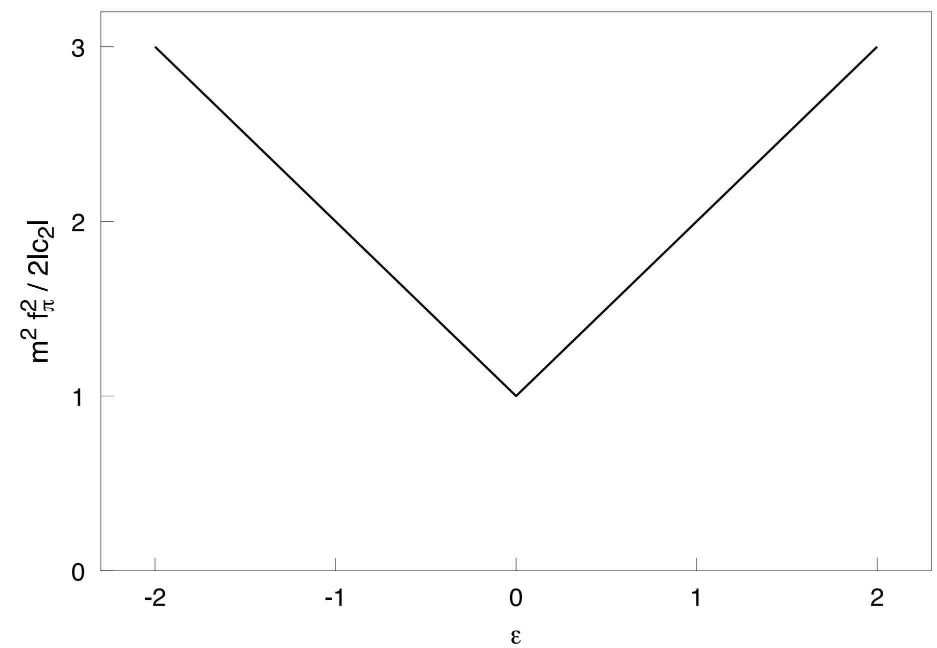
Defining equation for critical quark mass:

$$m_\pi\big(r, M_{\rm cr}(r)\big) = m_{\pi,\rm min}$$

ChPT: minimal value is unique

$$\Longrightarrow \quad M_{\rm cr}(-r) = -M_{\rm cr}(r)$$

ChPT result (untwisted mass)

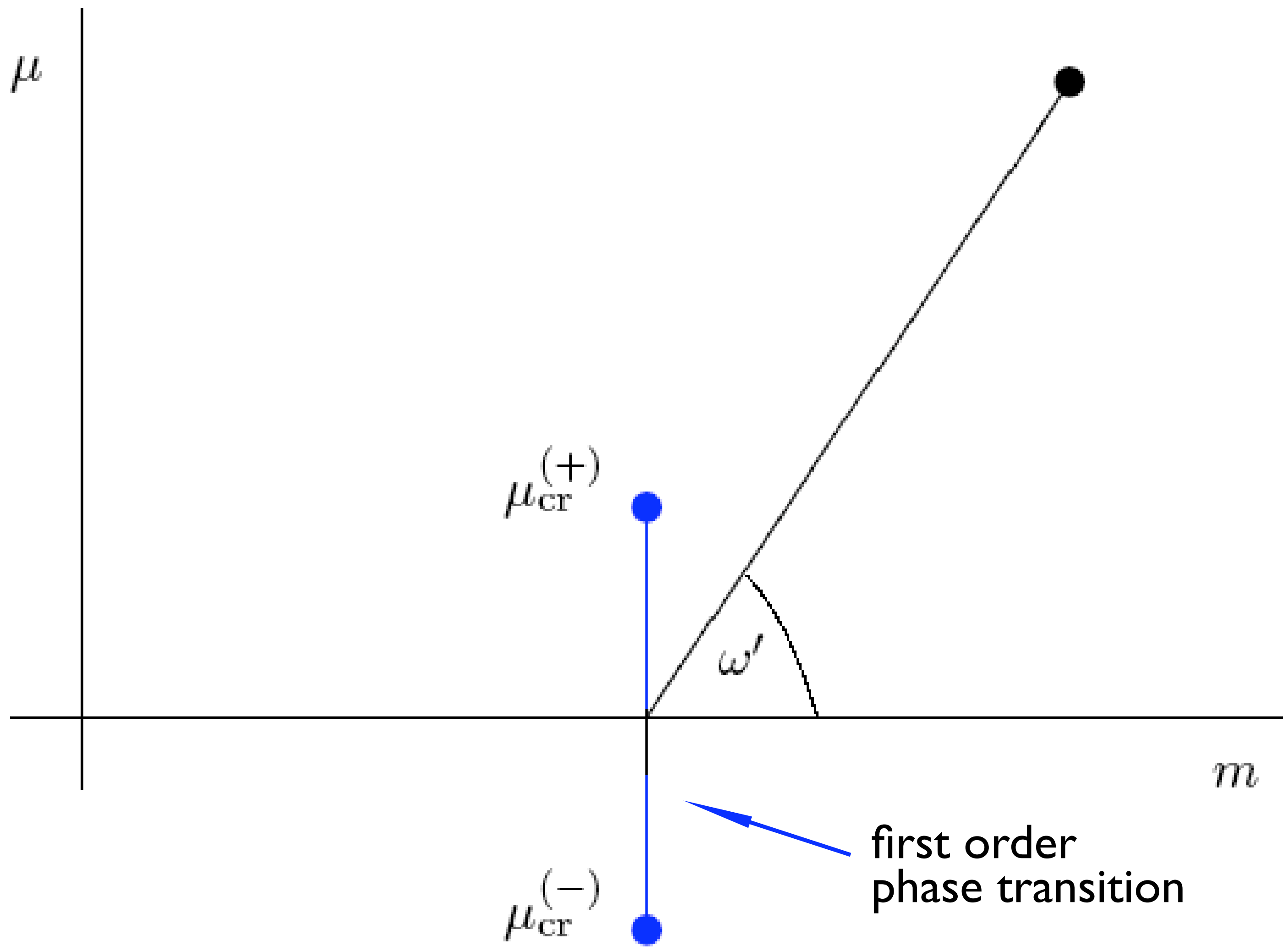


Sharpe, Singleton '98

$\Longrightarrow$ Automatic O(a) improvement for all m

Phase diagram for $c_2 < 0$

Muenster 2004



Endpoints of the phase transition line:

$$\mu_{\rm cr}^{(\pm)} = \pm \frac{c_2}{B} a^2$$

Automatic O(a) improvement fails for $|\mu| < \mu_{\rm cr}^{(+)}$

O(a) improvement a la Symanzik

Quark mass bounds for automatic O(a) improvement at maximal twist:

1. $c_2 > 0$ arbitrary $\mu > 0$
2. $c_2 < 0$ $\mu > a^2 \frac{c_2}{B} = a^2 \Lambda_{\rm QCD}^3$

Results are independent of O(a) Symanzik improvement !

but: Adding a clover term may change the value of c_2 $\Longrightarrow$ indirect dependency

Recent paper by S. Sharpe and J. Wu: Same conclusion

Sharpe, Wu
hep-lat/0411021

Conclusion

- Automatic O(a) improvement should work for twisted quark masses

  down to $\mu \approx a^2 \Lambda^3_{\rm QCD}$ (or even $\mu > 0$)

- The two quark mass bounds given by Frezzotti / Rossi

  1. $\mu \gg a \Lambda^2_{\rm QCD}$
  2. $\mu \gg a^2 \Lambda^3_{\rm QCD}$

  are too strong

- Automatic O(a) improvement requires $M_{\rm cr}(-r) = -M_{\rm cr}(r)$

  This is not automatically guaranteed !

What next ?

Scaling analysis of tm LQCD at maximal twist

- Confirm the a^2-scaling for small quark masses

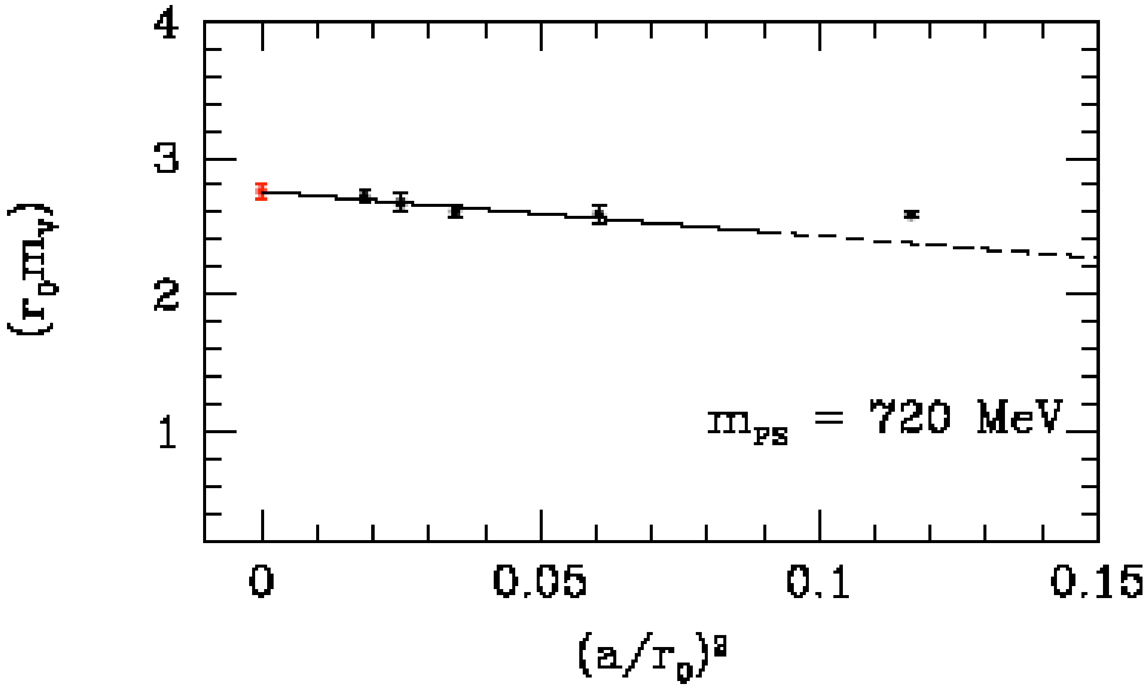
- Confirm the scaling violation for "wrong" twist angle.

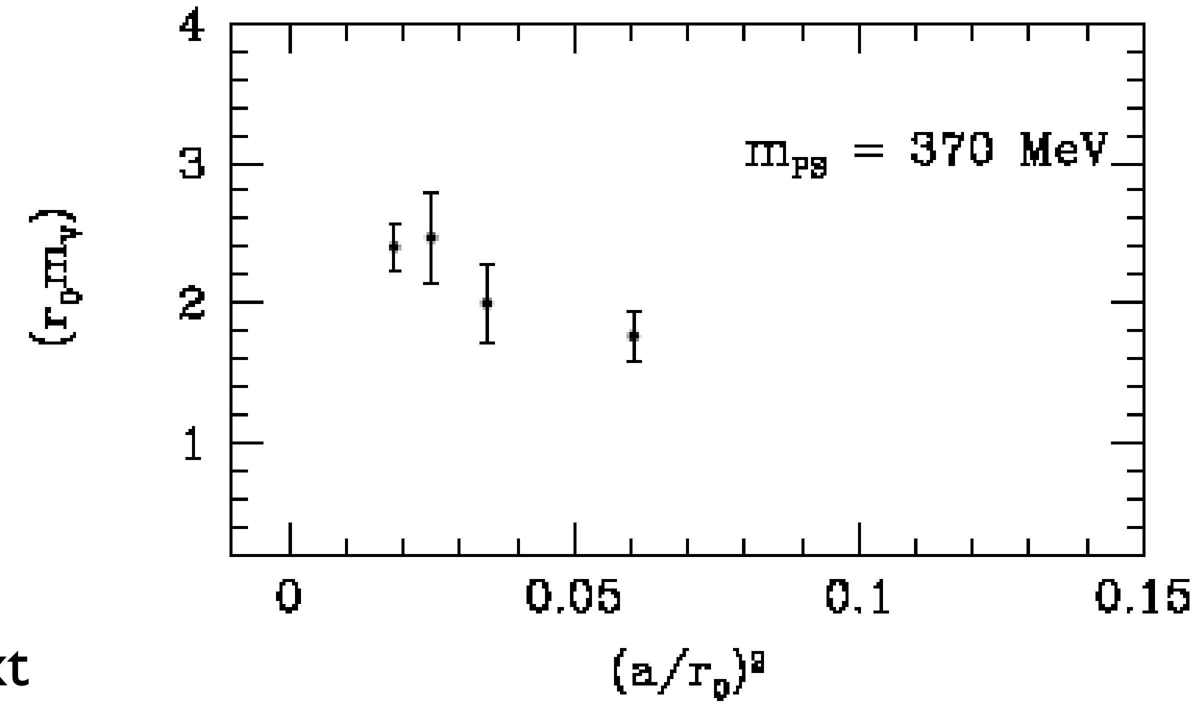
  Preliminary results by the $\chi_L F$ Collaboration (Izu workshop Sep '04)

  They do find deviations for small quark masses

  Does the effective theory describe the scaling violation correctly ?

Scaling of the vector meson mass





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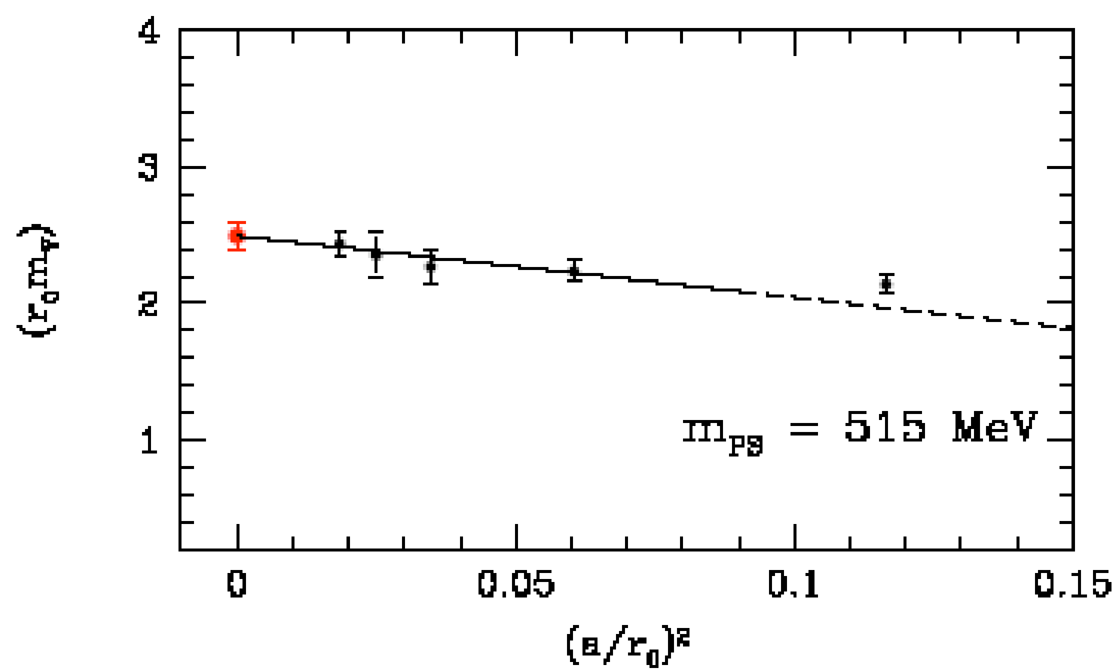

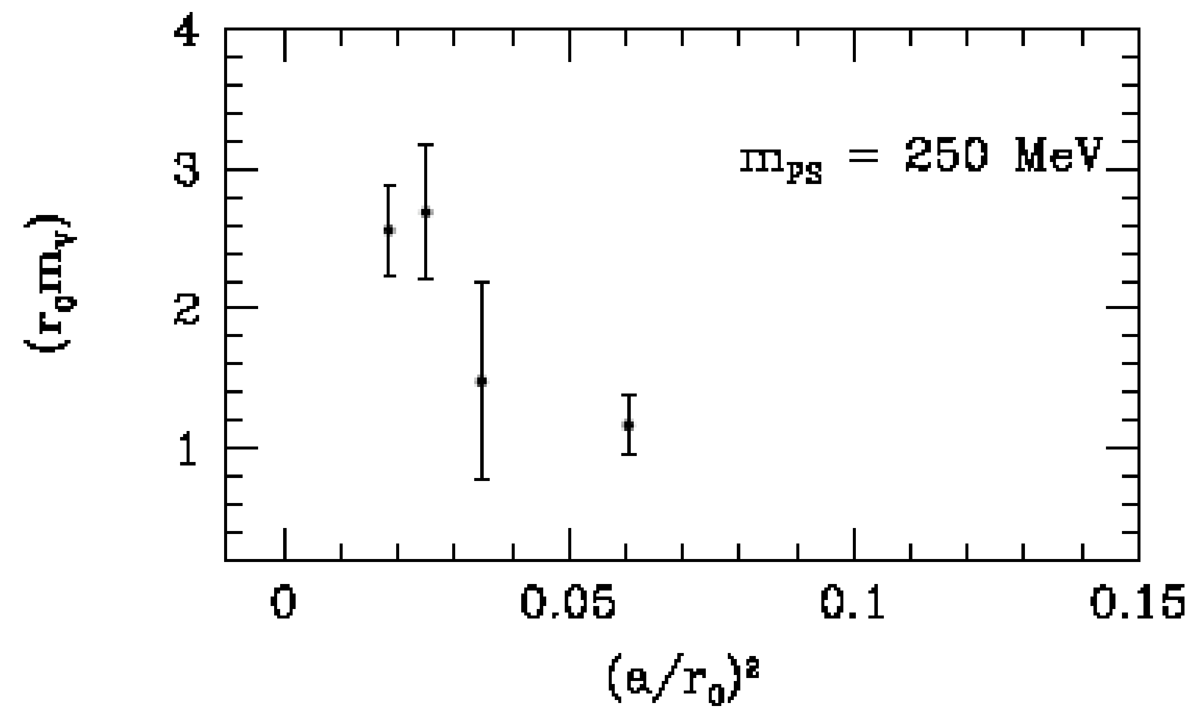

χLF Collaboration

PRELIMINARY RESULTS !

plots by A. Shindler, Sep 2004