I=2 $\pi\pi$ scattering phase shift

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1. Introduction

Isospin $I = 2$ S-wave $\pi\pi$ scattering phase shift

1. with dynamical u, d quark effect ($N_f = 2$ fullQCD)
2. center of mass and laboratory frames
3. continuum limit

Motivations :

(i) Understanding of hadron dynamics

- scattering length a_0

  previous work

  – Sharpe, Gupta, and Kilcup, 1992
  – Gupta, Patel, and Sharpe, 1993
  – Kuramashi *et. al.*, 1993
  – JLQCD Collaboration, 1999
  – Lin, Zhang, Chen, and Ma, 2001

- scattering phase shift $\delta(p)$

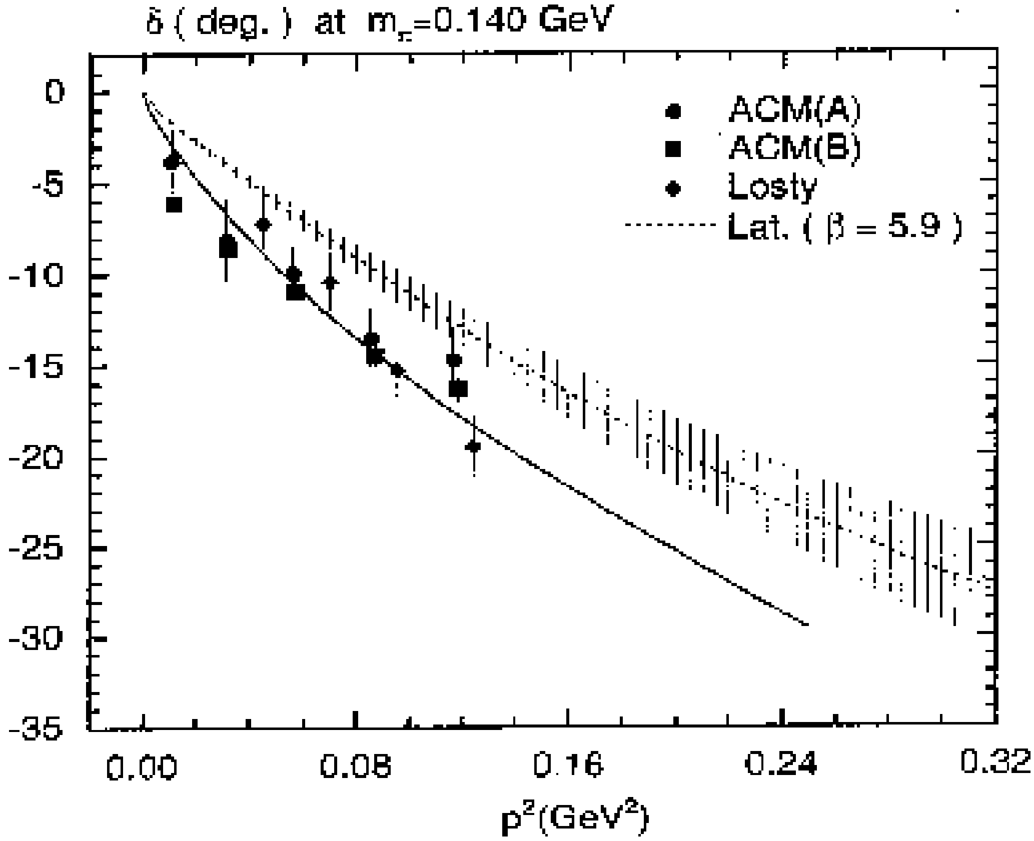
  – Fiebig, Rabitsch, Markum, and Mihály, 2000 from $\pi\pi$ potential
  – CP-PACS Collaboration, 2002

(ii) First step for study of hadron decay

$$\rho \to \pi\pi, \quad K \to \pi\pi, \quad \text{etc.}$$

CP-PACS Collaboration result

hep-lat/0209124



1. without dynamical quark effect
   (quenched approximation)

2. center of mass frame

3. lattice spacing a is finite

Contents

2. Method

2.1 $\delta(p)$ from lattice calculation

(i) Lüscher's formula

Commmun. Math. Phys. 105, 153(1986)
Nucl. Phys. B354, 531(1991)

wave function of two-particle

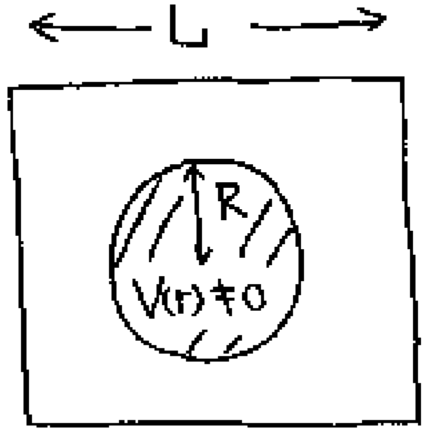
$$\phi(\mathbf{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}(\theta,\varphi)\psi_{lm}(r)$$

$$\psi_{lm}(r) \propto \alpha_l(p) j_l(pr) - \beta_l(p) n_l(pr)$$

$j_l(\rho)$: spherical Bessel function
$n_l(\rho)$: spherical Neumann function

$$\tan(\delta_l(p)) = \frac{\beta_l(p)}{\alpha_l(p)}$$

- center of mass frame
- finite volume L^3
- periodic boundary condition
- interaction range R : $L > 2R$



$V(r) \neq 0 \quad 0 < r < R$

$V(r) = 0 \quad r > R$

two-particle energy on finite volume $E_{\pi\pi}$

$$\begin{aligned} E_{\pi\pi} &= \overline{E}_{\pi\pi} + \Delta E \\ & \quad \overline{E}_{\pi\pi} = 2\sqrt{m_\pi^2 + \overline{p}^2} \\ & \quad \overline{p} = \frac{2\pi}{L} n, \quad n = 0, \cdots, L-1 \\ &= 2\sqrt{m_\pi^2 + p^2} \end{aligned}$$

Lüscher's formula

$$\tan(\delta(p)) = \frac{\pi^{3/2} q}{Z_{00}(1;q^2)}, \quad q = \frac{Lp}{2\pi}, \ q \neq n$$

$$Z_{00}(1;q^2) = \frac{1}{\sqrt{4\pi}} \sum_{\mathbf{n} \in \mathbf{Z}^3} \frac{1}{\mathbf{n}^2 - q^2}$$

Scattering phase shift $\delta(p)$ can be obtained from two-particle energy on finite volume.

$$E_{\pi\pi} \to \delta(p)$$

(ii) Generalized Lüscher's formula

Rummukainen and Gottlieb, Nucl. Phys. B450 397(1995)

laboratory frame $\qquad \pi_{\overline{p}_1} \longrightarrow \quad \longleftarrow \pi_{\overline{p}_2} \quad (\overline{p}_1 + \overline{p}_2 = P)$

$$E^L_{\pi\pi} = \overline{E}^L_{\pi\pi} + \Delta E^L$$

$$\overline{E}^L_{\pi\pi} = \sqrt{m_\pi^2 + \overline{p}_1^2} + \sqrt{m_\pi^2 + \overline{p}_2^2}$$

$$\Downarrow \text{ boost}$$

center of mass frame $\qquad \pi_{\overline{p}'_1} \rightarrow \quad \leftarrow \pi_{\overline{p}'_2} \quad (\overline{p}'_1 + \overline{p}'_2 = 0)$

$$\left(E^{CM}_{\pi\pi}\right)^2 = \left(E^L_{\pi\pi}\right)^2 - P^2 = \left(2\sqrt{m_\pi^2 + p^2}\right)^2$$

Generalized Lüscher's formula

$$\tan(\delta(p)) = \frac{\gamma\pi^{3/2} q}{\mathcal{Z}^P_{00}(1;q^2)}, \quad q = \frac{Lp}{2\pi}$$

$$\mathcal{Z}^P_{00}(1;q^2) = \frac{1}{\sqrt{4\pi}} \sum_{\mathbf{r}\in \mathbf{P}_d} \frac{1}{\mathbf{r}^2 - q^2}, \ \gamma = \frac{1}{\sqrt{1-v^2}}, \ v = \frac{P}{E^L_{\pi\pi}}, \ \mathbf{d} = \frac{LP}{2\pi}$$

$$\mathbf{P}_d = \{\mathbf{r} \in \mathbf{R}^3 | \mathbf{r} = \gamma^{-1}(\mathbf{n} + \mathbf{d}/2), \mathbf{n} \in \mathbf{Z}^3\}$$

$$E^L_{\pi\pi} \rightarrow \delta(p)$$

advantages of laboratory frame

- simultaneous measurement

  Center of mass and laboratory frames

- intermediate energy (momentum)

$$E^{CM}_{\pi\pi}(0) < E^{L}_{\pi\pi}(0) < E^{CM}_{\pi\pi}(1)$$

- number of frame

$$P = \frac{2\pi}{L}, \sqrt{2}\frac{2\pi}{L}, \cdots \quad (P = \overline{p}_1 + \overline{p}_2)$$

2.2 Diagonalization method

Problem on lattice (center of mass frame)

$\pi\pi$ 4-point function ($\Omega_p = \pi(p)\pi(-p)$)

$$\begin{aligned} C_{pp}(t) &= \langle 0|\Omega_p^\dagger(t)\Omega_p(0)|0\rangle \\ &= \sum_k |V_{kp}|^2 e^{-E_k t} + \cdots \\ &\to |V_{0p}|^2 e^{-E_0 t} \quad t \to \infty \end{aligned}$$

$$V_{kp} = \langle \overline{\Omega}_k|\Omega_p|0\rangle, \quad E_k = 2\sqrt{m_\pi^2 + k^2}$$

We cannot obtain E_p from single exp. fit.

E_0 is extracted from single exp. fit.

Diagonalization of $C_{pq}(t)$

Lüscher and Wolff, Nucl. Phys. B339 222(1990)

$N \times N$ 4-point function matrix

$$C_{pq}(t) = \sum_\nu^N V_{\nu p}^T \cdot \Delta_\nu(t) \cdot V_{\nu q}, \quad \Delta_\nu(t) = e^{-E_\nu t}$$

$$D(t, t_0) w_\nu = \lambda_\nu(t, t_0) w_\nu$$

$D(t, t_0) = C^{-1/2}(t_0) C(t) C^{-1/2}(t_0)$, t_0 : reference time

$$\lambda_\nu(t, t_0) = \frac{\Delta_\nu(t)}{\Delta_\nu(t_0)} = e^{-E_\nu(t - t_0)}$$

independent of operator normalization

In laboratory frame $\lambda_\nu(t, t_0)$ is obtained with diagonalization as well as center of mass frame.

3. Set up and parameters

(i) $\pi\pi$ 4-point function

CP-PACS Collaboration, hep-lat/0209124

- sink

$$\Omega_p(t) = \sum_{R\in \mathrm{Oh}} \pi(R(p),t)\pi(R(-p),t)$$

Oh : cubic group

$\sum_{R\in \mathrm{Oh}}$: projection to A_1^+ (S-wave)

- source

$$\widetilde{\Omega}_p(t) = \tilde{\pi}(p,t)\tilde{\pi}(-p,t)$$

$$\tilde{\pi}(p,t) = \left(\sum_x \overline{q}(x)\eta^*(x)\right)\gamma_5\left(\sum_y e^{-ipy}\eta(y)q(y)\right)$$

$\eta(x)$: U(1) noise, $\sum_\eta \eta^*(x)\eta(y) = \delta^3(x-y)$

- 4-point function

$$C_{pq}(t) = \sum_\eta \langle 0|\Omega_p^\dagger(t)\widetilde{\Omega}_q(t_s)|0\rangle$$

$p \leftrightarrow q$ symmetric

t_s : source point

(ii) momentum setup $(d = d_1 + d_2)$

I. center of mass frame

CM $d = (0,0,0)$ 0, 1, 2

n	d_1	d_2	W
0	(0, 0, 0)	(0, 0, 0)	1.40
1	(1, 0, 0)	(-1, 0, 0)	1.60
2	(1, 1, 0)	(-1,-1, 0)	1.78

II. laboratory frame 1

L=1 $d = (1,0,0)$ 0, 1, 2, 3

n	d_1	d_2	W
0	(1, 0, 0)	(0, 0, 0)	1.45
1	(1, 1, 0)	(0,-1, 0)	1.65
2	(2, 0, 0)	(-1, 0, 0)	1.81
3	(1, 1, 1)	(0,-1,-1)	1.83

III. laboratory frame 2

L=2 $d = (1,1,0)$ 0, 1, 2, 3

n	d_1	d_2	W
0	(1, 1, 0)	(0, 0, 0)	1.49
1	(1, 0, 0)	(0, 1, 0)	1.51
2	(1, 1, 1)	(0, 0,-1)	1.69
3	(1, 0, 1)	(0, 1,-1)	1.70

$$W = \sqrt{E^2 - \left(\frac{2\pi}{L}d\right)^2} \text{ at } m_\pi = 0.7, L = 16$$

$$E = \sqrt{m_\pi^2 + \left(\frac{2\pi}{L}d_1\right)^2} + \sqrt{m_\pi^2 + \left(\frac{2\pi}{L}d_2\right)^2}$$

index $p \to n$, $\Omega_p \to \Omega_n$, $C_{pq} \to C_{nm}$

(iii) Simulation parameters

$N_f = 2$ fullQCD configuration

CP-PACS Collaboration, Phys. Rev. D65, 054505(2002)

- improved action

  gauge : Iwasaki action

  fermion : clover action

- parameter

β	C_{sw}	a^{-1}[GeV]	$L^3 \cdot T$
1.80	1.60	0.918	$12^3 \cdot 24$
1.95	1.53	1.27	$16^3 \cdot 32$

$La \sim 2.5$[fm]

- pion mass and number of configurations

$\beta = 1.80$				
m_π [MeV]	1060	900	760	490
# of conf.	650	520	730	400
$\beta = 1.95$				
m_π [MeV]	1130	930	760	540
# of conf.	590	690	680	490

- boundary condition

  spatial direction : periodic boundary

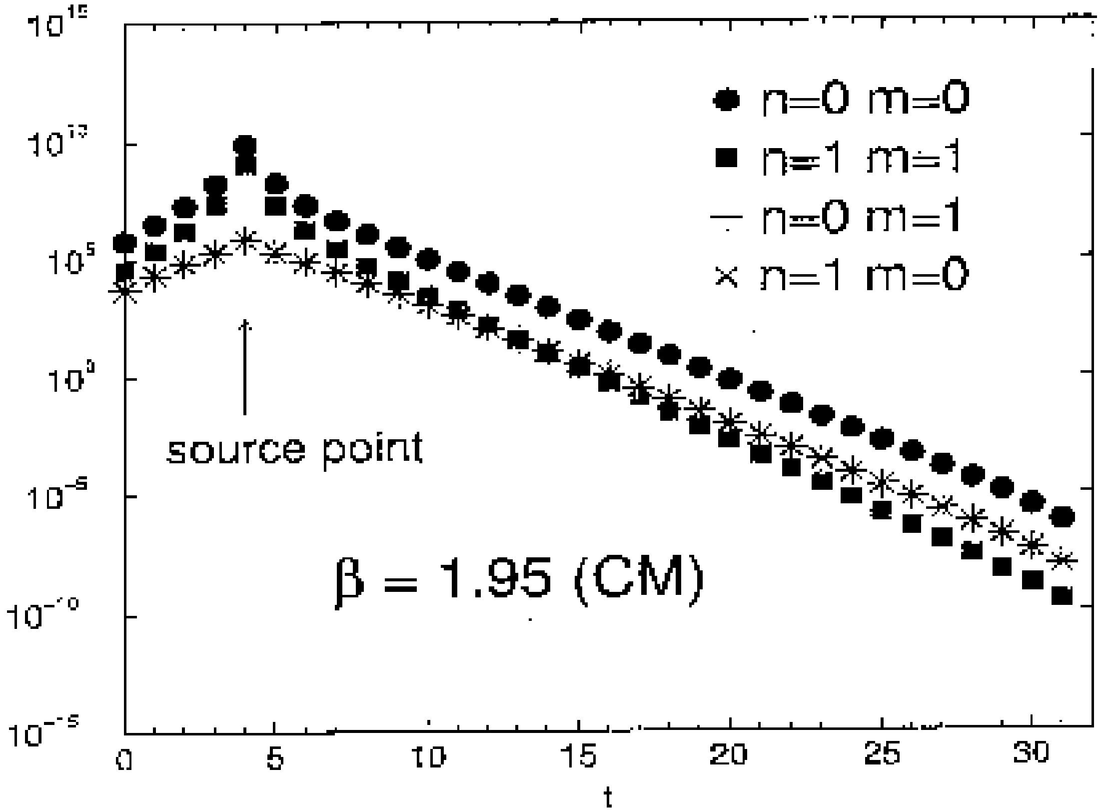
  time direction : Dirichlet boundary

- # of U(1) noise ; 2 / conf.

4. Results

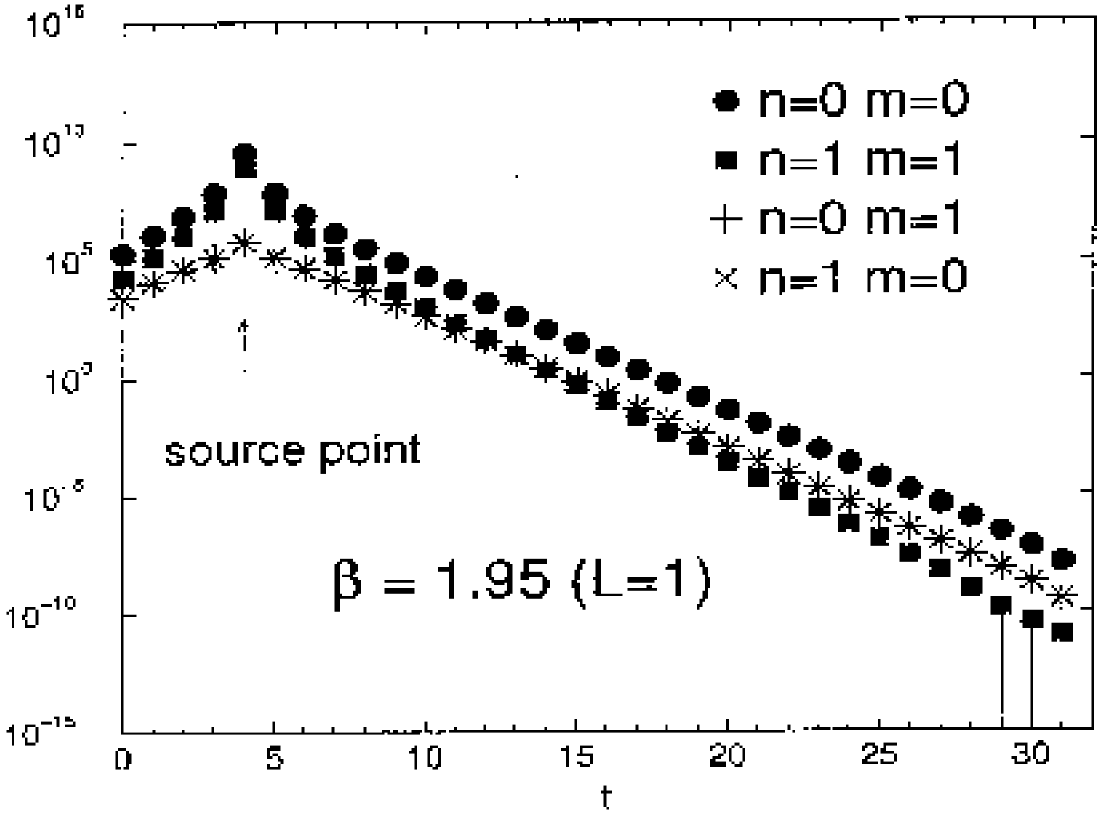
4.1 4-point function $C_{nm}(t)$

(i) center of mass frame ($m_\pi = 760$[MeV])



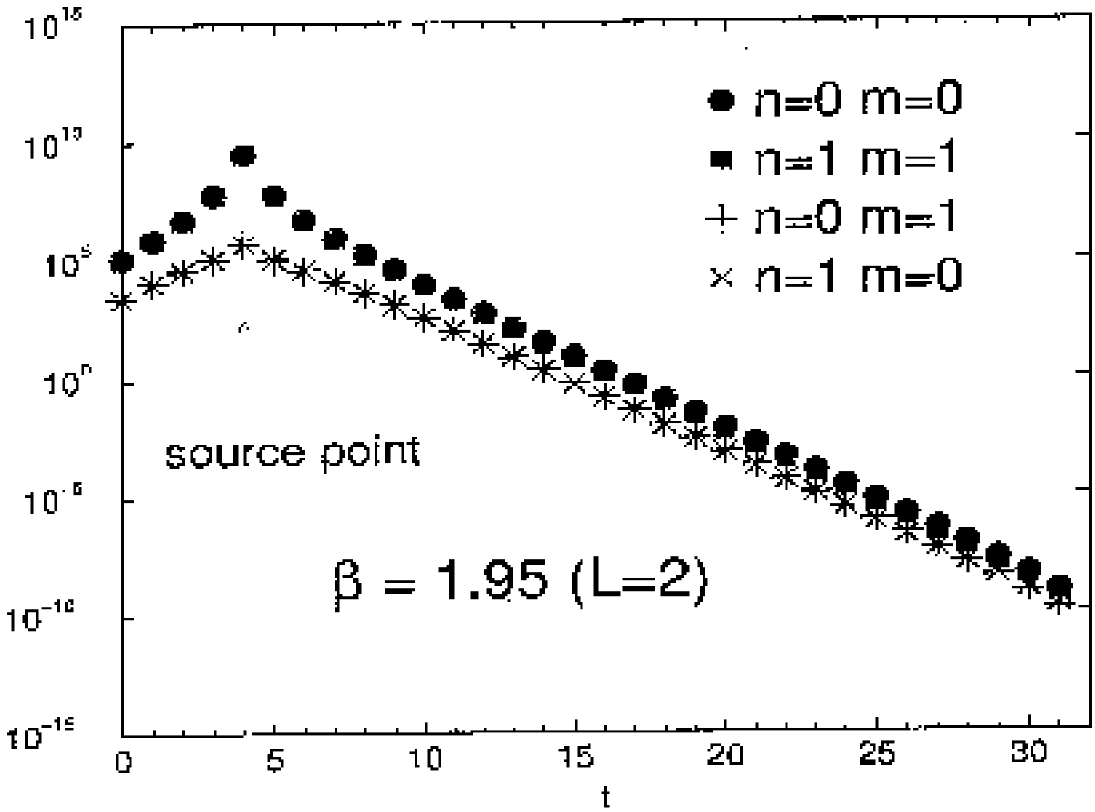
Off-diagonal part : finite contribution
symmetric $n \leftrightarrow m$

(ii) laboratory frame 1 ($m_\pi = 760$[MeV])



Off-diagonal part : finite contribution
symmetric $n \leftrightarrow m$
same behavior as CM

(iii) laboratory frame 2 ($m_\pi = 760$[MeV])



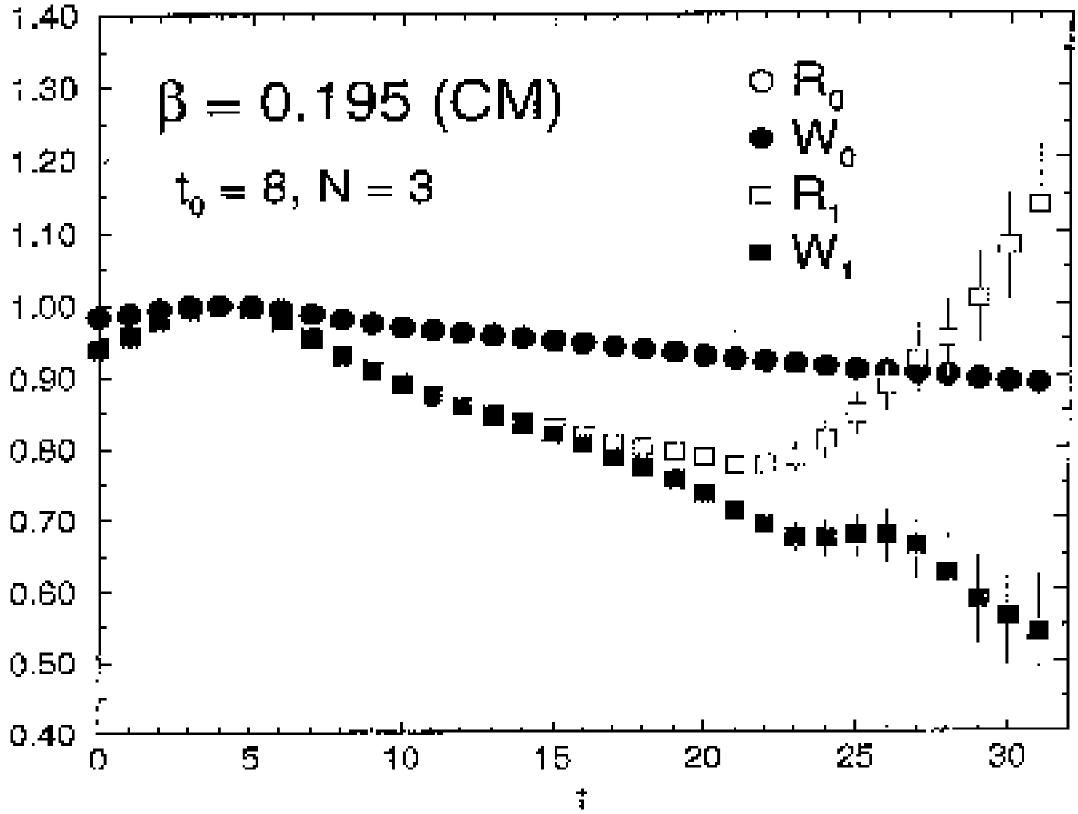
diagonal part : same exp. decay

Off-diagonal part : finite contribution

symmetric $n \leftrightarrow m$

4.2 diagonalization

(i) center of mass frame ($m_\pi = 760$[MeV])



4-point function ratio R_n (open symbol)

$$R_n(t) = \frac{C_{nn}(t)}{G_n^1(t)G_n^2(t)} \rightarrow e^{-(E_0-\overline{E}_n)(t-t_S)} \quad (t \rightarrow \infty)$$

$G_n^i(t) = \langle 0|(\pi^i(p_j,t)\pi(p_j,t_S)|0\rangle \sim \exp(-\sqrt{m_\pi^2 + p_j^2})$

$\overline{E_n}$: 2-pion energy without interction

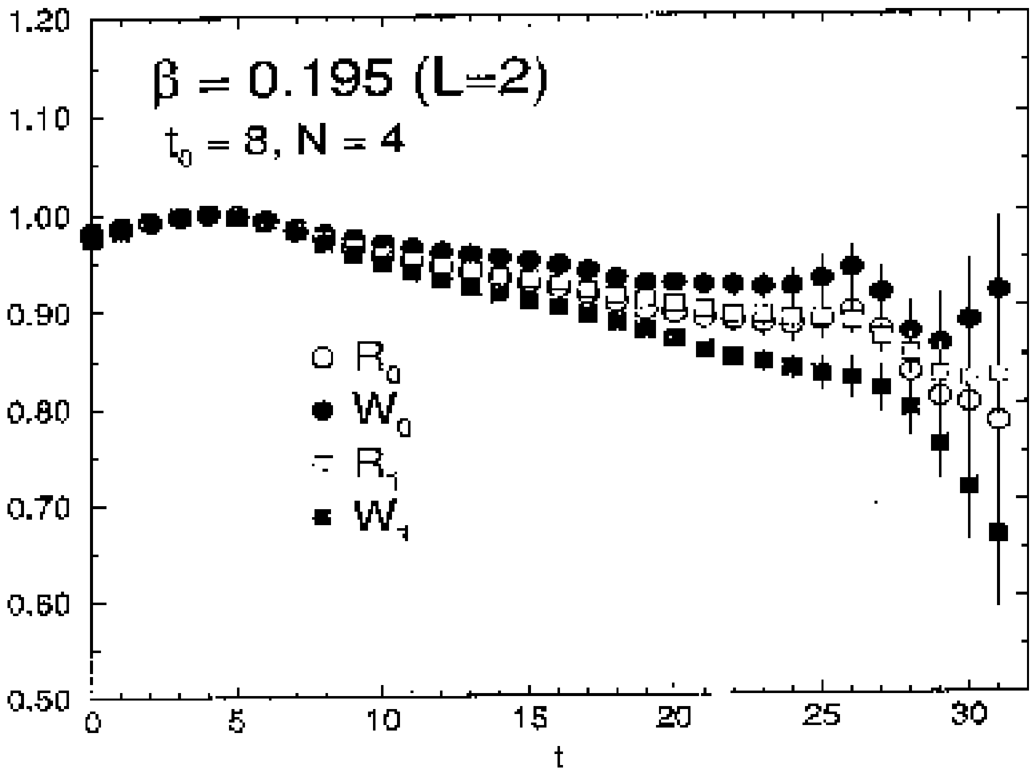
eigenvalue ratio W_n (closed symbol)

$$W_n(t) = N_n \frac{\lambda_n(t,t_0)}{G_n^1(t)G_n^2(t)} \propto e^{-\Delta E_n(t-t_0)}$$

$$\Delta E_n = E_n - \overline{E}_n$$

Diagonalization is needed to extract E_1.

(ii) laboratory frame 2 ($m_\pi = 760$[MeV])

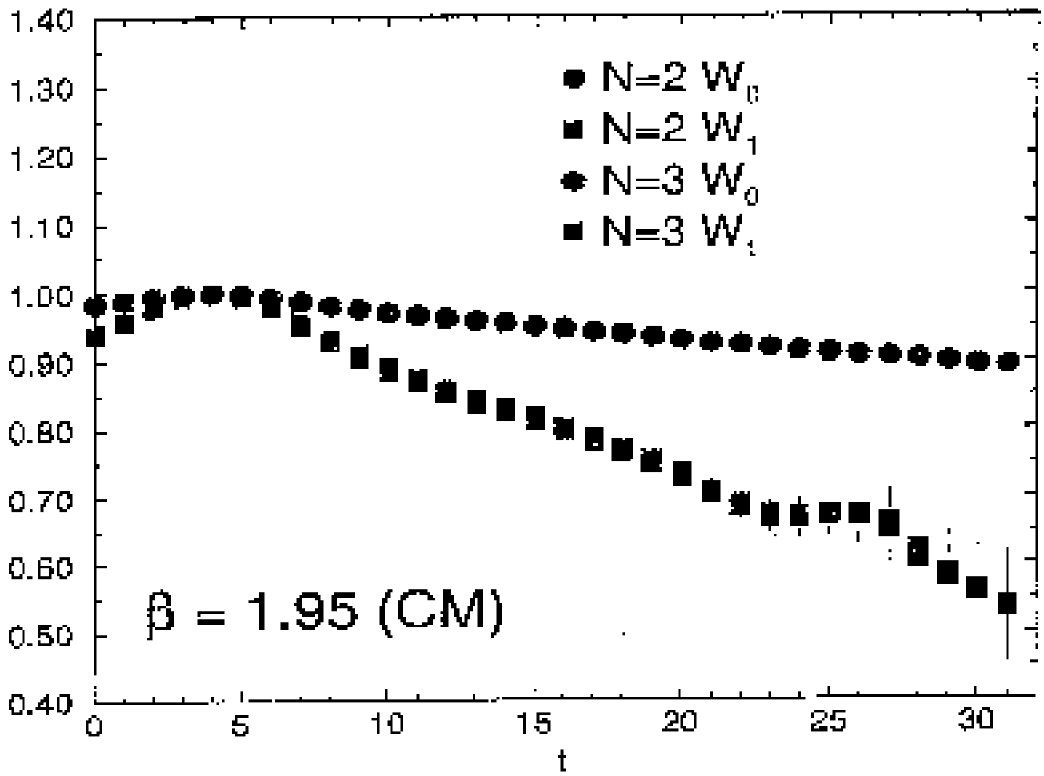


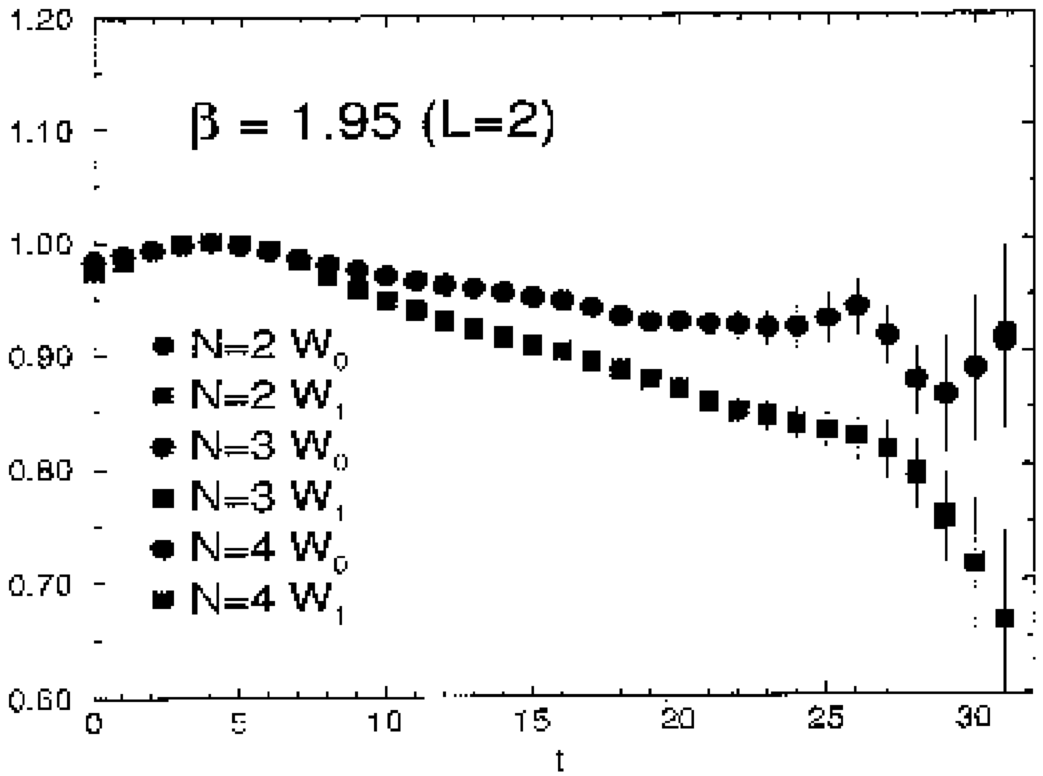
E_0 and E_1 are split after diagonalization.

$$\begin{pmatrix} A & B \\ B & A+\Delta A \end{pmatrix} X = EX$$

$$\begin{aligned} E &= A + \frac{\Delta A}{2} \pm \sqrt{B^2 + \frac{(\Delta A)^2}{4}} \\ &\sim A + \frac{\Delta A}{2} \pm B \quad (B \gg \Delta A) \end{aligned}$$

(iii) reliability ($N \times N$ matrix diagonalization)





Results for $n = 0, 1$ states are stable.

4.3 chiral extrapolation

$m_\pi > 500$[MeV] $\rightarrow m_\pi = 140$[MeV]

amplitude $T(p)$

global fit for m_π^2, p^2 with all data at each β

$$
\begin{aligned}
T(p) &= \frac{\tan(\delta(p))}{p} E_\pi, \quad E_\pi = \sqrt{m_\pi^2 + p^2} \\
&= \sum_{j,k} A_{jk} \, (m_\pi^2)^j \, (p^2)^k \\
&= A_{10} m_\pi^2 + A_{20} m_\pi^4 + A_{01} p^2 \\
&\quad + A_{11} m_\pi^2 p^2 + A_{02} p^4 \\
&\rightarrow a_0 m_\pi \quad p \rightarrow 0, \quad a_0 \text{ : scattering length}
\end{aligned}
$$

preliminay result

β	1.80	1.95
A_{10}[GeV^{-2}]	-0.943(52)	-1.050(60)
A_{20}[GeV^{-4}]	0.494(50)	0.447(53)
A_{01}[GeV^{-4}]	-0.74(25)	-1.37(38)
A_{11}[GeV^{-4}]	0.67(19)	0.99(28)
A_{02}[GeV^{-4}]	0.72(67)	1.33(96)
χ^2/d.o.f	1.4	1.8

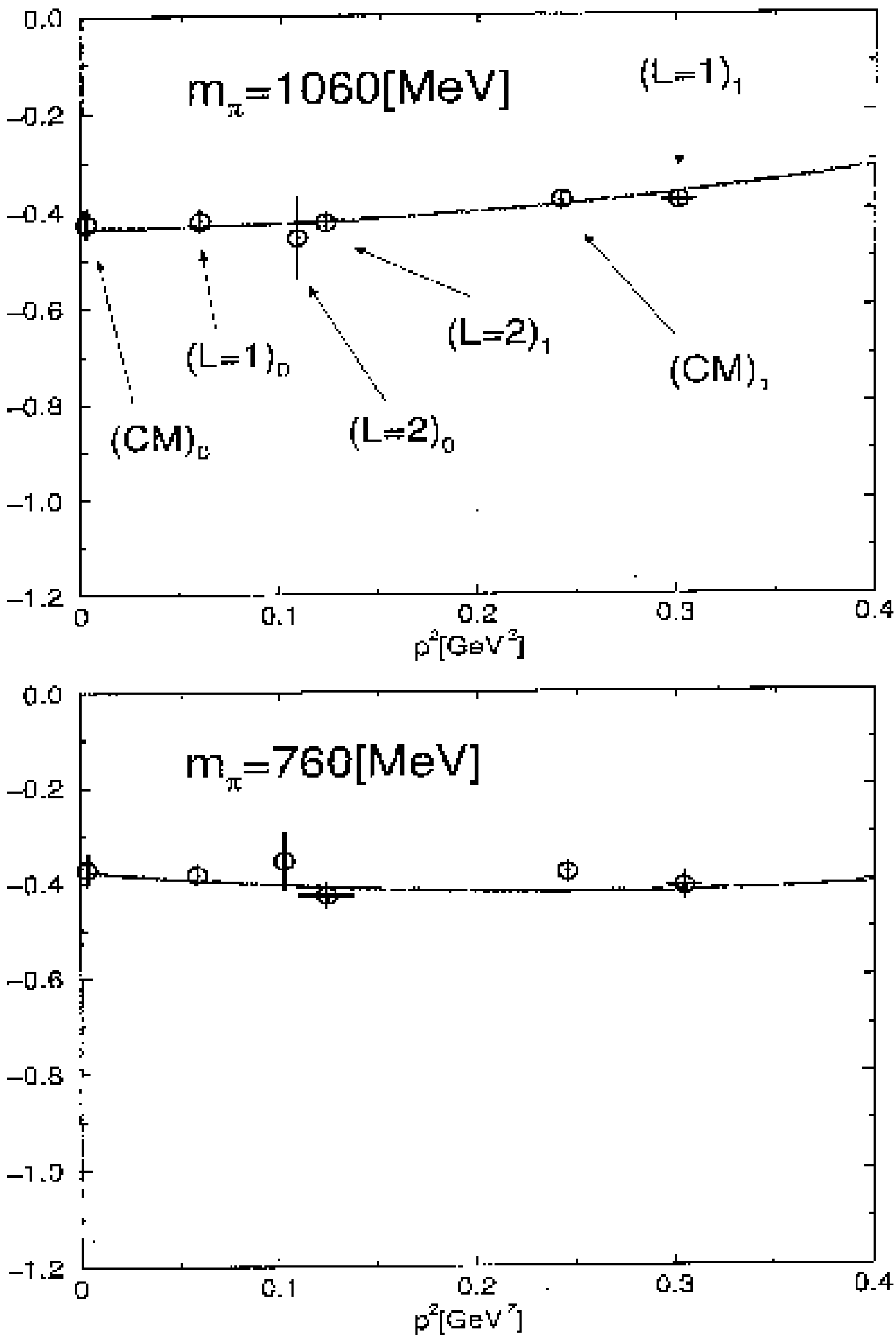
$m_\pi = 140$[MeV]

$a_0 m_\pi$ -0.0182(10) -0.0204(11)

-0.0444(10) (CHPT)

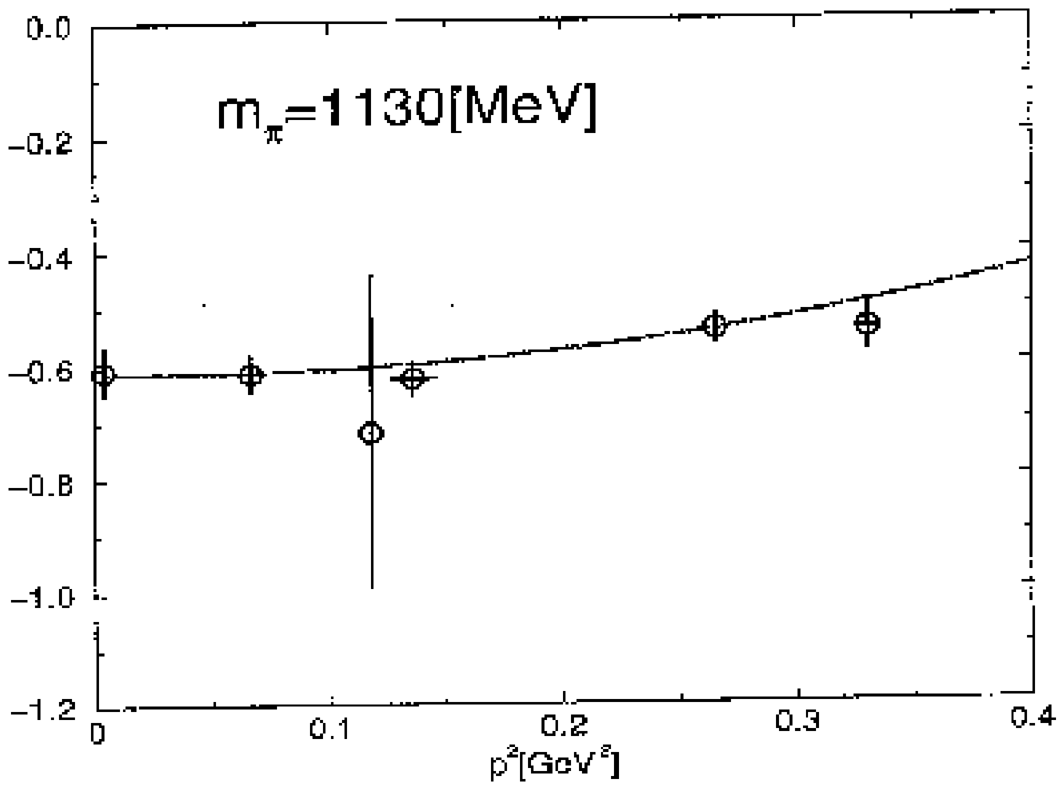
large $O(a)$ effect

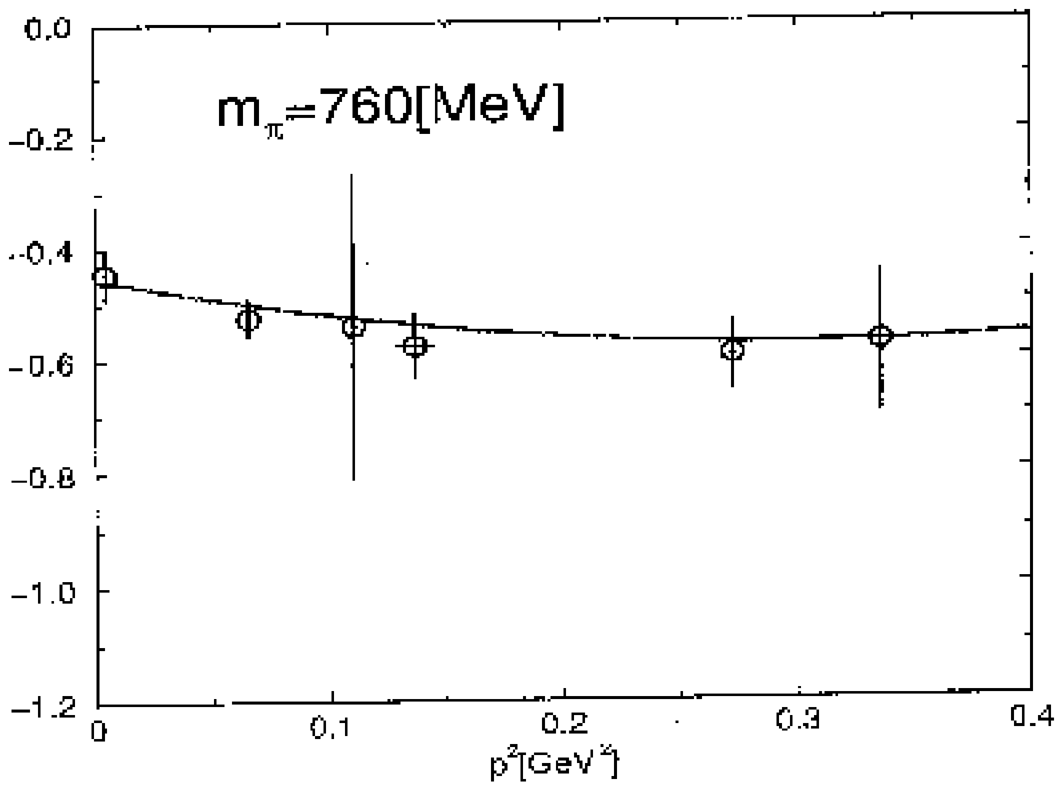
(i) $T(p)$ at $\beta = 1.80$



Reasonable fit are possible with results from different frames.

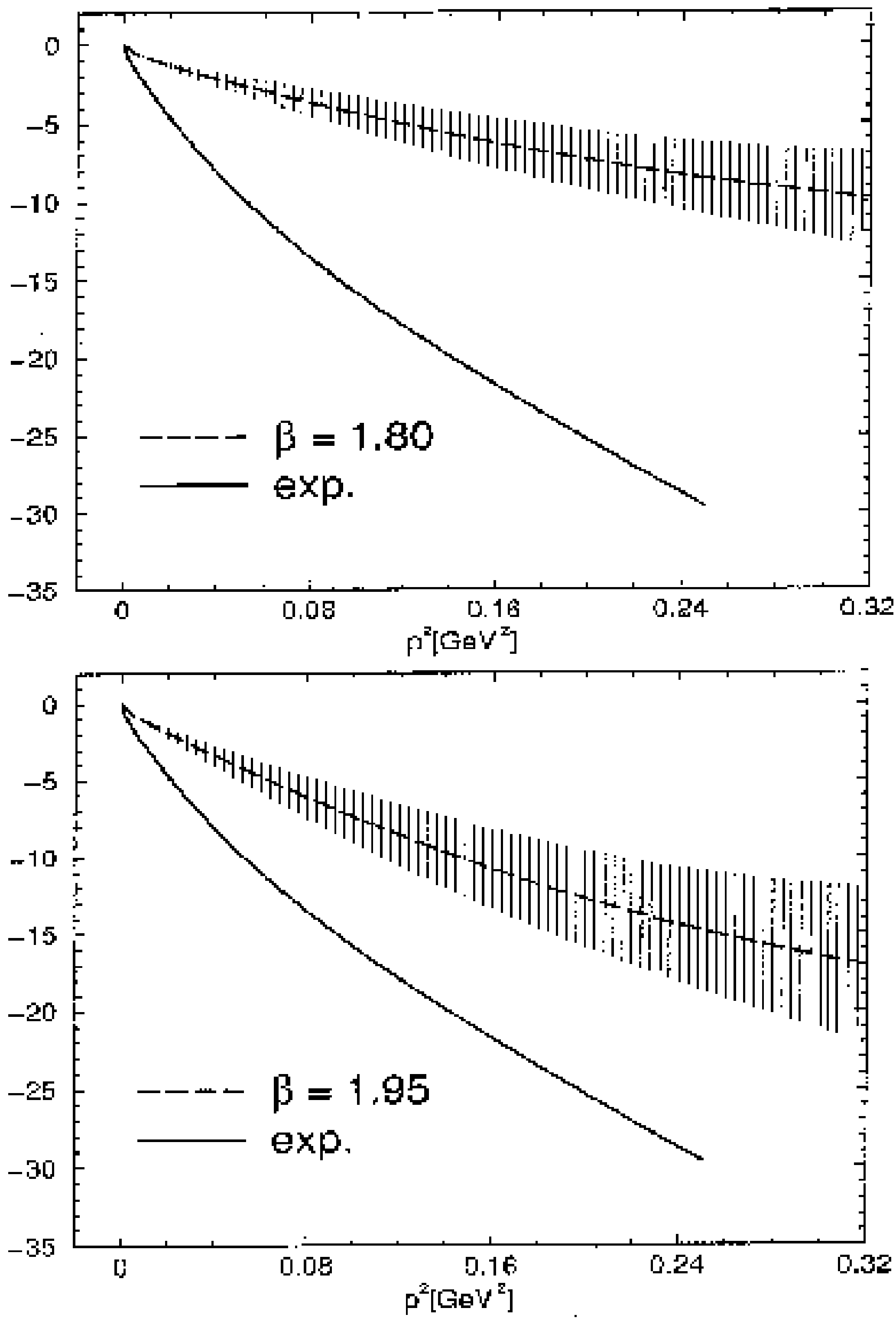
(ii) $T(p)$ at $\beta = 1.95$





Reasonable fit are possible.

phase shift at $m_\pi = 140[\mathrm{MeV}]$
(preliminary result)



$|\delta(p)|$ increases as a decreases.
$O(a)$ effect exists.

5. Conclusion

We calculate $I = 2$ $\pi\pi$ scattering phase shift with $N_f = 2$ fullQCD.

- Energy of $\pi\pi$ state are extracted by diagonalization method.
- We can obtain scattering phase shift in laboratory frame as well as center of mass frame.
- Scattering length $a_0 m_\pi$ (preliminay result)

$\beta = 1.80$	-0.0182(10)
$\beta = 1.95$	-0.0204(11)
CHPT	-0.0444(10)

  $O(a)$ effect exists.
- Scattering phase shift (preliminay result)
  $|\delta(p)|$ increases as a decreases.
  $O(a)$ effect exists.

future work

1. $I = 2$ $\pi\pi$ scattering phase shift in $a \to 0$
2. $I = 1$ $\pi\pi$ resonance scattering phase shift
   $\rho \to \pi\pi$ decay
3. $I = 0$ $\pi\pi$ scattering phase shift
4. $K \to \pi\pi$ decay
5. check of interaction range $R < L/2$
   $V(r) \neq 0$ $(0 < r < R)$