# U(1) axial symmetry at high temperature

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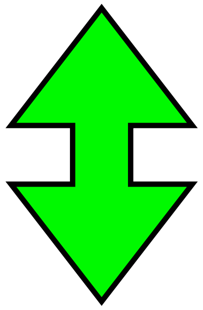
# 1. Introduction

Symmetries of QCD at high temperature

Restoration of non-singlet chiral symmetry $U(1)_B \otimes SU(N_f)_L \otimes SU(N_f)_R$

Theoretical questions

1. U(1)_A symmetry at high T ?



relation ?

2. Eigenvalue distribution of Dirac operator

$\rho(\lambda)$ $\lambda$: eigenvalue of Dirac operator

## Eigenvalue density

$$\rho(\lambda) = \sum_n \rho_n \frac{\lambda^n}{n!}$$

$$\lim_{m\to 0} \langle \bar{\psi}\psi \rangle = \pi\rho(0)$$

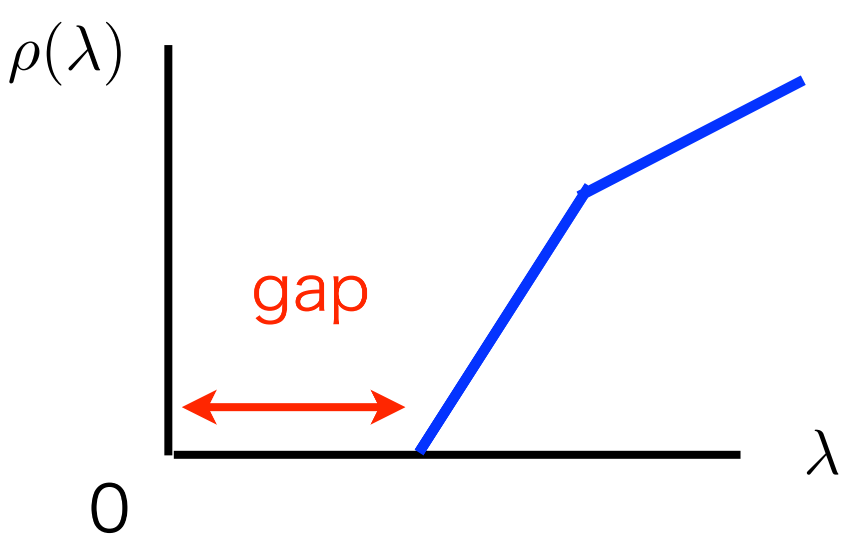
Banks-Casher relation

$\rho(0) = \rho_0 = 0$ if chiral symmetry is restored.

If $\rho(\lambda)$ has a gap

Anomalous $U(1)_A$ symmetry is fully restored.

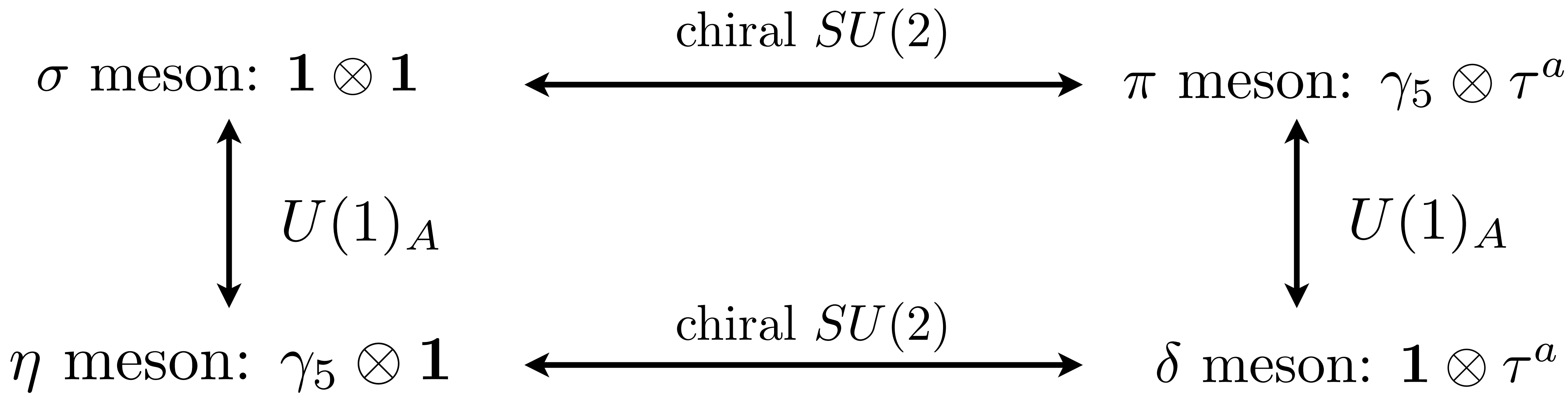
(See later.)



Susceptibility

$$\chi_\Gamma^A = \frac{1}{V}\int d^4x \langle M_\Gamma^A(x) M_\Gamma^A(0)\rangle$$

$N_f = 2$



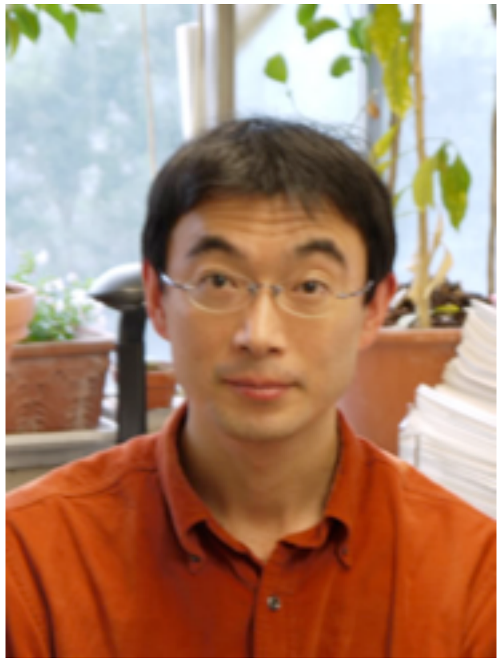

$U(1)_A$ susceptibilities

$$\chi^{\sigma-\eta} \equiv \chi^\sigma - \chi^\eta \qquad \chi^{\pi-\delta} \equiv \chi^\pi - \chi^\delta \qquad \chi^{\pi-\eta} \equiv \chi^\pi - \chi^\eta$$

If $U(1)_A$ is recovered, $\chi^{\sigma-\eta} = \chi^{\pi-\delta} = \chi^{\pi-\eta} = 0.$

# 2. Previous Theoretical Investigation

S.A, H. Fukaya, Y. Taniguchi,
“Chiral symmetry restoration, eigenvalue density of Dirac operator and axial U(1) anomaly at finite temperature”,
Phys. Rev D86(2012)114512.



# Set up

Lattice regularization with Overlap fermion, 2-flavor

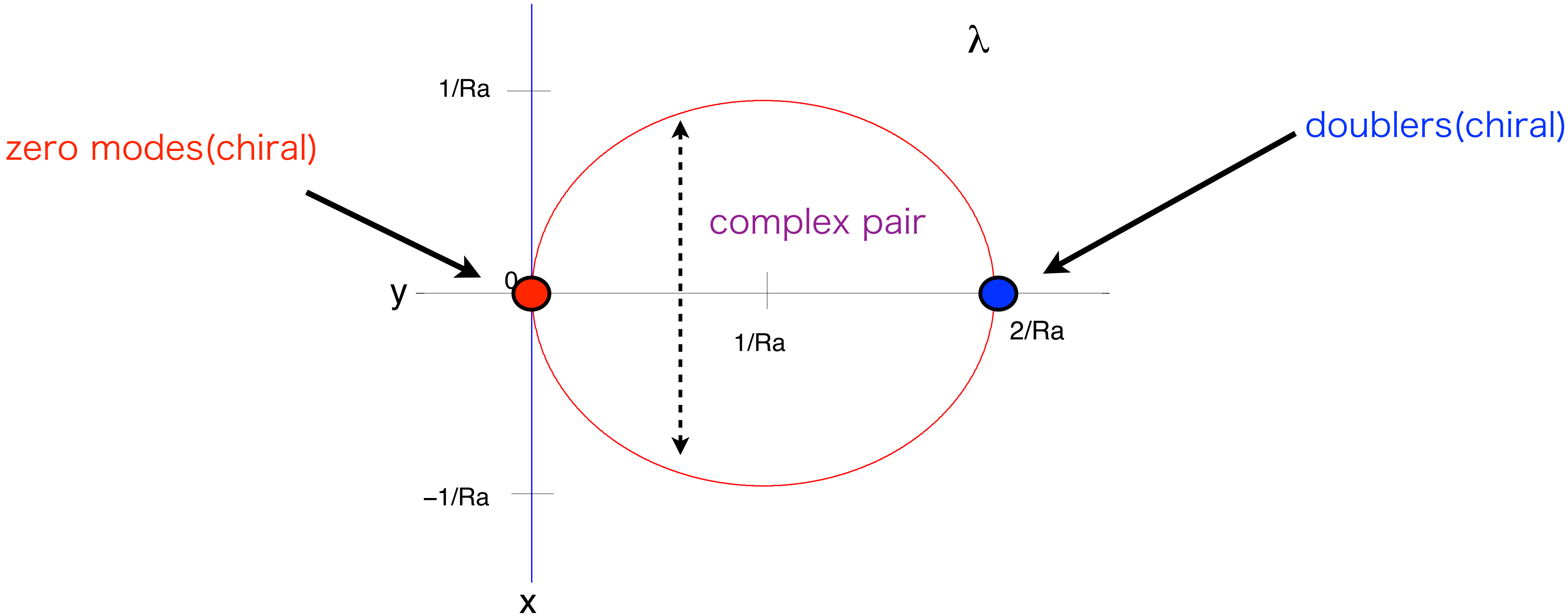
Exact “chiral” symmetry but explicit U(1)_A anomaly form Ginsparg-Wilson relation

$$D\gamma_5 + \gamma_5 D = aDR\gamma_5 D$$

Eigenvalue spectrum

$$\lambda_n^A + \bar{\lambda}_n^A = aR\bar{\lambda}_n^A \lambda_n^A$$

$A$: gauge configuration




$$D(A)\phi_n^A = \lambda_n^A \phi_n^A \quad \Rightarrow \quad D(A)\gamma_5\phi_n^A = \bar{\lambda}_n^A \gamma_5 \phi_n^A$$

## Propagator

$$S(x,y) = \sum_n \left[ \frac{\phi_n(x)\phi_n^\dagger(y)}{f_m\lambda_n - m} + \frac{\gamma_5\phi_n(x)\phi_n^\dagger(y)\gamma_5}{f_m\bar{\lambda}_n - m} \right] - \sum_{k=1}^{N_{R+L}} \frac{1}{m}\phi_k(x)\phi_k^\dagger(y) + \sum_{K=1}^{N_D} \frac{Ra}{2}\phi_K(x)\phi_K^\dagger(y)$$

bulk modes(non-chiral) zero modes(chiral) doublers(chiral)

$$f_m = 1 + \frac{Rma}{2}$$

## Measure

\# of doublers

$$P_m(A) = e^{-S_{YM}(A)}(-m)^{N_f N_{R+L}^A} \left(\frac{2}{Ra}\right)^{N_f N_D^A} \prod_{\Im\lambda_n^A > 0} (Z_m^2 \bar{\lambda}_n^A \lambda_n^A + m^2)$$

\# of zero modes

$$Z_m^2 = 1 - (ma)^2 \frac{R^2}{4}$$

positive definite and even function of $m \neq 0$ for even $N_f$

N_f=2 in this talk.

# Chiral symmetry is restored

$$\lim_{m\to 0}\langle \delta^a \mathcal{O}_{n_1,n_2,n_3,n_4}\rangle_m = 0$$

$$\mathcal{O}_{n_1,n_2,n_3,n_4} = (P^a)^{n_1}(S^a)^{n_2}(P^0)^{n_3}(S^0)^{n_4}$$

$$S^a = \int d^4x\, S^a(x), \quad P^a = \int d^4x\, P^a(x)$$

scalar

pseudo-scalar

chiral rotation at N_f=2

$$\delta^a S^b = 2\delta^{ab} P^0, \delta^a P^b = -2\delta^{ab} S^0$$

$$\delta^a S^0 = 2P^a, \delta^a P^0 = -2S^a$$

# Some assumptions

Assumption 1 non-singlet chiral symmetry is restored.

Assumption 2 if $\mathcal{O}(A)$ is $m$-independent $A$: gauge configuration

$$\langle \mathcal{O}(A) \rangle_m = f(m^2)$$

$f(x)$ is analytic at $x = 0$

(Too strong. We should loosen this condition.)

Note that this does not hold if the chiral symmetry is spontaneously broken.

Ex. $$\lim_{V\to\infty} \frac{1}{V} \langle Q(A)^2 \rangle_m = m \frac{\Sigma}{N_f} + O(m^2)$$

topological charge

Assumption 3 eigenvalues density can be expanded as

$$\rho^A(\lambda) \equiv \lim_{V\to\infty} \frac{1}{V} \sum_n \delta\left(\lambda - \sqrt{\bar{\lambda}_n^A \lambda_n^A}\right) = \sum_{n=0}^{\infty} \rho_n^A \frac{\lambda^n}{n!} \qquad \text{at } \lambda = 0 \ (\lambda < \epsilon\ )$$

More precisely, configurations whose eigenvalue density can not be expanded at the origin are "measure zero" in the configuration space.

(We should remove this assumption and use more general forms in future investigations.)

At finite lattice spacing, integrals over all eigenvalues are convergent, since

$$|\lambda| \leq \frac{2}{Ra}$$

# Analysis (some examples)

$$\lim_{m\to 0}\lim_{V\to\infty}\frac{\langle S_0\rangle_m}{V}=0 \quad \Leftarrow \quad \lim_{m\to 0}\langle\bar{\psi}\psi\rangle_m=0$$

$$\frac{S_0}{V}=-\frac{1}{V}\sum_n\left[\frac{F(\lambda_n)}{f_m\lambda_n-m}+\frac{F(\bar{\lambda}_n)}{f_m\bar{\lambda}_n-m}\right]+\frac{N^A_{R+L}}{Vm}$$

$V\to\infty$

$$I_1=\frac{1}{Z_m}\int_0^{\Lambda_R}d\lambda\,\rho^A(\lambda)g_0(\lambda^2)\frac{2m_R}{\lambda^2+m_R^2}=\pi\rho_0^A+O(m)$$

source of m singularity

$$\langle\rho_0^A\rangle_m=O(m^2)$$

$\Rightarrow$

$$\lim_{V\to\infty}\left\langle\frac{N_{R+L}}{V}\right\rangle_m=O(m^2)$$

$$\lim_{m\to 0} \chi^{\eta-\delta} = 0$$

$$\chi^{\eta-\delta} = \frac{1}{V}\langle P_0^2 - S_a^2 \rangle$$

$$\chi^{\eta-\delta} = N_f \left\langle \frac{1}{m^2 V}\{2N_{R+L} - N_f Q(A)^2\} + \frac{1}{Z_m}\left(\frac{I_1}{m_R} + I_2\right)\right\rangle_m$$

=0

from others

topological charge

$$Q(A) = N_R^A - N_L^A$$

$$I_2 = \frac{2}{Z_m}\int_0^{\Lambda_R} d\lambda\, \rho^A(\lambda)\, \frac{m_R^2 - \lambda^2 g_0(\lambda^2) g_m}{(\lambda^2 + m_R^2)^2}, \quad g_m = \frac{1}{Z_m^2}\left(1 + \frac{m^2}{2\Lambda_R^2}\right)$$

$$\frac{I_1}{m_R} + I_2 = \rho_0^A\left(\frac{\pi_m}{m} + \frac{2}{\Lambda_R}\right) + 2\rho_1^A + O(m),$$

$$\langle \rho_0^A \rangle_m = O(m^2)$$

$$\Rightarrow \quad \lim_{m\to 0} \frac{N_f^2 \langle Q(A)^2 \rangle_m}{m^2 V} = 2 \lim_{m\to 0} \langle \rho_1^A \rangle_m$$

repeat these analysis for higher susceptibilities.

# Final Results

$$\lim_{m\to 0}\langle \rho^A(\lambda)\rangle_m = \lim_{m\to 0}\langle \rho_3^A\rangle_m \frac{|\lambda|^3}{3!} + O(\lambda^4)$$

No constraints to higher $\langle \rho_n^A\rangle_m$

$\langle \rho_3^A\rangle_m \neq 0$ even for "free" theory.

$$\langle \rho_0^A\rangle_m = 0$$

$$\lim_{V\to\infty}\frac{1}{V^k}\langle (N_{R+L}^A)^k\rangle_m = 0, \quad \lim_{V\to\infty}\frac{1}{V^k}\langle Q(A)^{2k}\rangle_m = 0$$

# Consequences

Singlet susceptibility at high T

$$\lim_{V\to 0} \chi^{\pi-\eta} = \lim_{m\to 0} \lim_{V\to\infty} \frac{N_f^2}{m^2 V} \langle Q(A)^2 \rangle_m = 0$$

This, however, does not mean U(1)_A symmetry is recovered at high T.

$$\lim_{m\to 0} \chi^{\pi-\eta} = 0 \quad \Rightarrow \quad "m_\pi = m_\eta"$$

is necessary but NOT “sufficient” for the recovery of U(1)_A .

## More general Singlet WT identities

$$\langle J^0 \mathcal{O} + \delta^0 \mathcal{O} \rangle_m = O(m)$$

anomaly(measure)    singlet rotation

We can show for $\mathcal{O} = \mathcal{O}_{n_1,n_2,n_3,n_4} = (P^a)^{n_1}(S^a)^{n_2}(P^0)^{n_3}(S^0)^{n_4}$

$$\lim_{V\to\infty} \frac{1}{V^k} \langle J^0 \mathcal{O} \rangle_m = \lim_{V\to\infty} \left\langle \frac{Q(A)^2}{mV} \times O(V^0) \right\rangle_m = 0$$

where $k$ is the smallest integer which makes the $V \to \infty$ limit finite.

$S^0 \sim O(V), \ P^a, S^a, P^0 \sim O(V^{1/2})$

$$\lim_{m\to 0} \lim_{V\to\infty} \frac{1}{V^k} \langle \delta^0 \mathcal{O} \rangle_m = 0$$

Breaking of U(1)_A symmetry is invisible for these “bulk quantities”.

$$\mathrm{SU}(2)_L \otimes \mathrm{SU}(2)_R \otimes \mathrm{Z}_4$$

# Remarks

Important conditions

Large volume limit $V \to \infty$

chiral limit $m \to 0$

lattice chiral symmetry Ginsparg-Wilson relation $D\gamma_5 + \gamma_5 D = aDR\gamma_5 D$

Fractional power for the eigenvalue density

$$\rho^A(\lambda) \simeq c_A \lambda^\gamma, \ \gamma > 0$$

non-singlet chiral symmetry is recovered.

⇒ $\gamma \leq 2$ is excluded. ⇒ $\gamma > 2$

consistent with the integer case (n > 2)

Universal treatment ? (future investigations)

# 3. Recent Numerical Results

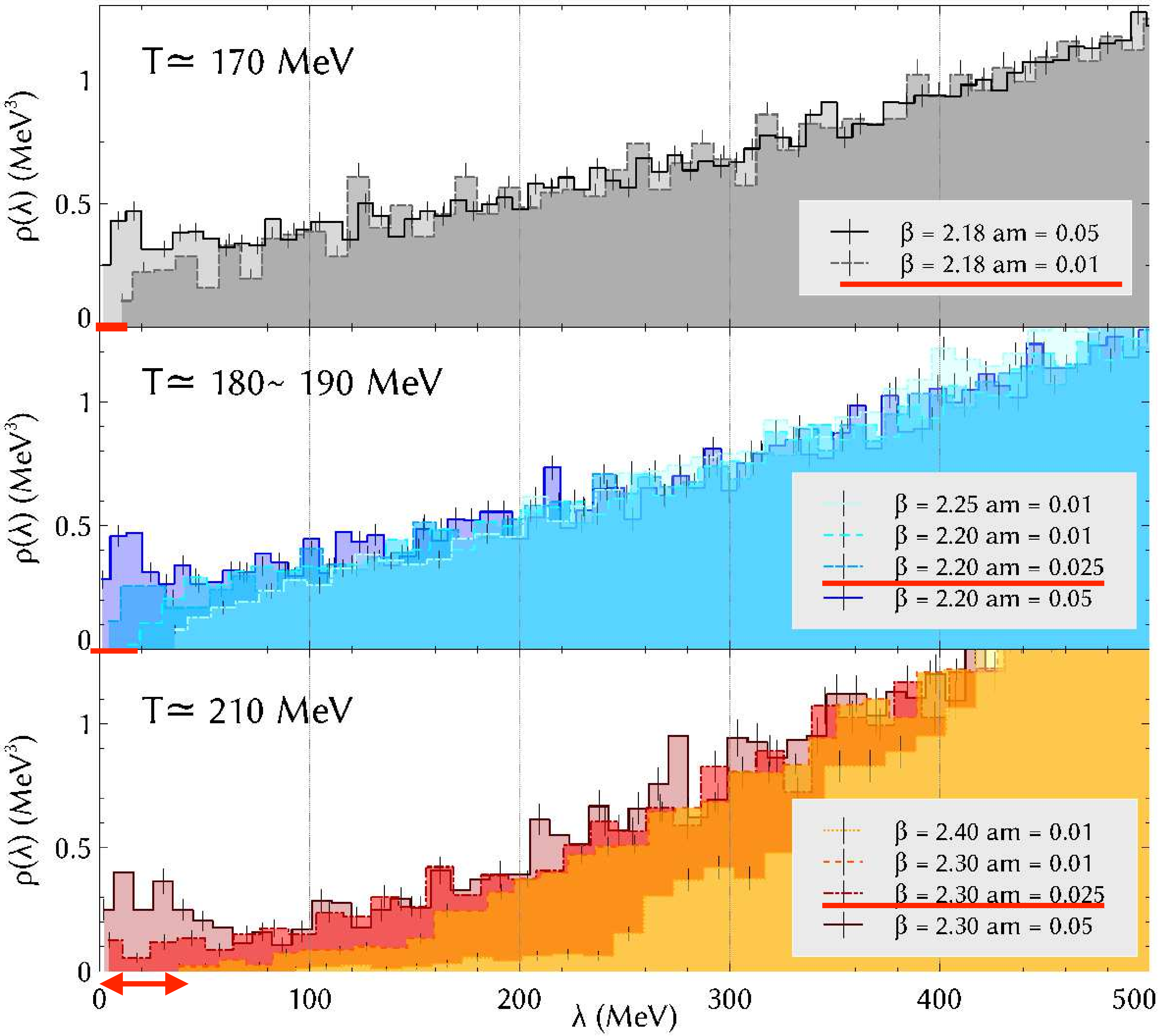
A. Tomiya et al. (JLQCD), Lat2015

G. Cossu et al. (JLQCD), Lat2015

# Eigenvalue densities

$$\rho(\lambda) = \lim_{V\to\infty} \frac{1}{V} \sum_n \delta(\lambda - \lambda_n)$$

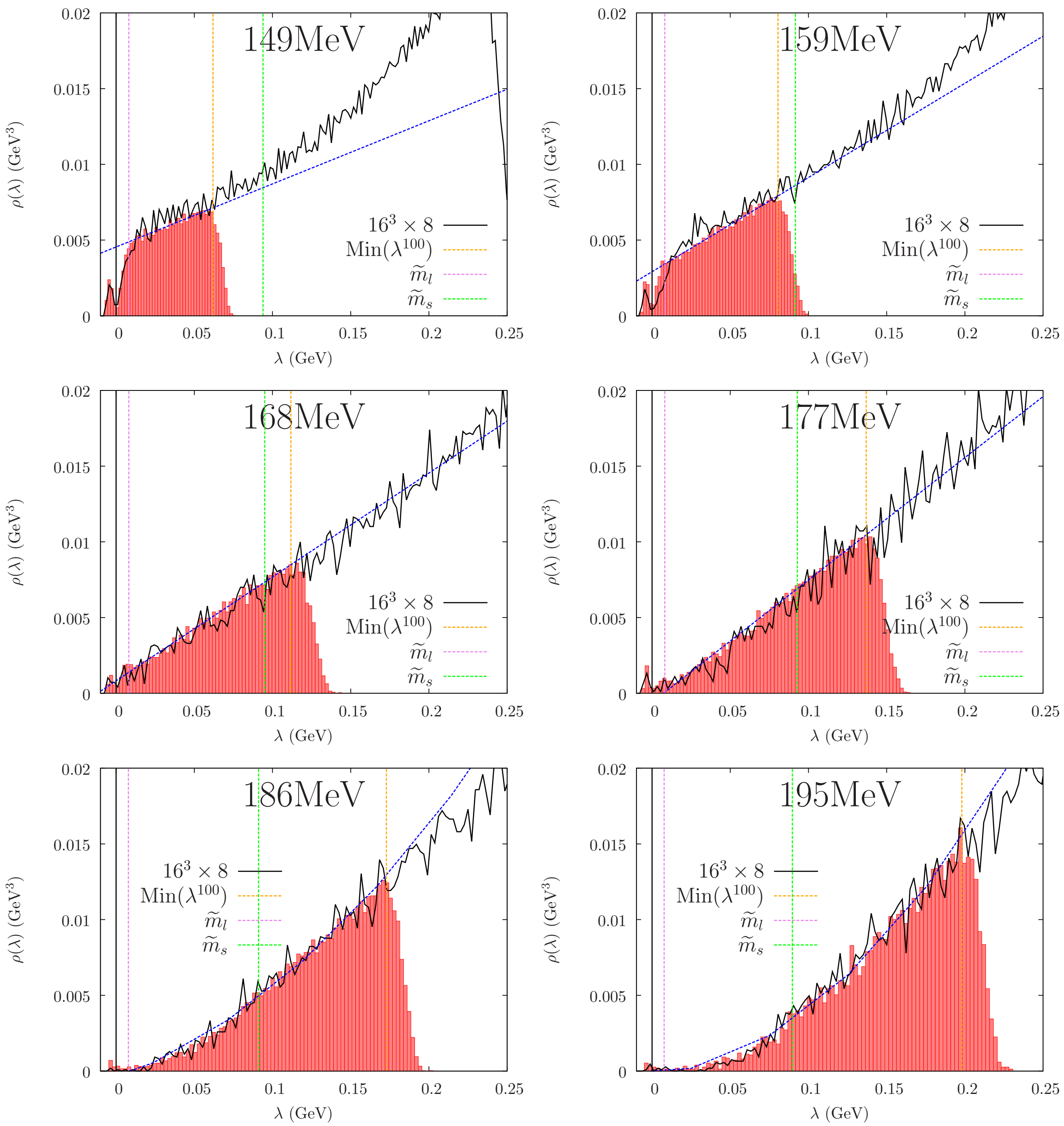
Cossu *et al.* (JLQCD), Overlap
Phys. Rev. D87 (2013) 114514




Gap seems to open at smaller quark mass.

$T_c \simeq 180$ MeV

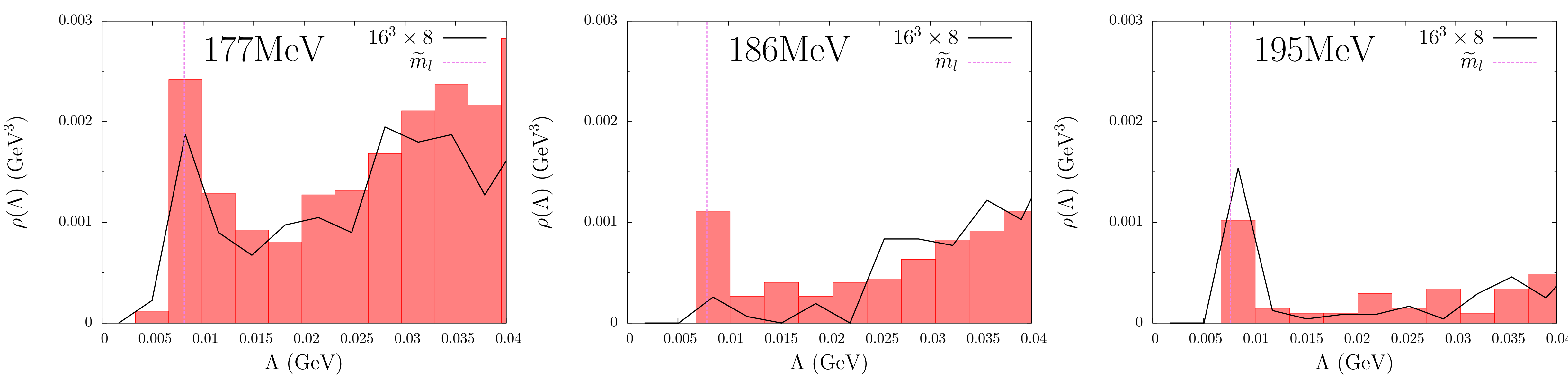
Buchoff *et al.* (LLNL/RBC), DomainWall, Phys. Rev. D89 (2014) 5,054514




Small eigenvalues appear.

Gap seems to close at or above critical temperature

$$T_c \simeq 180 \text{ MeV}$$



# Summary of recent results from chiral fermions.

| Group | Fermion | Size | Gap in the spectrum | $U_A(1)$ Correlator | $U(1)_A$ $T \gtrsim Tc$ |
|---|---|---|---|---|---|
| **JLQCD (2013)** | **Overlap (Top. fixed)** | 2 fm | **Gap** | Degenerate | **Restored** |
| TWQCD (2013) | Optimal domain-wall | 3 fm | **No gap** | Degenerate | **Restored** |
| LLNL/RBC, Hot QCD (2013,2014) | **(Möbius)-Domain-wall (W/ ov)** | 2, 4, 11 fm | **No gap** | No degeneracy | **Violated** |
| Viktor Dick et al (2015) | **OV on HISQ sea** | 3, 4 fm | **No gap** | No degeneracy | **Violated** |

What causes this difference ?

volume ? quark mass ? lattice chiral symmetry ?

JLQCD collaboration

Overlap: exact GW relation

LLNL/RBC collaborations

DomainWall: approximated GW relation

**Recent study by A. Tomiya et al. for JLQCD collaboration**

Preliminary

generate gauge configurations with an improved DomainWall quarks

very small violation of GW relation

(0)calculate eigenvalue distribution of DW operator on these configurations

**original**

(1)calculate eigenvalue distribution of overlap operator on these configurations

**partially quenched**

(2)reweighting factor from the improved DW to Overlap is introduced to obtain the full eigenvalue distribution

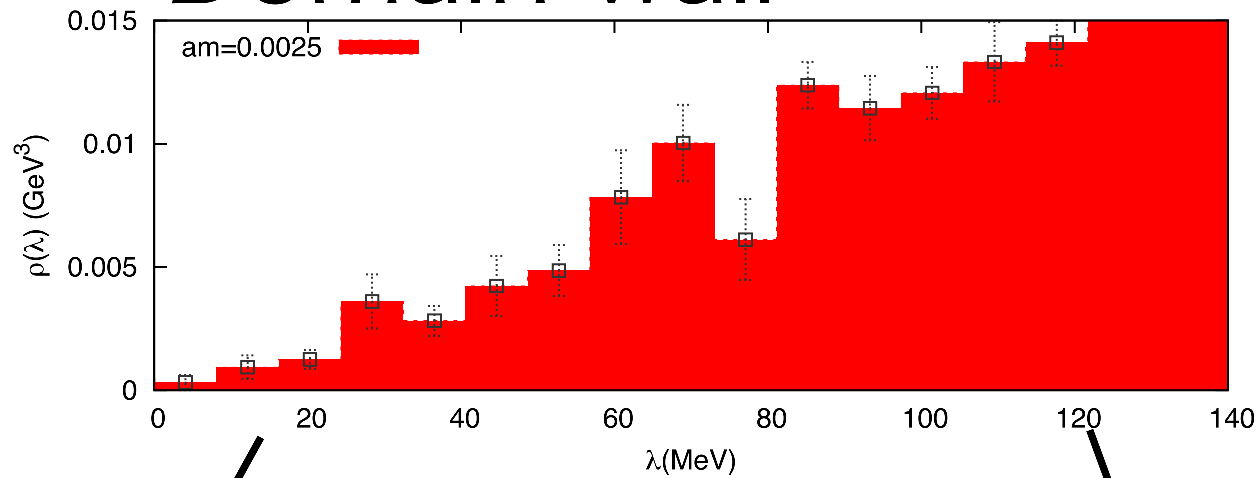
**full Overlap**

# T=190 MeV for L=3 fm, T=1.05 $T_c$

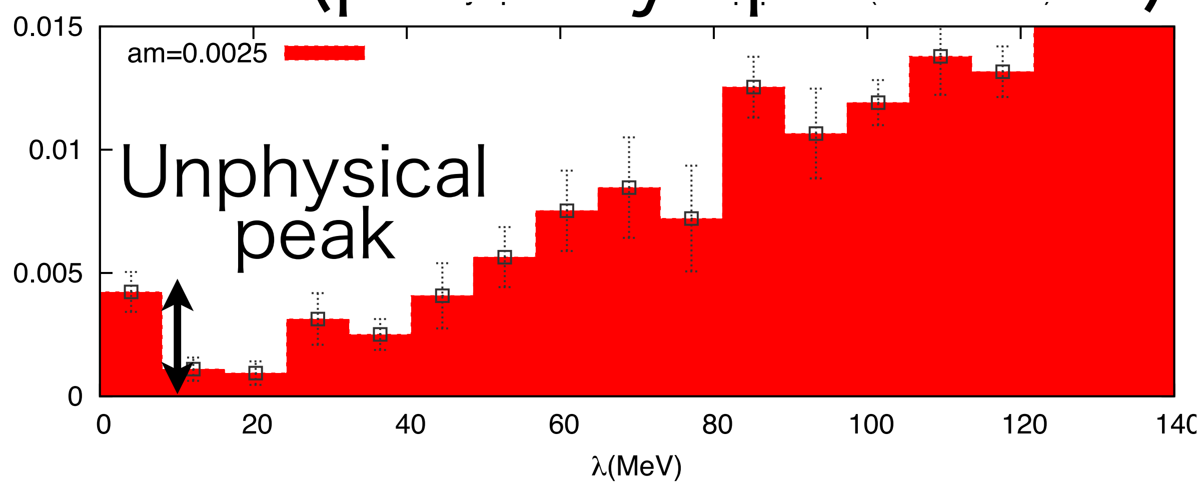
A. Tomiya et al. (JLQCD), Lat2015

Preliminary

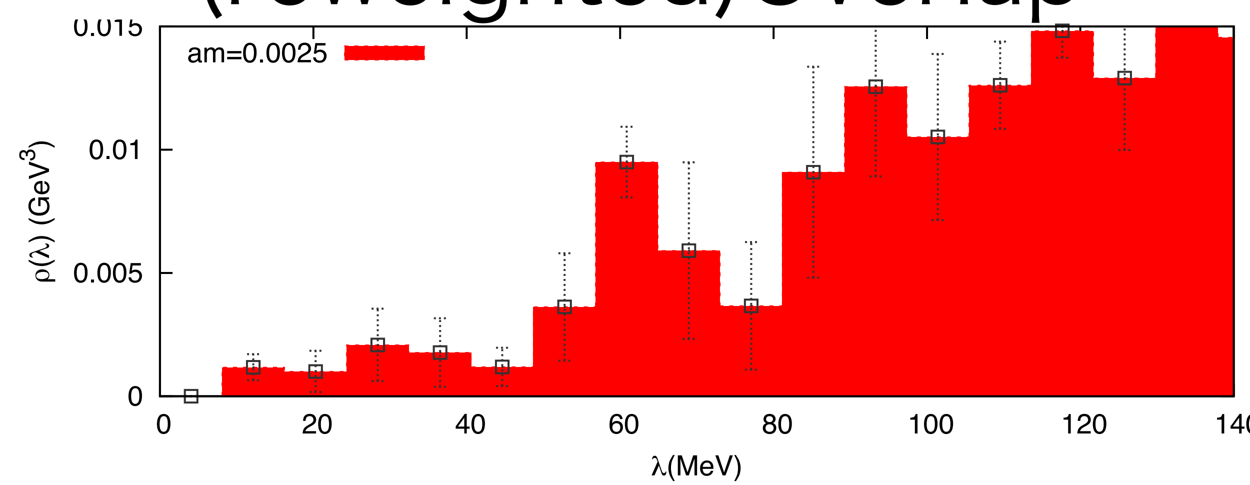
Domain-wall




Overlap on domain-wall sea (partially quenched)




(reweighted)Overlap




After the reweighting, small eigenvalues in PQ disappear, and the gap seems to open in full Overlap.

An exact lattice chiral symmetry is essential. A tiny violation of the chiral symmetry may destroy the theoretically expected relation.

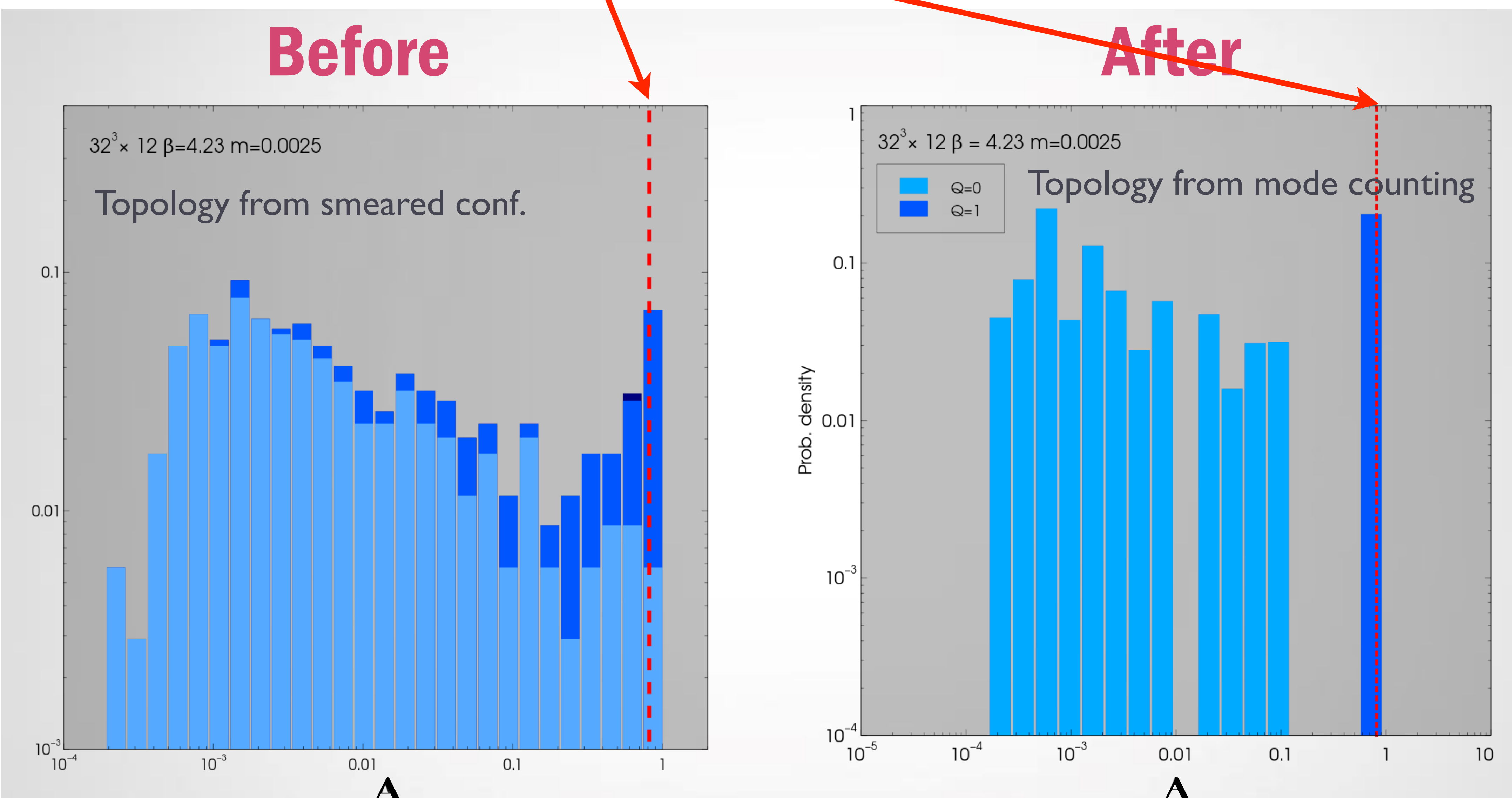
# U(1)_A susceptibility

$$\Delta := \chi^{\pi} - \chi^{\delta}$$

G. Cossu et al. (JLQCD), Lat2015

$$\Delta = \boxed{\frac{2N_{R+L}}{Vm^2}} + \sum_{\lambda\neq 0} \frac{2m^2}{V(\lambda^2+m^2)^2}$$

zero-modes



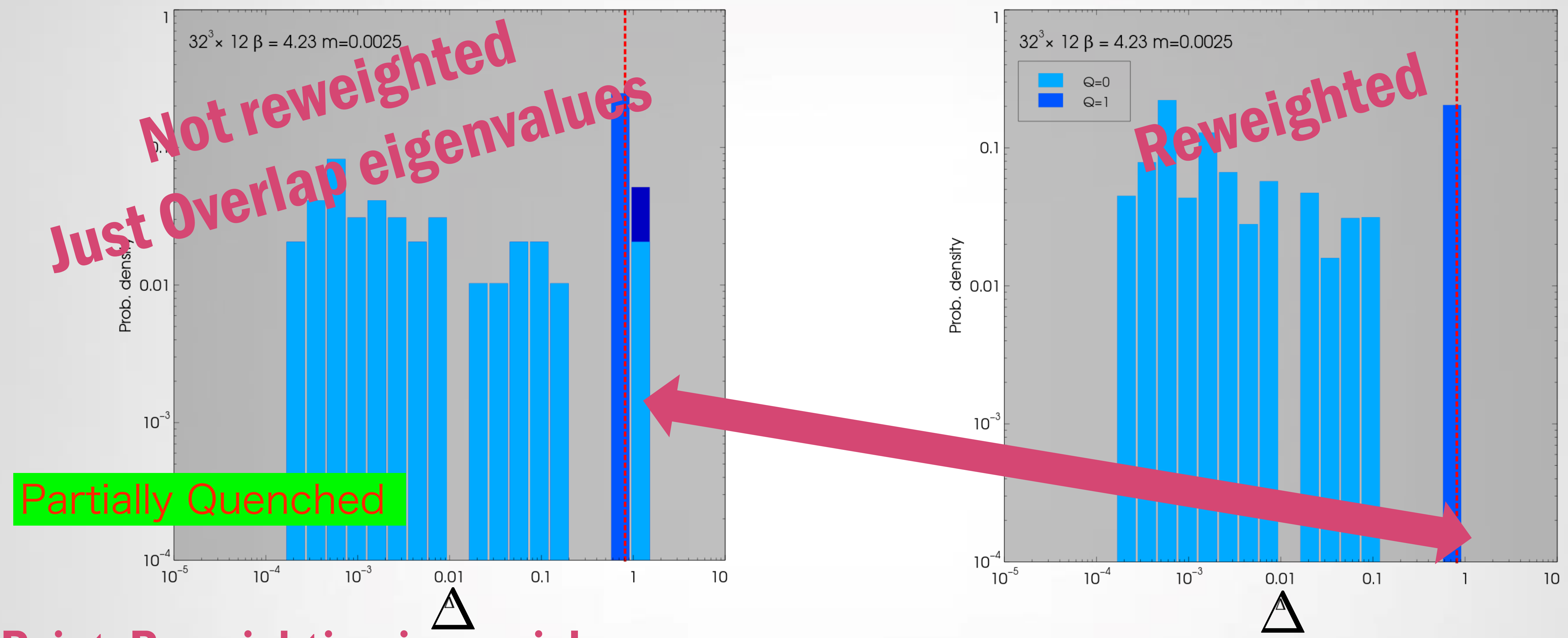

$\Delta$ $\Delta$

DomainWall

reweighted Overlap

G. Cossu et al. (JLQCD), Lat2015

# REWEIGHTING IS CRUCIAL




**Point: Reweighting is crucial**

Partially quenched results show accumulation of unphysical near zero modes

If the gap opens, the effective symmetry is

$$SU(2)_L \otimes SU(2)_R \otimes U(1)_A$$

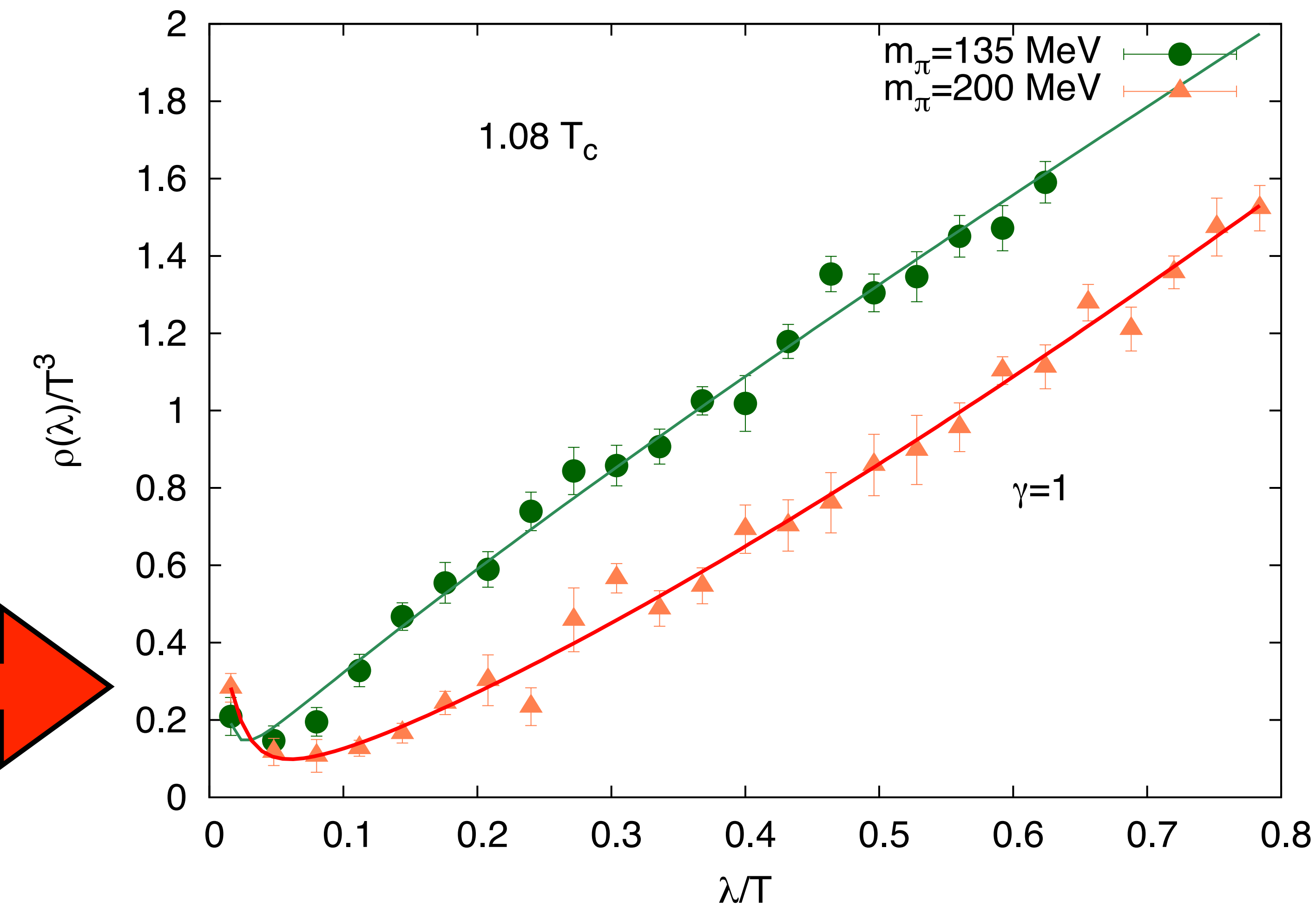
S. Sharma, V. Dick, F. Karsch, E. Laermann, S. Mukherjee, Lattice2015

Eigenvalues density of Overlap on DomainWall (partially quenched !)

From Sharma's talk@Lat2015,

- General features: Near zero mode peak +bulk
- We fit to the ansatz: $\rho(\lambda) = \frac{A\epsilon}{\lambda^2+A} + B\lambda^\gamma$
- Bulk rises linearly as $\lambda$,no gap seen.
- No gap even when quark mass reduced!

This is an artifact due to PQ !



# 4. Discussion

## Possible loopholes for the theoretical argument ?

**Assumption 2** $\langle \mathcal{O}(A) \rangle_m = f(m^2)$ $f(x)$ is analytic at $x = 0$

May this be violated ? (This condition may be too strong.)

Note that non-analyticity comes from valence quarks, not from determinant, even at zero temperature, where non-analyticity appears due to the spontaneous chiral symmetry breaking.

What kind of physics implies non-analyticity at m=0 ?

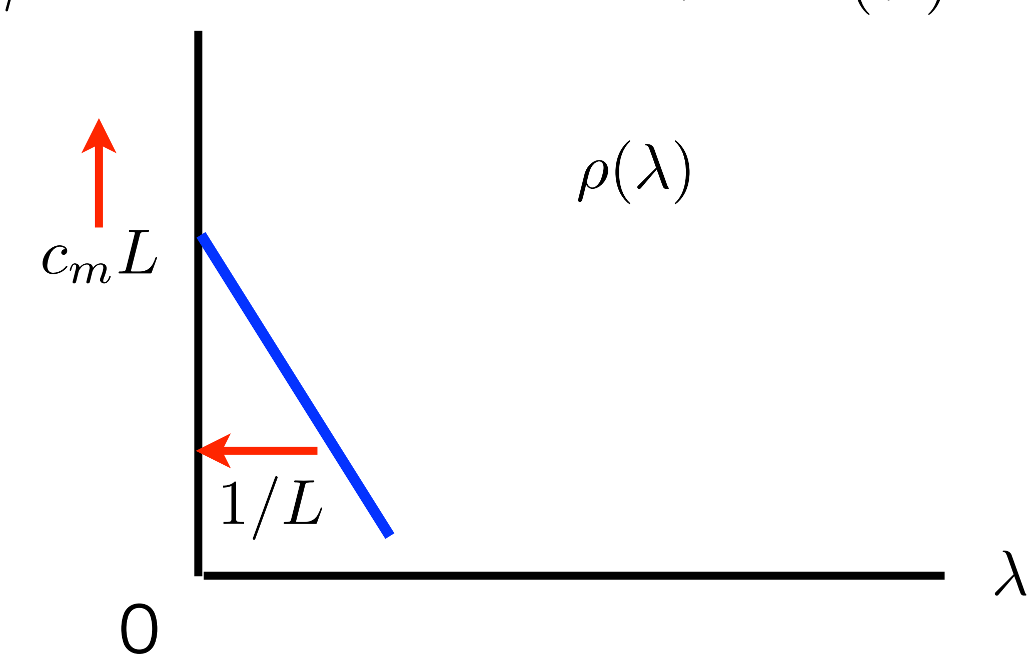
**Assumption 3** eigenvalues density can be expanded at the origin ?

Ex. claim by LLNL/RBC: accumulation of near-zero modes leads to

$$\rho(\lambda) \simeq c_m \delta(\lambda) + \cdots$$

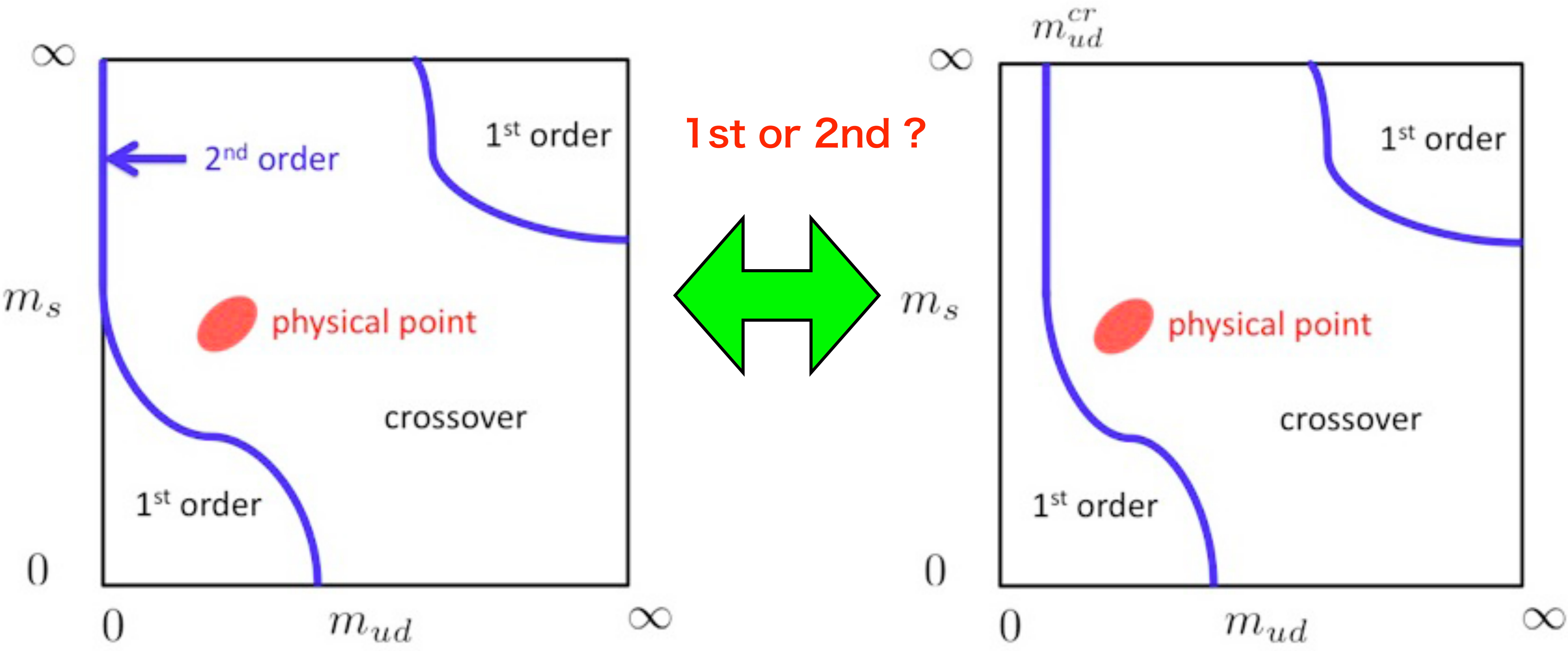
$$\Rightarrow \quad \pi\delta(\lambda) \sim \frac{1/L}{\lambda^2 + 1/L^2}$$

# of near zero-modes $\propto VL = (V)^{5/4}$




Is this possible ?

# Order of phase transition in 2-flavor QCD




gapless density $\mathrm{SU}(2)_L \otimes \mathrm{SU}(2)_R \otimes \mathrm{Z}_4$

density with gap $\mathrm{SU}(2)_L \otimes \mathrm{SU}(2)_R \otimes \mathrm{U}(1)_A$

Conformal bootstrap method predicts IR fixed point for these cases.

Even if the phase transition is of 2nd order, its universality class should be different from O(4).