Matrix Product State 法の Schwinger 模型への応用

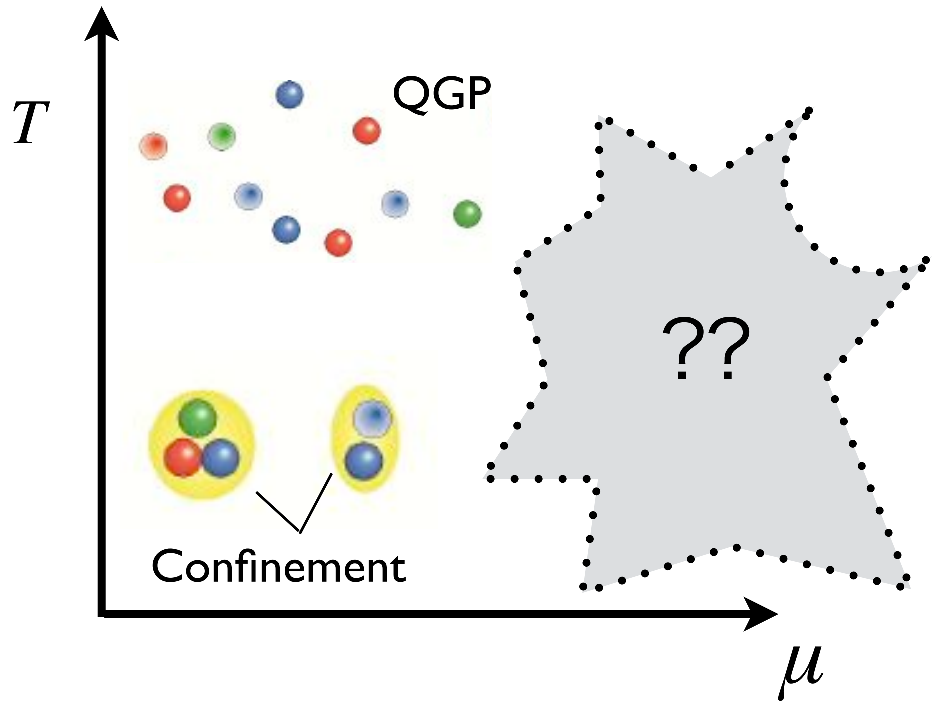
齋藤　華
(CCS, 筑波大)

with M. C. Bañuls, K. Cichy, J. I. Cirac and K. Jansen

H. Saito et al. PoS LATTICE2014, 302, 2014, arXiv:1412.0596
M. C. Bańuls et al, arXiv:1505.00279

QCD Phase diagram

- QCD Phase diagram : non-perturbative aspect
- Lattice QCD simulation
- A lot of interests in dense QCD
  - critical point
  - unknown phases
- Lattice QCD at finite chemical potential μ : sign problem
- **“Is there another method?”**

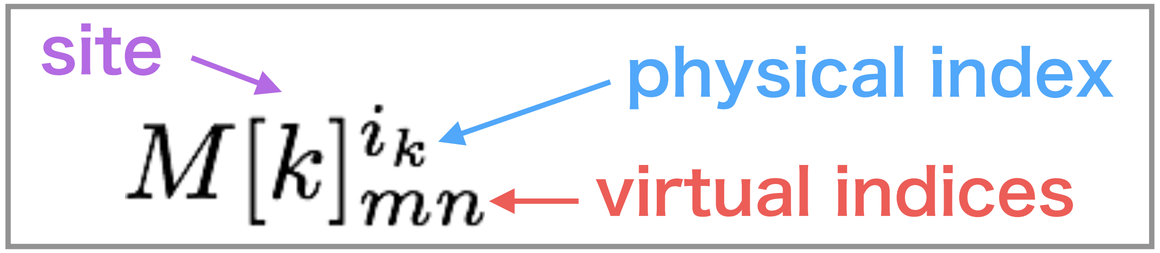


Tensor network (TN)

- An efficient approximation of quantum many-body state from quantum information
- Hamiltonian formalism   cf. Lagrangian formalism

  Y. Shimizu, Y. Kuramashi PRD90, 074503, PRD90, 014508
- **Matrix product state (MPS)** : TN for 1d

$$|\psi\rangle \approx \sum_{i_1,\cdots,i_N} \mathrm{Tr}\left[M[1]^{i_1}\cdots M[N]^{i_N}\right]|i_1\cdots i_N\rangle$$



$$M[k]^{i_k}_{mn}$$

i_k: physical indices at site k,
m, n $(=1,\ldots, D)$: indices from this approximation, **D : bond dimension**

An example of MPS

- 1/2-spin 2 particle system: $i_k = \uparrow$ or $\downarrow$ for $k = 1, 2$
- Supposing $D = 2$,

$$M[1]^{i_1=\uparrow} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}, \quad M[1]^{i_1=\downarrow} = \begin{pmatrix} 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \quad M[2]^{i_2=\uparrow} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad M[2]^{i_2=\downarrow} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

- By computing trace of products $\mathrm{Tr}\left[M[1]^{i_1} M[2]^{i_2}\right]$

$$\boxed{\mathrm{Tr}\left[M[1]^{i_1} M[2]^{i_2}\right] = \begin{cases} \dfrac{1}{\sqrt{2}} & \text{for } (i_1, i_2) = (\uparrow, \downarrow), (\downarrow, \uparrow) \\ 0 & \text{for the others} \end{cases}}$$

$$\sum_{i_1, i_2} \underline{\mathrm{Tr}\left[M^{i_1} M^{i_2}\right]} |i_1 i_2\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

Variational search

- For ground (and some excited) state search
- Ground state derived by searching minimum of trial energy computed by trial MPS state:

$$E = \min E_{\text{trial}} = \min \frac{\langle\psi| H |\psi\rangle}{\langle\psi \mid \psi\rangle}$$

$|\psi\rangle$: a trial MPS state

- the minimum searched with variational approach

$$\frac{dE_{\text{trial}}}{dM[n]^{i_n}_{k_n k_{n+1}}} = 0$$

with fixing the other elements

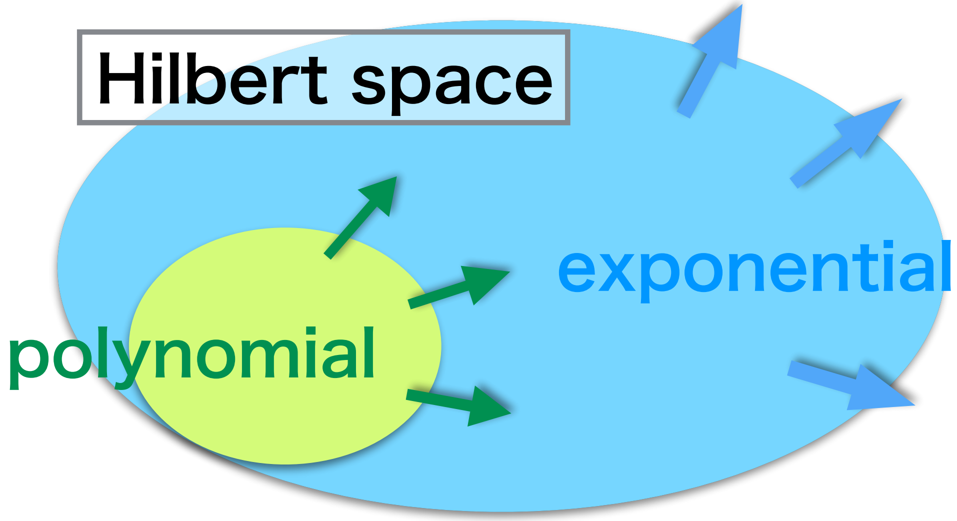
Bond dimension D

- “the relevant tiny corner” of Hilbert space

Ex.) 1d spin system

size of the full Hilbert space ⇔ MPS with finite D :

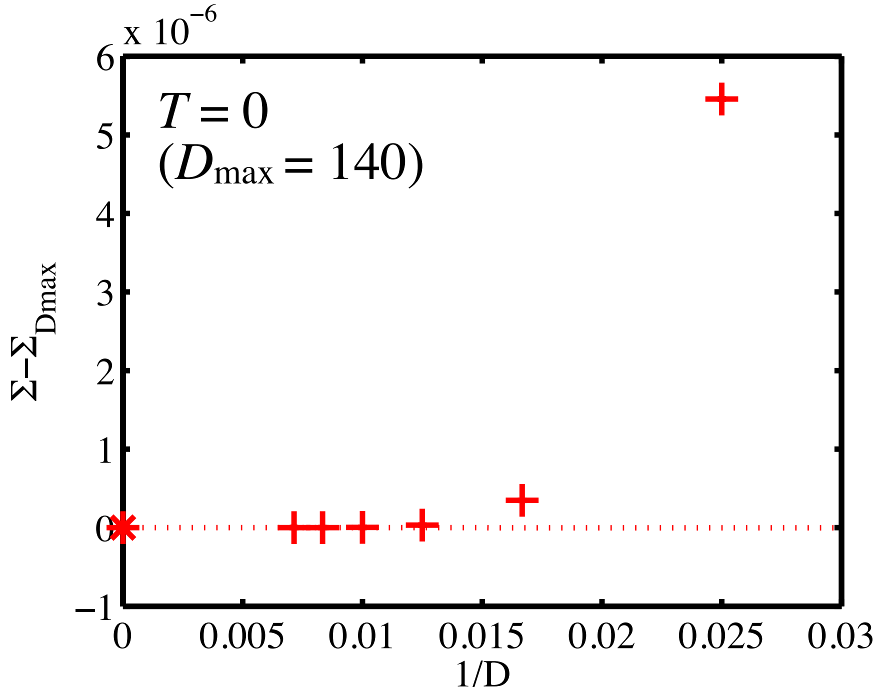
d^N ⇔ NdD^2



The full Hilbert space by $D = (d^{N\text{-}1}/N)^{1/2} \sim d^{N/2}$

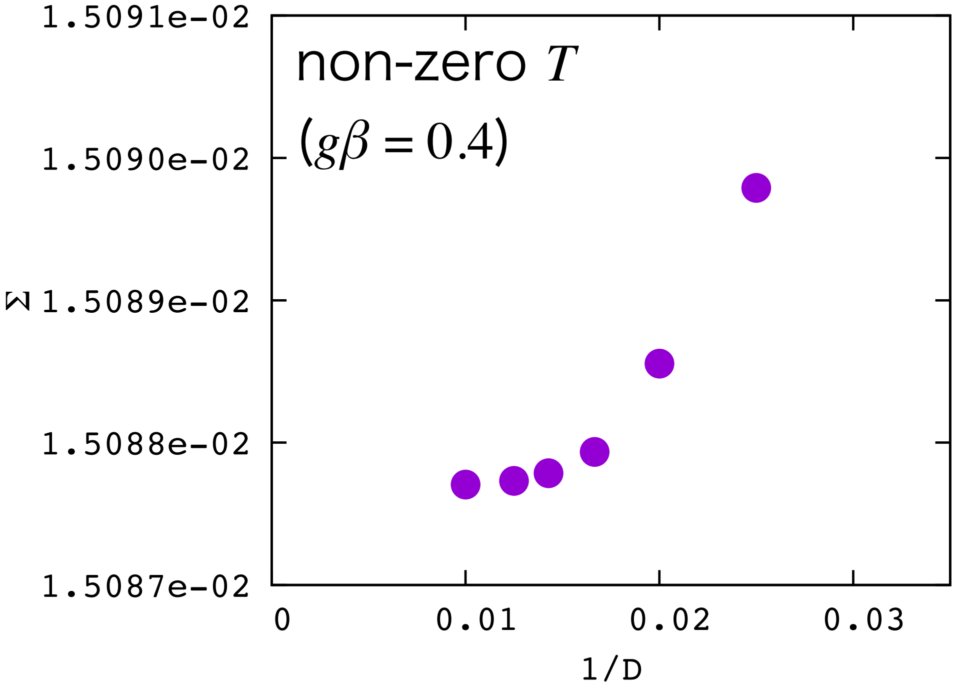
- Approximation improved systematically
- From quantum information, $D << d^{N/2}$

Convergence in D



- **Pure state** M. C. Bañuls et al, LAT2013, 332 (2013)

$N = 300 \Longrightarrow d^{N/2} = 2^{150} \sim O\,(10^{45})$



- **Mixed state**

$(N = 40)$

Schwinger model for $N_{\mathrm{f}} = 1$

J. Schwinger Phys.Rev. 128 (1962)

- 1+1 dimensional QED model

N. L. Pak and P. Senjanovic, Phys.Let.B71, 2 (1977),
K. Johnson Phys.Let. 5, 4(1963)

✳ not QCD, but **similar to QCD** :

confinement, chiral symmetry breaking (via anomaly for $N_{\mathrm{f}}=1$)

✳ exactly solvable in massless case ⇒ a good test case

- Hamiltonian of Schwinger model (staggered discrieization) in spin language

hopping term

mass term

$$H = x \sum_{n=0}^{N-2} \left[\sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+\right] + \frac{\mu}{2} \sum_{n=0}^{N-1} \left[1 + (-1)^n \sigma_n^z\right]$$

$$+ \sum_{n=0}^{N-2} \left[l + \frac{1}{2} \sum_{k=0}^{n} \left((-1)^k + \sigma_k^z \right) \right]^2$$

$$= H_{\mathrm{hop}} + H_{\mathrm{mass}} + H_{\mathrm{g}}$$

T. Banks, L. Susskind and
J. Kogut, PRD13, 4 (1973)

gauge part

Gauss law

where inverse coupling $x=1/a^2g^2$, dimensionless mass $\mu=2m/ag^2$ and $l = L(0)$

Additional tools:
Matrix Product Operator (MPO)

Ex.) MPO of one hopping term with $N = 4$, open boundary

$$H_{\mathrm{hop,half}} = x \sum_{n=0}^{3} \sigma_n^+ \sigma_{n+1}^- \qquad \text{:hopping}$$

$$= \sum_{i_0, i_1, i_2, i_3} \mathrm{Tr}\left[A_0^{i_0} A_1^{i_1} A_2^{i_2} A_3^{i_3}\right] \left(\tilde{\sigma}_0^{i_0} \otimes \tilde{\sigma}_1^{i_1} \otimes \tilde{\sigma}_2^{i_2} \otimes \tilde{\sigma}_3^{i_3}\right)$$

where $A_0^0 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$, $A_0^1 = \begin{pmatrix} 0 & x & 0 \end{pmatrix}$, for left boundary

$$\underbrace{A_n^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_n^1 = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_n^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},}_{\text{for bulk } n = 2,3}$$

$$\underbrace{A_3^0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, A_3^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{\text{for right boundary}}$$

$$\tilde{\sigma}^p = (1_{2\times2}, \sigma^+, \sigma^-)$$

Our previous study

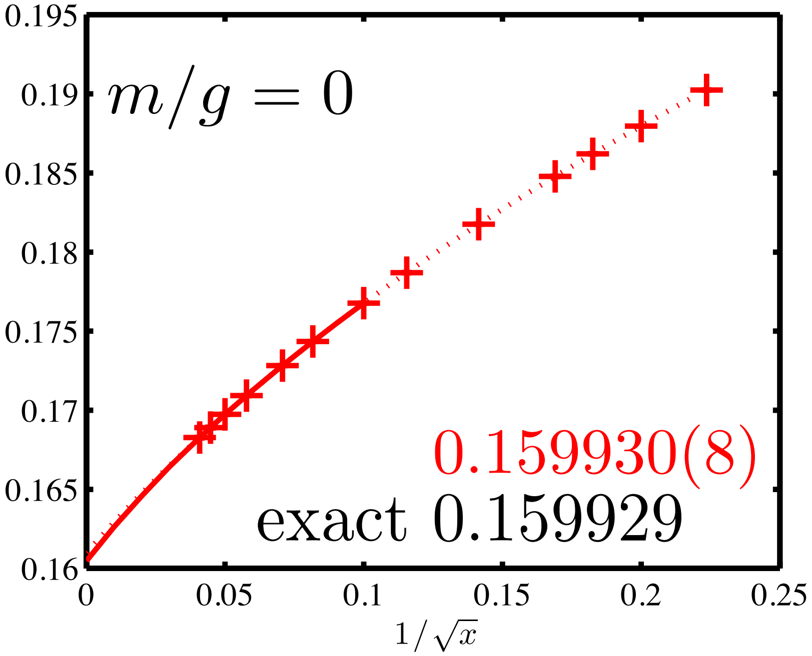
M. C. Bańuls et al JHEP 1311, 158, LAT2013, 332

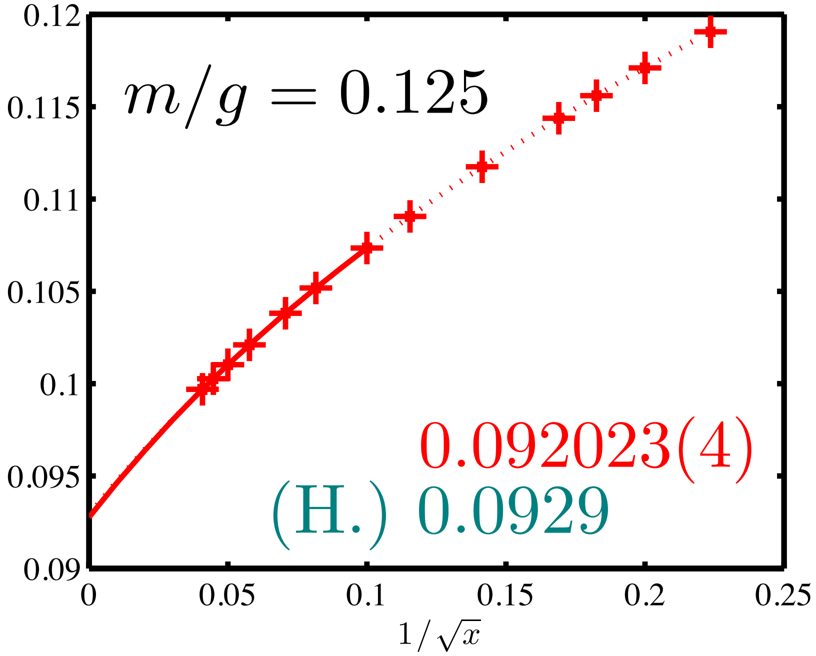
- Schwinger model with MPS method
- With variational method, computing:
  - ✳ spectrum
  - ✳ (subtracted) chiral condensate: $\frac{\bar{\psi}\psi}{g} = \frac{\sqrt{x}}{N}\sum_n (-1)^n \left[\frac{1+\sigma_n^z}{2}\right]$

in spin language

- Continuum limit: $1/\sqrt{x} \to 0$

  with inverse coupling $x = 1/g^2a^2$





(H.) Y. Hosotani arXiv:9703153

Fit function:

$$f(x) = A + F\frac{\log(x)}{\sqrt{x}} + B\frac{1}{\sqrt{x}} + C\frac{1}{x}$$

Logarithmic correction from analytic form of free theory

Lattice gauge theory (LGT) with TN approach

- Earlier Study: critical behavior of Schwinger model with Density Matrix Renormalization Group

T. Byrnes, et al. PRD.66.013002 (2002)

- Nowadays: various branches
  - ✴ Our previous studies

    M. C. Bańuls et al JHEP 1311, 158, LAT2013, 332 (2013)
  - ✴ LGT with TN on higher dimension
  - ✴ Real time evolution

    B. Buyens, et al. arXiv:1312.6654
  - ✴ Quantum link model

    P. Silvi, et al arXiv:1404.7439
    E. Rico, et al. PRL112, 201601 (2014)
  - ✴ Tensor Renormalization Group

    Y. Shimizu, Y. Kuramashi arXiv:1403.0642 (With Lagrangian)

This study

Chiral symmetry restoration of Schwinger model for $N_\mathrm{f} = 1$

- Chiral symmetry breaking at $T = 0$ (via anomaly) ⇔ At high T, the symmetry restoration
- Order parameter : chiral condensate $\langle\bar\psi\psi\rangle$
- Analytic formula

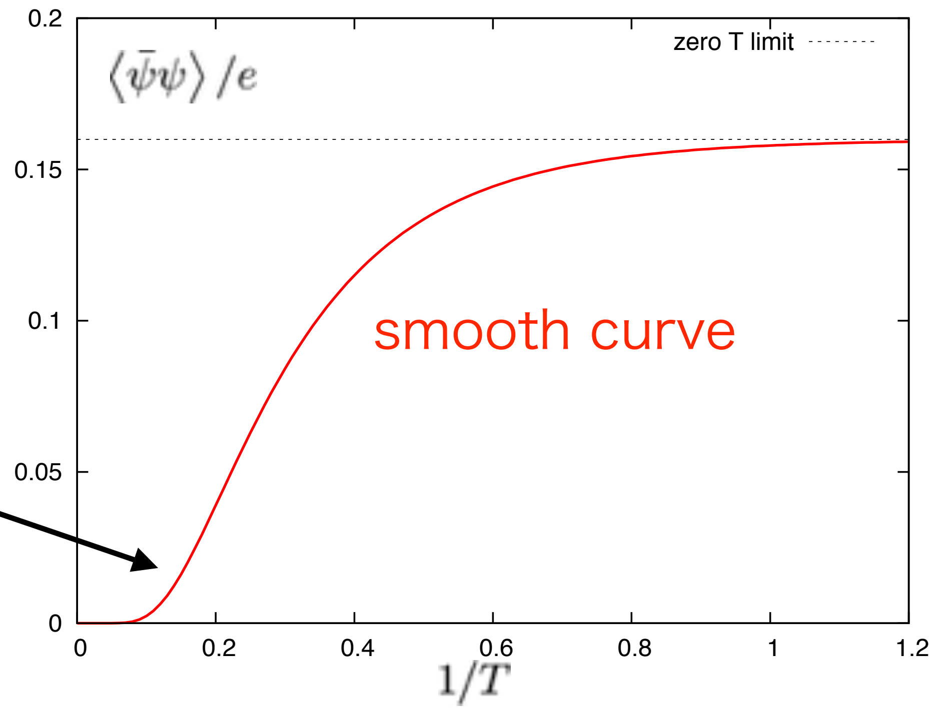
I. Sachs and A. Wipf, arXiv:1005.1822

$$\langle\bar\psi\psi\rangle = \frac{m_\gamma}{2\pi}e^{\gamma}e^{2I(\beta m_\gamma)}$$

$$= \begin{cases} \frac{m_\gamma}{2\pi}e^{\gamma} & \text{for} \quad T \to 0 \\ 2Te^{-\pi T/m_\gamma} & \text{for} \quad T \to \infty \end{cases}$$

where $I(x) = \int_0^\infty \frac{1}{1-e^{x\cosh(t)}}dt$ Euler constant $\gamma = 0.57721\ldots$

$m_\gamma = e/\sqrt{\pi}$



Thermal calculation

- Expectation value at finite T $\qquad \langle \mathcal{O} \rangle_\beta = \frac{\mathrm{tr}\left[\mathcal{O}\ \rho(\beta)\right]}{\mathrm{tr}\left[\rho(\beta)\right]}$
- How to calculate the $\rho(\beta)$

$$\rho(\beta) \approx e^{-\delta H} \cdots e^{-\delta H}$$

where $e^{-\delta H} \approx \left[1 - \frac{\delta}{2} H_g\right] e^{-\frac{\delta}{2} H_e} e^{-\delta H_o} e^{-\frac{\delta}{2} H_e} \left[1 - \frac{\delta}{2} H_g\right]$

$\boxed{\textbf{high } T \rightarrow \textbf{low } T}$

Ex.) For fixed δ, larger N_{step} $(= \beta / \delta)$ corresponds to lower T

- **Approximation to $\rho(\beta)$** (MPO ansatz, global opt.)

In $\beta = 0,\ \rho\ (\beta)$ is identity.

1st step $\qquad \langle i_1 \cdots i_N | \left[1 - \frac{\delta}{2} H_g\right] | j_1 \cdots j_N \rangle \approx \mathrm{Tr}\left[M[1]^{i_1 j_1} \cdots M[N]^{i_N j_N}\right]$

d^N x d^N elements

as variational search, updating tensors $M[k]$ in size of d^2 x D^2

2nd step

$$\langle i_1 \cdots i_N | e^{-\frac{\delta}{2} H_e} \left[1 - \frac{\delta}{2} H_g\right] | j_1 \cdots j_N \rangle \approx \langle i_1 \cdots i_N | e^{-\frac{\delta}{2} H_e} | k_1 \cdots k_N \rangle \, \mathrm{Tr}\left[M[1]^{k_1 j_1} \cdots M[N]^{k_N j_N}\right]$$

$$\approx \mathrm{Tr}\left[M'[1]^{i_1 j_1} \cdots M'[N]^{i_N j_N}\right]$$

Simulation

- MPO approximation for $\rho(\beta)$

$$\rho(\beta) \approx \sum_{\substack{i_1,\cdots,i_N \\ j_1,\cdots,j_N}} \mathrm{Tr}\left[M[1]^{i_1 j_1} \cdots M[N]^{i_N j_N}\right] |i_1 \cdots i_N\rangle \langle j_1 \cdots j_N|$$

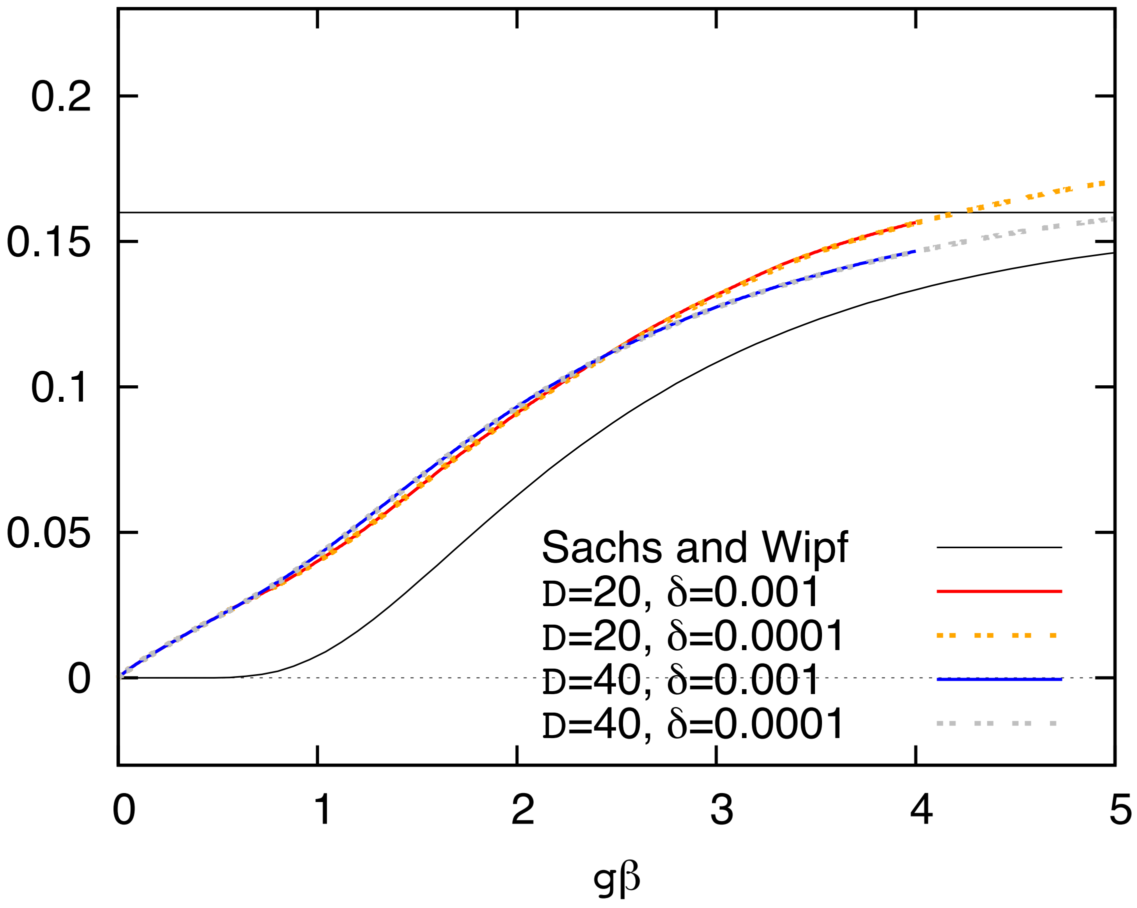
- Open Boundary condition
- Four simulation parameters

1. From MPS approx., bond dimension D $D \to \infty$
2. From T evol., step size δ $\delta \to 0$
3. chain length N $N \to \infty$
4. inverse coupling x $1/\sqrt{x} \to 0$

- To avoid large finite size effect: $N/\sqrt{x} > 15$

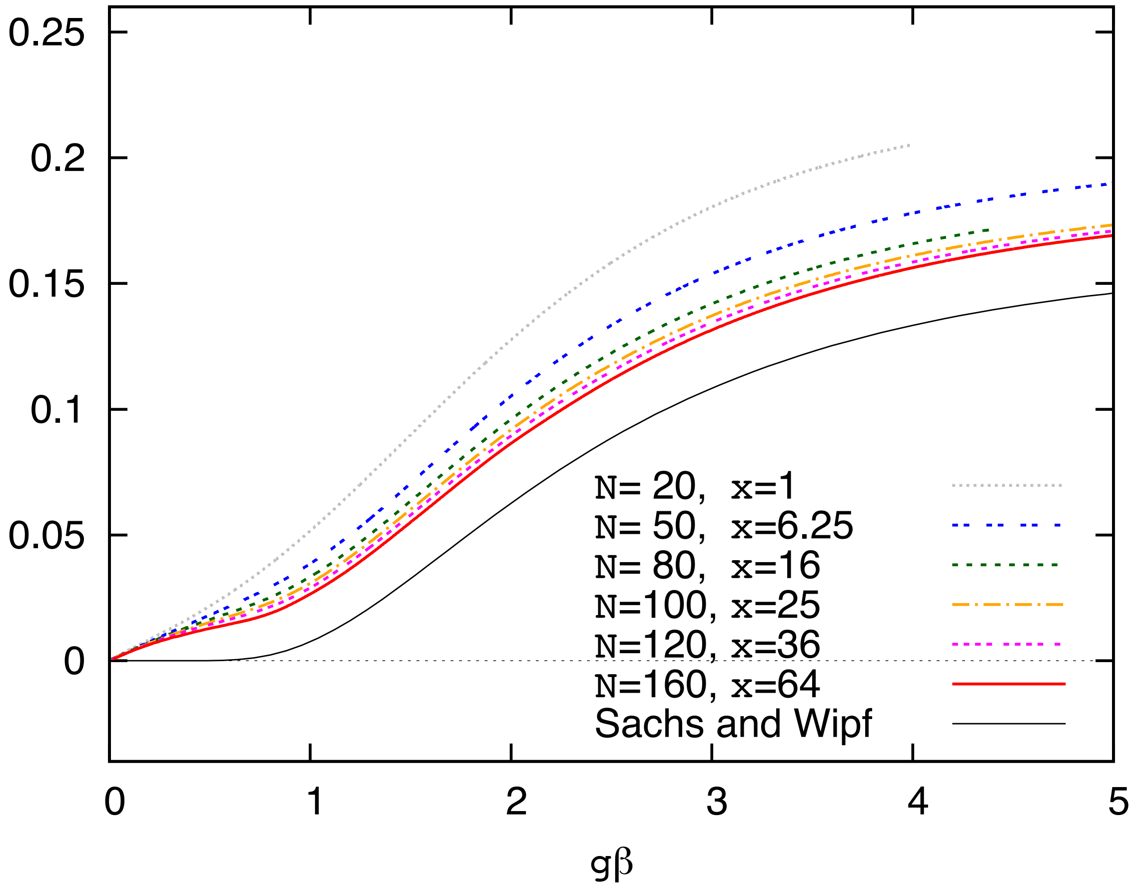
Four systematic errors

From bond dimension D, step size δ



From chain length N, inverse coupling x

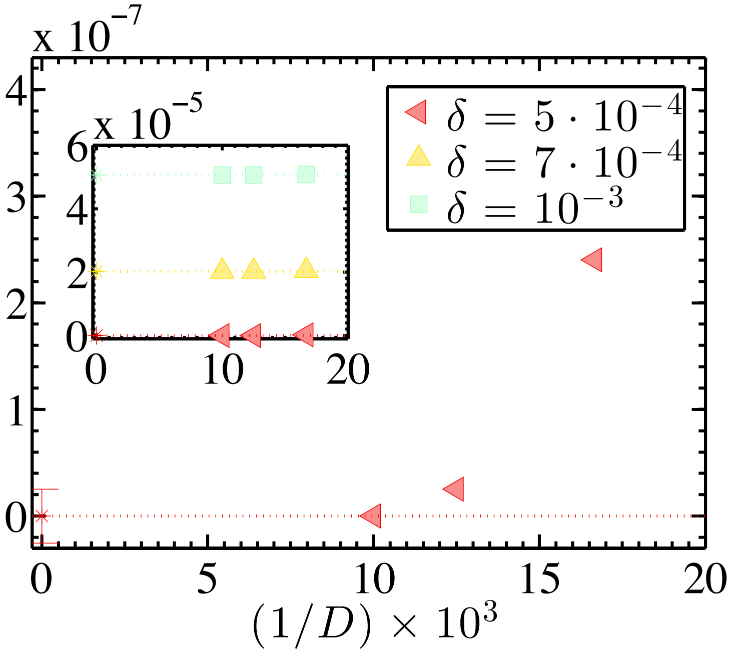
continuum limit with fixed physical length $N/\sqrt{x} = 20$



Extrapolations ① ② ③

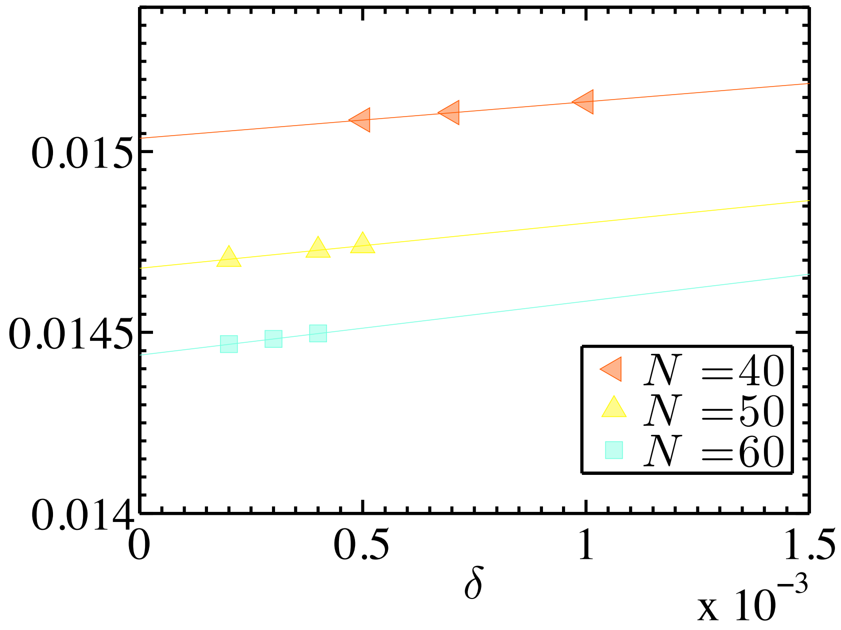
① $D \to \infty$ with

fixed δ, N, x



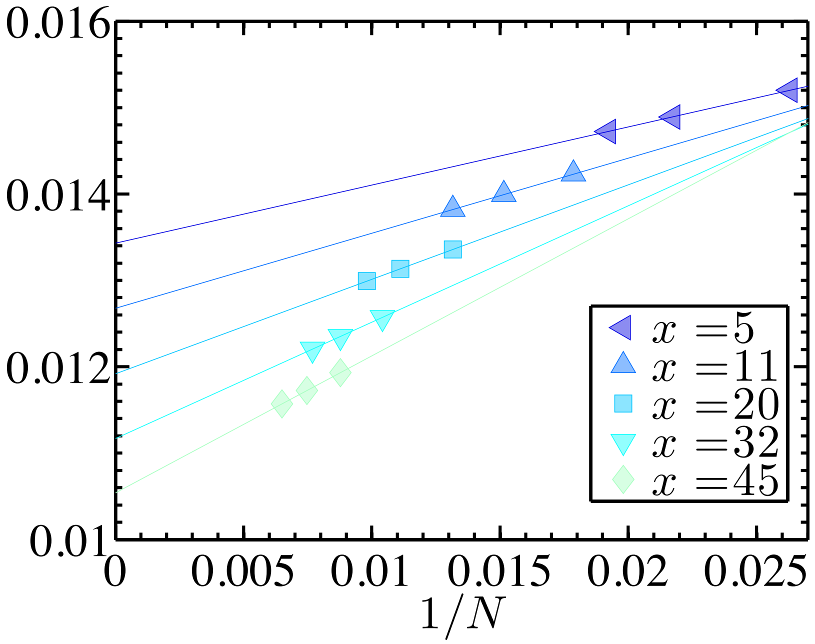
② $\delta \to 0$ with

fixed N, x



③ $N \to \infty$ with

fixed x

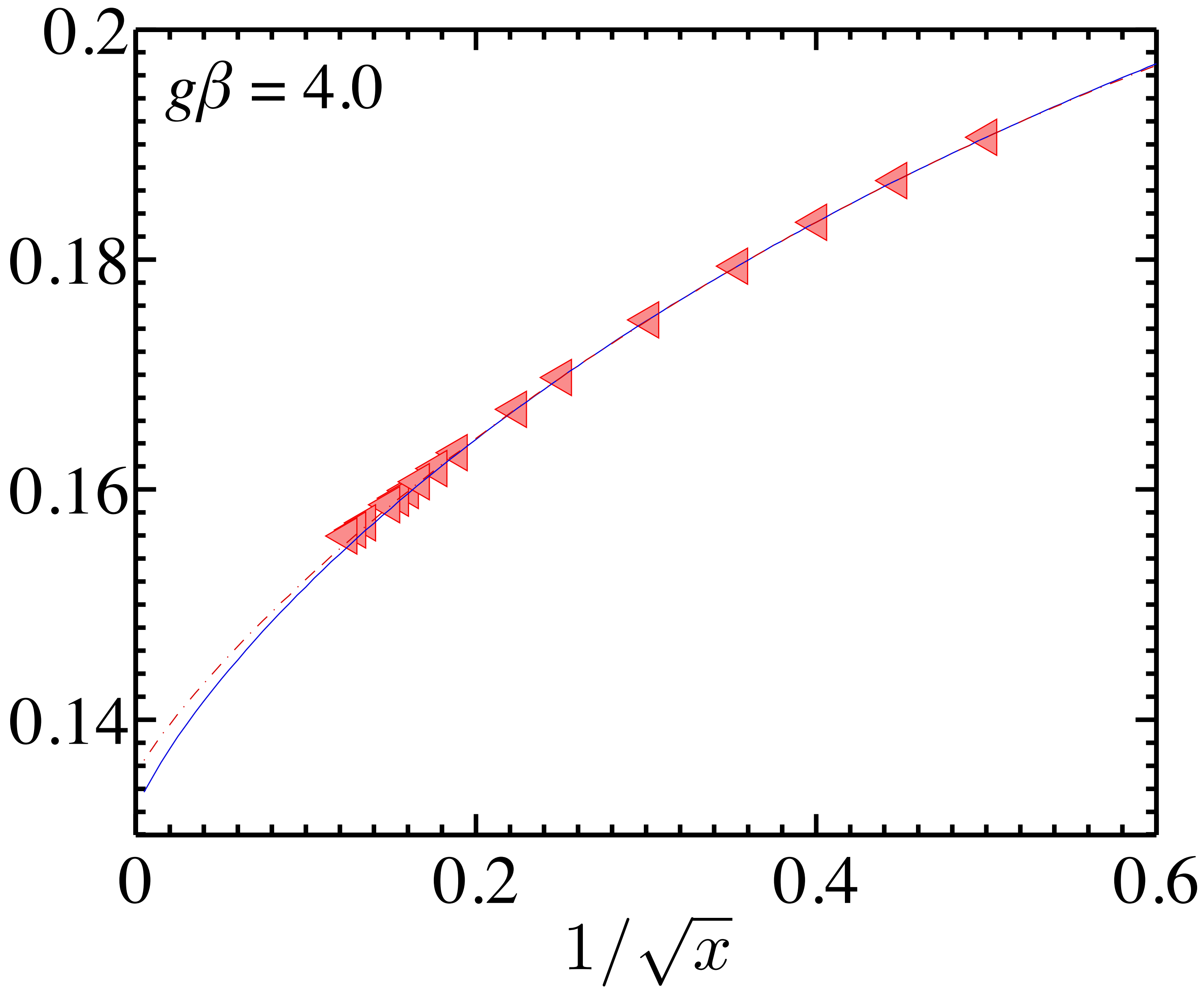


at $g\beta = 0.4$

Extrapolation ④

at each T

④ continuum extrapolation $1/\sqrt{x} \to 0$



(**solid blue**)

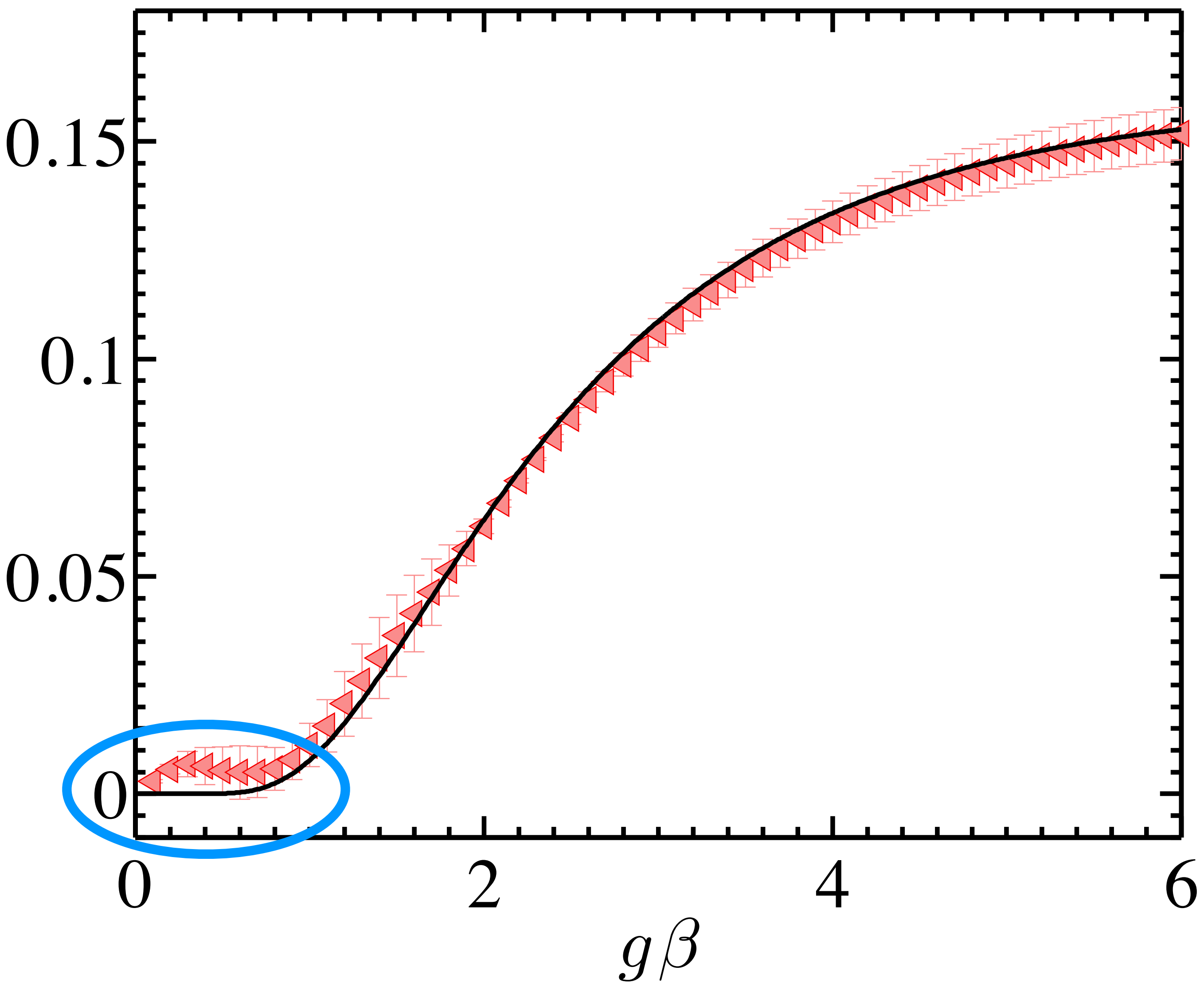
$$\Sigma = \Sigma_{\rm cont} + \frac{a_1}{\sqrt{x}}\log(x) + \frac{b_1}{\sqrt{x}}$$

(**dashed red**)

$$\Sigma = \Sigma_{\rm cont} + \frac{a_2}{\sqrt{x}}\log(x) + \frac{b_2}{\sqrt{x}} + \frac{c_2}{x}$$

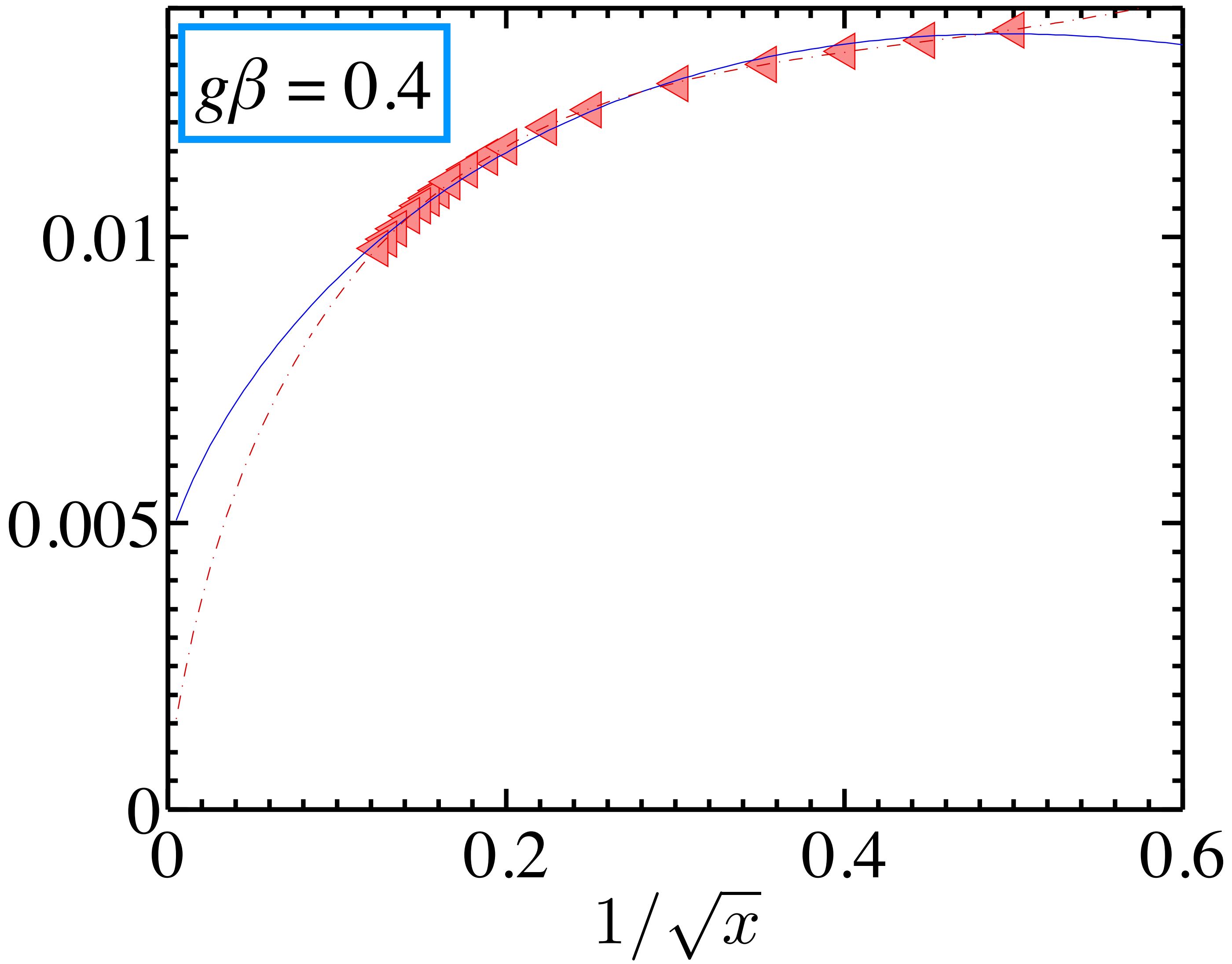
Chiral condensate in continuum limit

After eliminating those systematic errors …



Cut-off effect

- continuum extrapolation in high T



$$\Sigma = \Sigma_{\rm cont} + \frac{a_1}{\sqrt{x}}\log(x) + \frac{b_1}{\sqrt{x}}$$ (solid)

$$\Sigma = \Sigma_{\rm cont} + \frac{a_2}{\sqrt{x}}\log(x) + \frac{b_2}{\sqrt{x}} + \frac{c_2}{x}$$ (dashed)

large x required

→ large N for finite size effect

→ large norm of H

→ extremely small δ for approx.

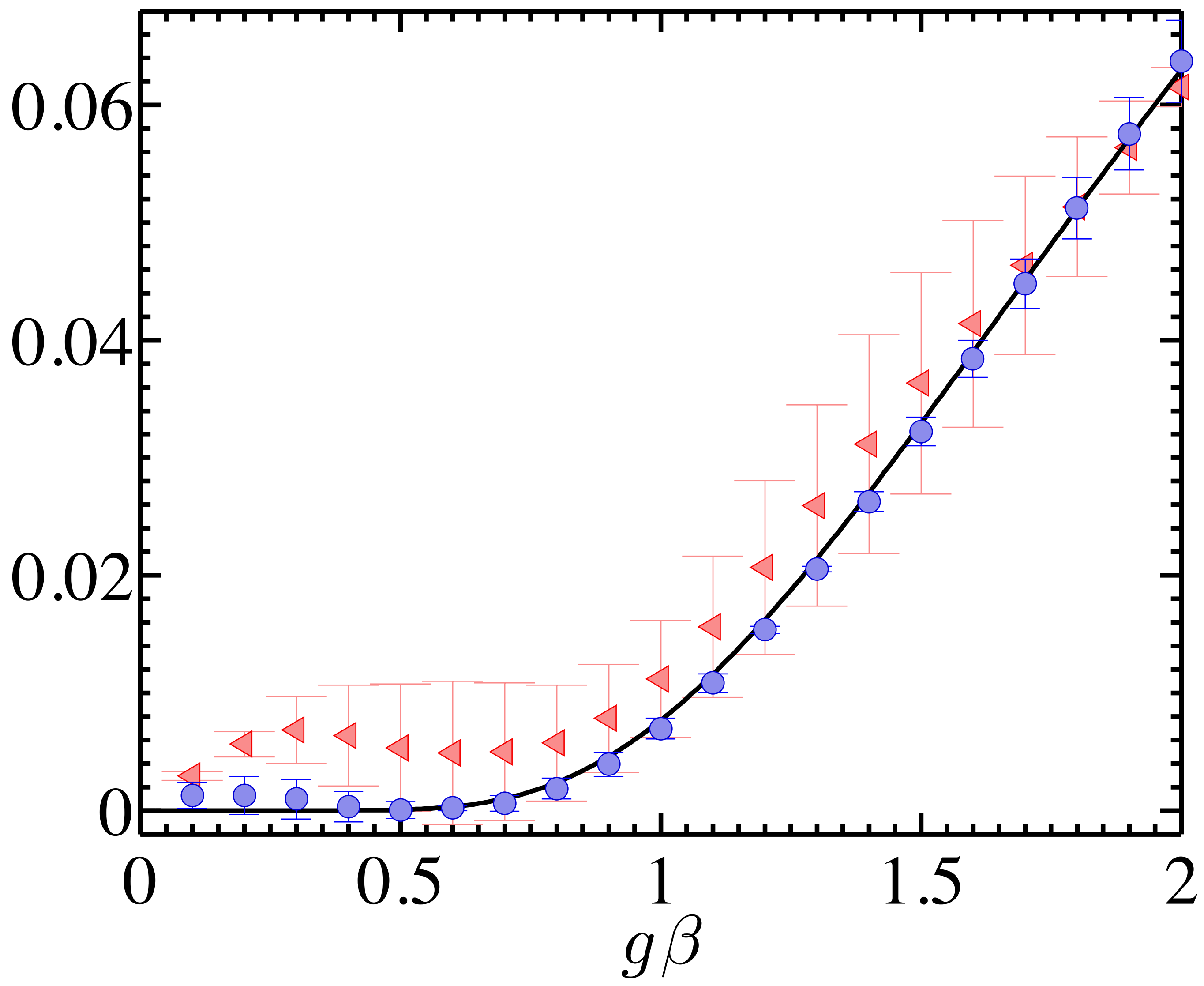
$\exp(-\delta H_z) \sim 1 - \delta H_z$

→ the huge number of steps

in Suzuki-Trotter exp.

Another approach

- chiral condensate at high T



- large electric flux is exponentially suppressed in thermal state:

$$e^{-\sum L_n^2}|\psi\rangle \approx 0 \quad \text{for} \quad l_n \geq 10$$

- additional extrapolation needed

Summary

- Computing chiral condensate at finite T in Hamiltonian formalism with tensor network methods
- Evaluating dependence of parameters: bond dimension, step size, system size, inverse of coupling
- By taking continuum limit, we obtained results consistent with an analytic formula. I. Sachs and A. Wipf, arXiv:1005.1822
- Future plans …