格子QCDによる
有限温度・密度解析

有限密度の計算方法を中心に

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今日の話

- イントロダクション（最近の話題のレビュー)
- 高密度でのシミュレーションの向けた試み

  （Density of state method + Reweighting method）
- カノニカル法（クォーク数を固定した計算法）

超高温高密度での量子色力学（QCD）

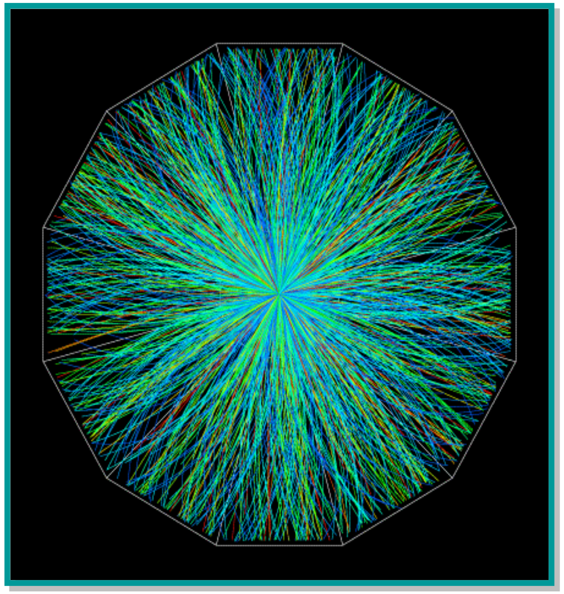
QCDが本質的な役割を果たす世界

- 温度1兆度の世界
  - 宇宙の進化の過程：宇宙誕生 0.0001秒後
  - カイラル相転移： 物質が質量を獲得したとき
  - 閉じ込め相転移： クォーク・グルーオン プラズマ状態
- 原子核の標準密度より高い密度の世界
  - 中性子星の内部（星の進化の最終段階）
  - 太陽の100兆倍以上の密度
- QCDの解析が困難。宇宙論構築の際の不確定要素。

高温高密度での格子QCDの数値シミュレーションが重要

１兆度の世界を地上で実現する
重イオン衝突実験 RHIC

- 金の原子核を衝突させて、瞬間的に超高温状態を作る。
- アメリカ・ブルックヘブン国立研究所（BNL)、RHIC ('01より)



- より高温の状態を目指してスイス・CERNのLHC、
- より高密度の状態を目指してドイツ・GSIのFairが予定されている。

超高密度物質の世界：中性子星、クォーク星

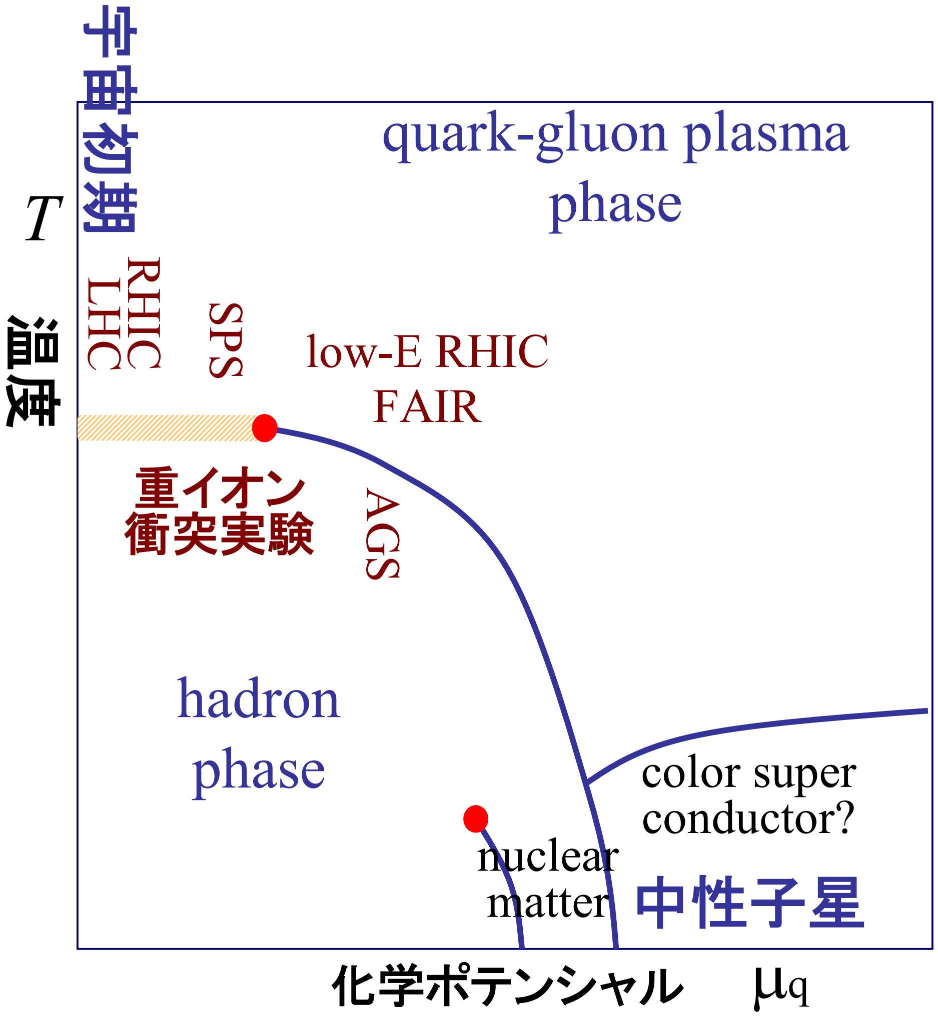
- 観測技術の進歩により中性子星の質量と半径が測定できるようになった。

- 星の硬さ（状態方程式、圧力と密度の関係）がわかれば、 星の内部構造が解明できる。
- 中性子星の内部、超高密度でのQCD物質の状態を、天体観測と理論の両側面から検証できる可能性。

格子QCDの数値シミュレーション

理論の第一原理計算が非常に重要

数値シミュレーションによる
高温高密度QCDの相構造の研究



- 格子QCDの数値シミュレーションで高温高密度の状態を調べる

⬇

- 重イオン衝突実験を支援
- 天体観測と比較

⬇

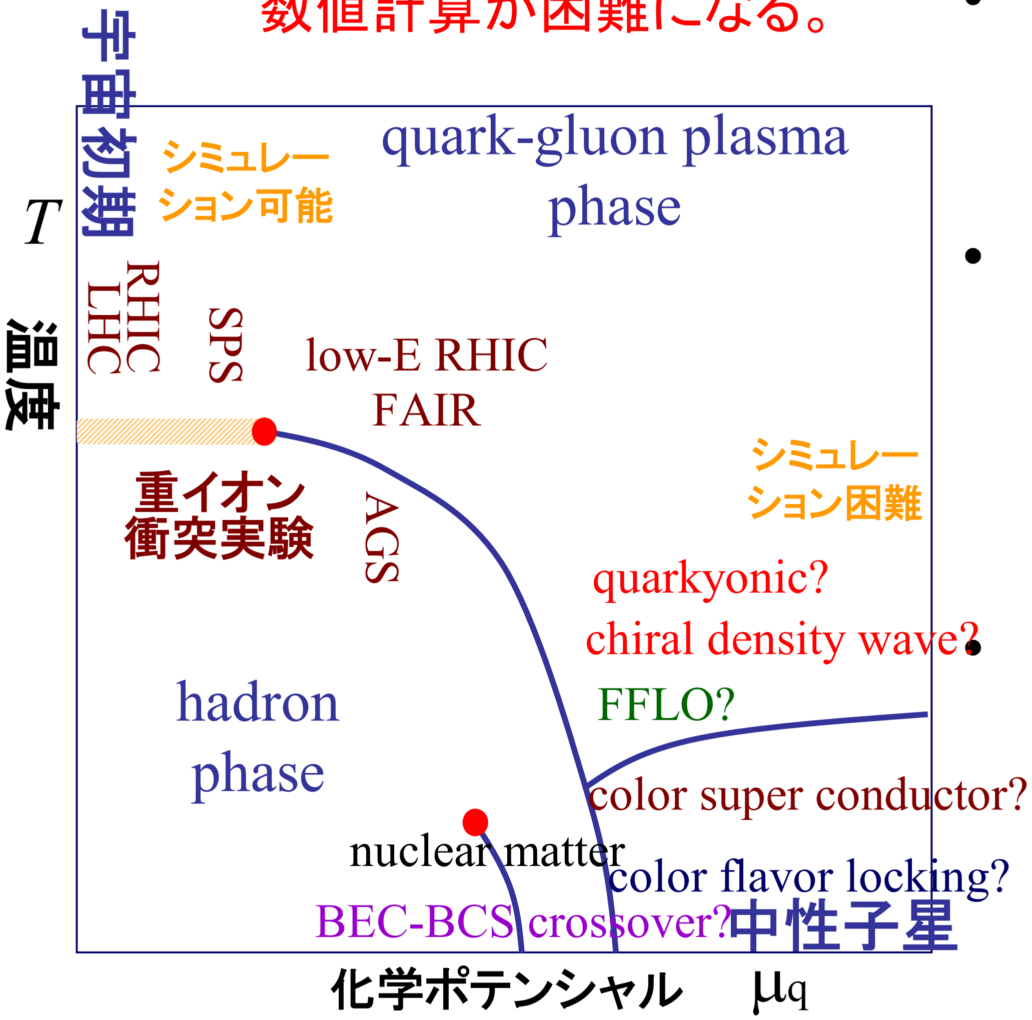
- 物質の極限状態を解明

⬇

- 宇宙論の構築

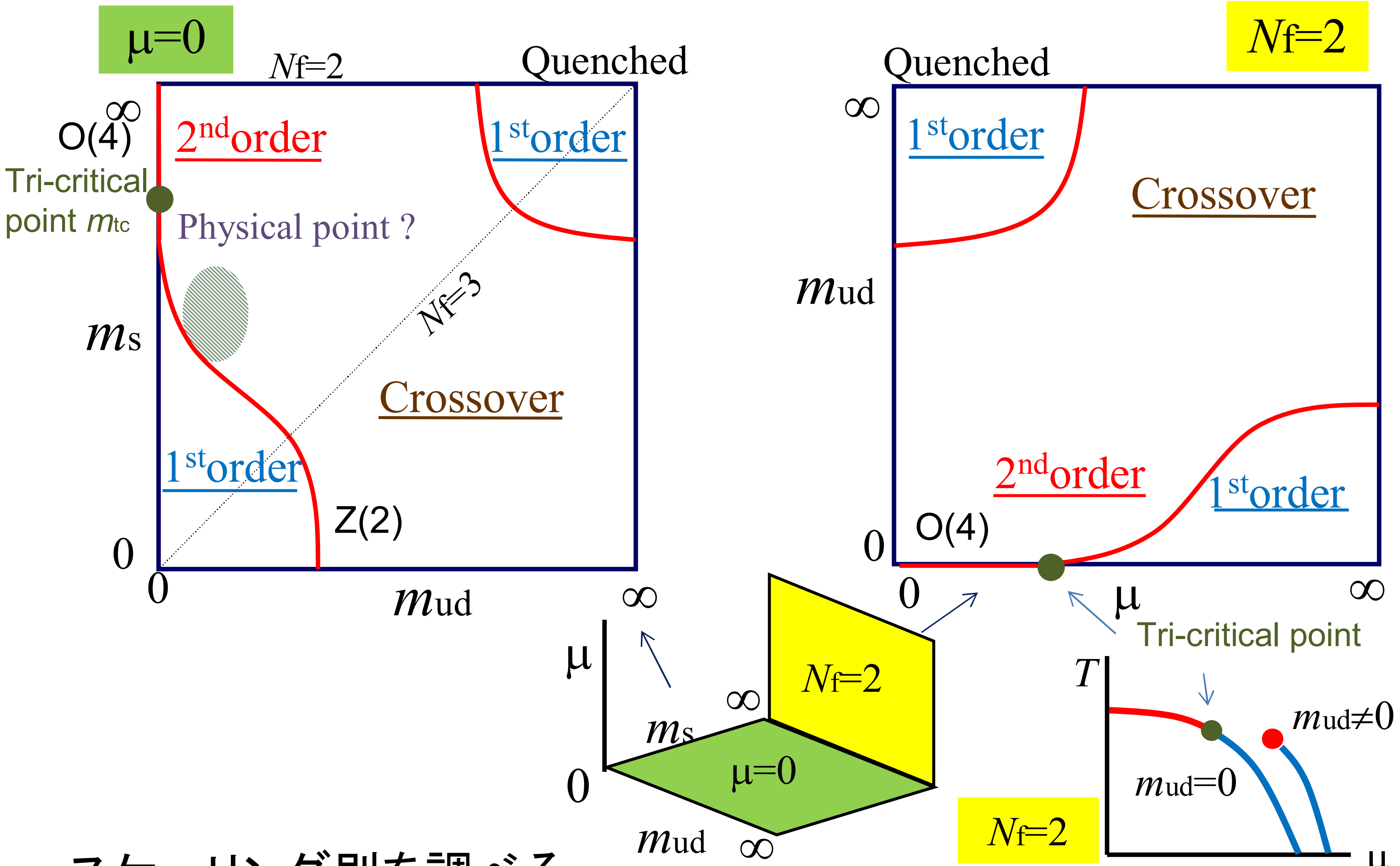
高温高密度QCDの~~相構造~~ 勢力分布

μ/T が大きくなるにつれて
数値計算が困難になる。



- 平和な高温低密度
  - 格子QCDの数値シミュレーションが重イオン衝突実験をサポート。
- 低温高密度は戦国時代
  - 現象論的な研究から、いろいろな状態が予想されている。
  - しかし、決定的な計算方法がない。
  - 数値シミュレーションも困難。
- 戦乱を鎮めるのは
  - <u>格子QCD？</u>
  - 天体観測？
  - 新たな画期的な実験？

相転移のクォーク質量依存性



- スケーリング則を調べる。
- 相転移線をユニバーサリティークラスで分類

有限温度・密度QCDの最近の研究成果

- 高温低密度領域
  - 宇宙初期でのクォーク・グルーオン・プラズマからハドロンに変化する相転移
  - 重イオン衝突実験 RHIC
    - 興味深い実験データが得られている。
    - 実験データが複雑。理論からのインプットが必要。
- 格子QCDによる数値シミュレーション

  QCDの第一原理計算のための最も強力な手法。

  密度が高くなるにつれて計算が困難になる。

- 相転移温度、状態方程式、輸送係数、有限密度での臨界点
  - QCDの特徴を実験と理論で比較・検証できる可能性。

状態方程式（熱力学量の温度・密度依存性）

- 有限バリオン数密度ではボルツマンの重みが複素数。
  - 直接モンテカルロシミュレーションできない。
- Taylor展開法 [Bielefeld-Swansea, Phys.Rev.D68(2003)014507; D71(2005)054508]
  - 物理量をTaylor展開。その係数を$\mu_q=0$で計算: 複素位相の問題と無関係。
  - 重イオン衝突実験で興味があるのは低密度領域。
    (RHICでは、μ_q/T~0.1, SPSでは、μ_q/T~0.6)

$$\frac{p}{T^4}=\frac{1}{VT^3}\ln Z=c_0+c_2\left(\frac{\mu_q}{T}\right)^2+c_4\left(\frac{\mu_q}{T}\right)^4+c_6\left(\frac{\mu_q}{T}\right)^6+\cdots$$

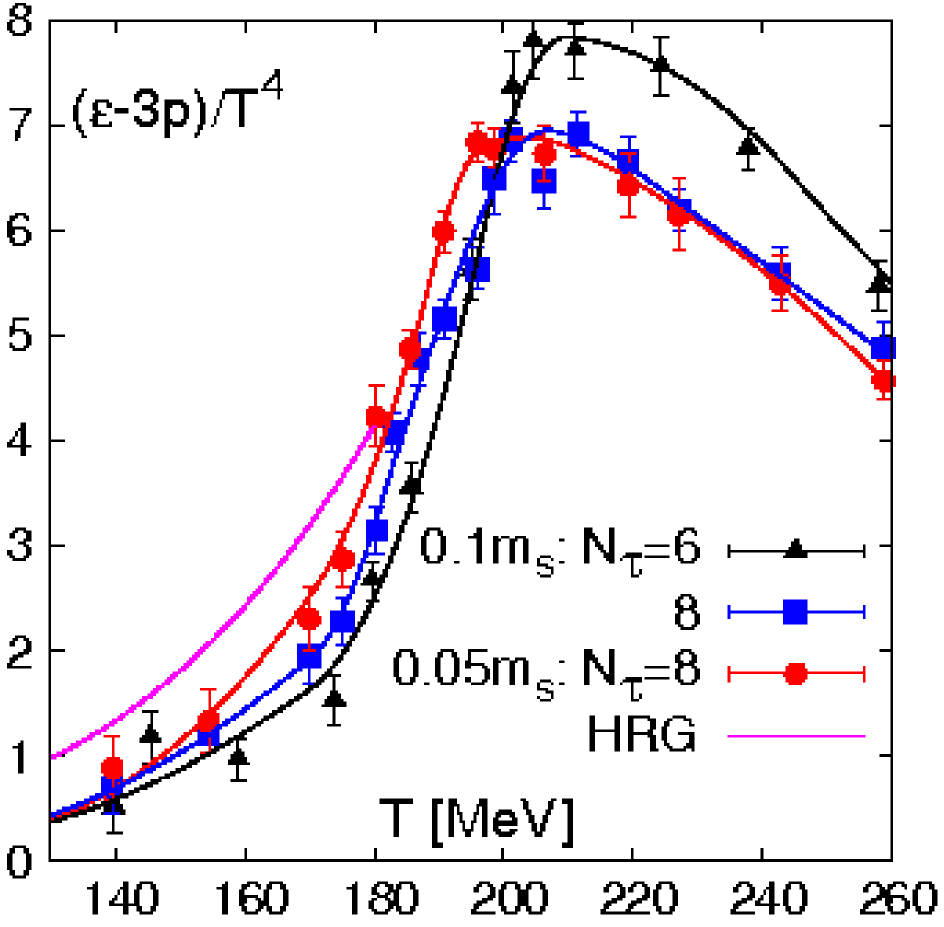
  - lnZの微分の計算: 大規模数値シミュレーションと相性がいい。
- 現在、ほぼ現実のクォーク質量でシミュレーションが行われている。
  RBC-Bielefeld Collab., MILC Collab., hotQCD Collab.
  Wuppertal-Budapest Collab.

➡ スタッガード・クォークによる研究: ゴールまで間近

➡ ウィルソン・クォークよる確認も重要 WHOT-QCD, QCDSF-DIK

相転移温度付近での状態方程式

hotQCD Collab., PRD81, 054504(2010)



ゼロ密度

低温相で
ハドロン・ガスの予言と比較

*N*τ=8: *a*≈0.14fm

0.1ms: *m*π ≈ 220MeV
0.05ms: *m*π ≈ 150MeV

- マゼンタの線: 質量2.5GeV以下のハドロン励起ガス模型の予言
- シミュレーションの結果(*m*π ≈ 150MeV)とよく一致

粘性 (viscosity)

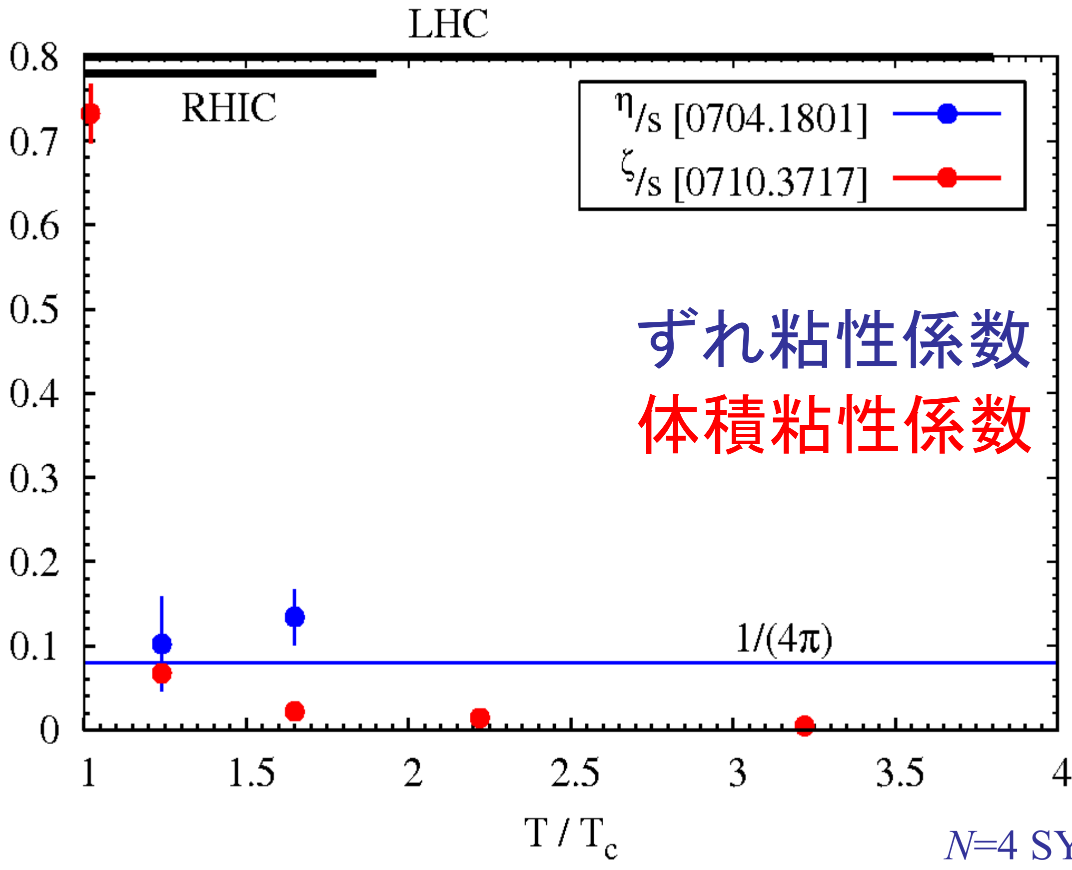
- QGPの粘性が小さいことはQCDから説明できるか。
  - 粘性を無視した流体計算が実験結果をよく説明できるが

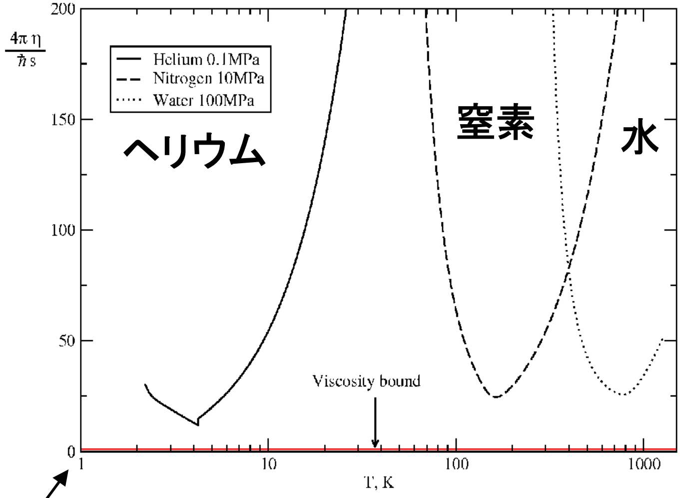
- 格子QCDのシミュレーション

  クエンチ・シミュレーション（動的クォークを無視）
  - Karsch, Wyld, PRD35, 2518 (1987): 最初の試み
  - Nakamura, Sakai, PRL94, 072305 (2005)：粘性が小さいことを証明
  - Meyer, PRD76, 101701(R)(2007); PRL100, 162001 (2008)：最近の結果

SU(3)ゲージ理論（クエンチ近似）での粘性係数の結果

H. Meyer, PRD76, 101701(R)(2007); PRL 100, 162001 (2008)





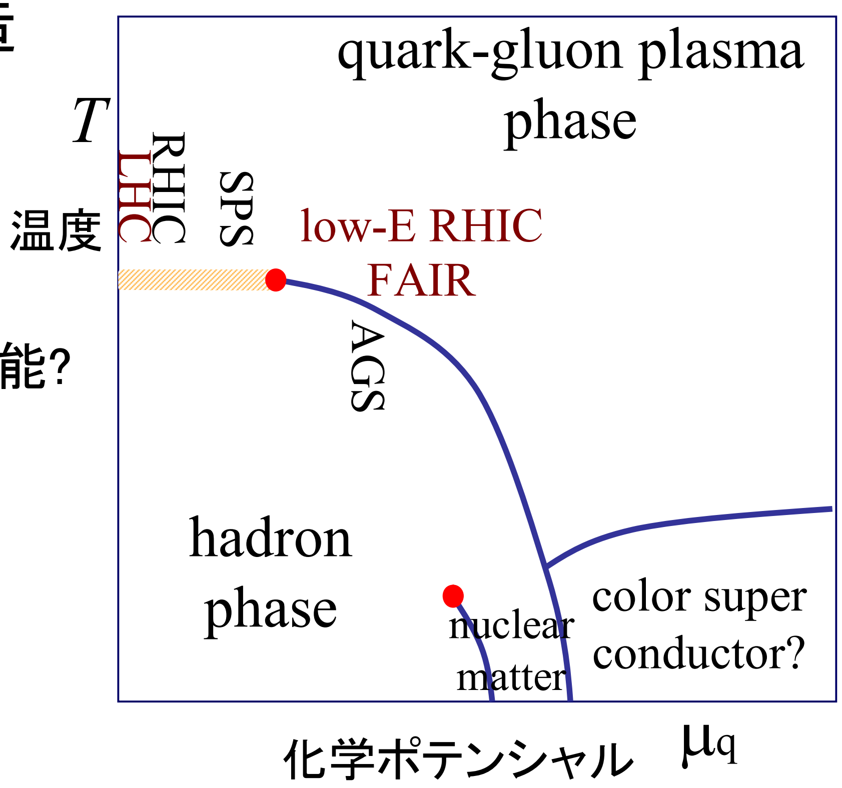
Kovtun, Son, Starinets, PRL 94, 111601 (2005) より

N=4 SYM theory: Policastro, Son, Starinets ’01

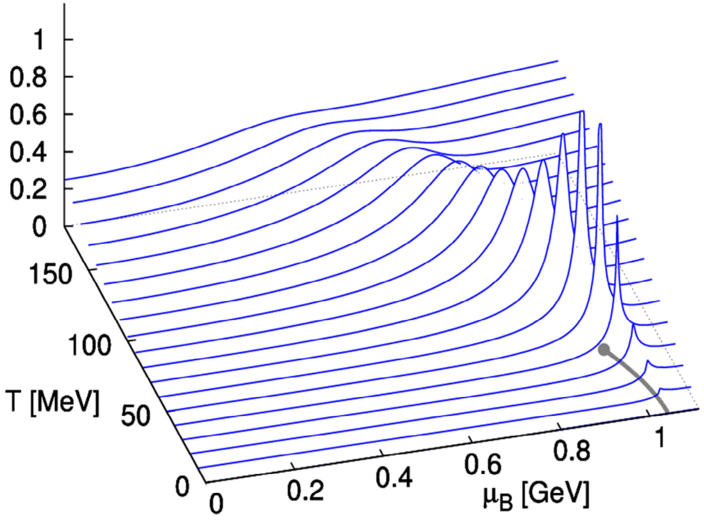
- 粘性は非常に小さい。
- 今後は Full QCD での計算が重要。

有限密度での臨界点

- 有限温度・密度でのQCDの相構造
- 有限密度での臨界点
  （一次相転移の終点）
  - バリオン数に大きな揺らぎ？
  - イベントごとの揺らぎの観測で測定可能?



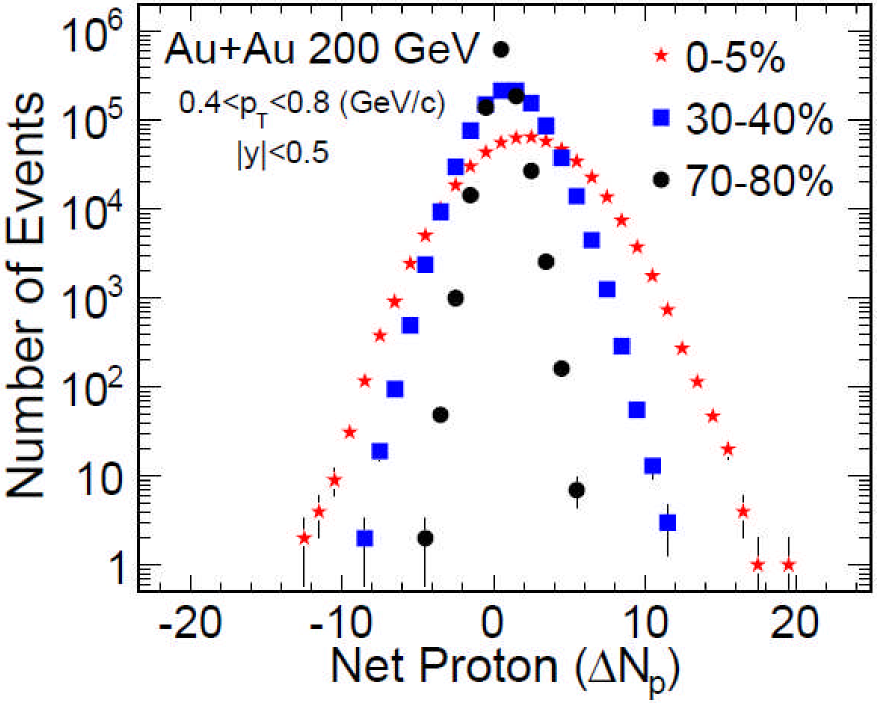
バリオン数の揺らぎ、NJLモデル
Asakawa, SE, Kitazawa, Phys.Rev.Lett.103



有限バリオン数密度での臨界点
有限温度QCDの最も顕著な特徴
有限密度でのシミュレーションが重要

イベントごとの生成粒子数の分布

STAR, ’10



- イベントごとの生成粒子数のヒストグラムが実験から得られている。
- 臨界点に近づくと分布の幅が広がる。
- 最近では、分布のガウス分布からのずれが注目されている。
- Lattice計算やモデル計算から、バリオン数に関する高次のキュムラントがT,μの関数として特徴的な値をとる。
  - 2次のキュムラント：分布の幅
  - それ以上：ガウス分布からのずれ

クォーク数に関する確率分布はカノニカル分布関数と直接関係がある。

その他の重要な課題

- チャーモニウムの融解温度
- 電気伝導度 (electric conductivity)
- 強い磁場中でのQCDの相転移

Toward the numerical study of lattice QCD at high density

- Avoiding the sign problem in finite-density QCD
- Density of state method + Reweighting method

Problems in simulations at μ≠0

- Problem of Complex Determinant at μ≠0
  - Boltzmann weight: complex at μ≠0
    - Monte-Carlo method is not applicable.
    - Configurations cannot be generated.

- Density of state method (Histogram method)

  X: order parameters, total quark number, average plaquette etc.

$$Z(m,T,\mu)=\int dX\, \underbrace{W(X,m,T,\mu)}_{\text{histogram}}$$

$$W(\overline{X},m,T,\mu)\equiv\int DU\,\delta(X\text{-}\overline{X})(\det M(m,\mu))^{N_{\mathrm{f}}}\ \mathrm{e}^{-S_g}$$

- Expectation values

$$\langle O[X]\rangle_{(m,T,\mu)}=\frac{1}{Z}\int dX\, O[X]W(X,m,T,\mu)$$

- The density of state method is useful if we combine the reweighting method.

Order of phase transitions
Distribution function (histogram)

- First order phase transition

  Two phases coexists at T_c

  e.g. SU(3) Pure gauge theory

- Average plaquette (1x1 Wilson loop): P

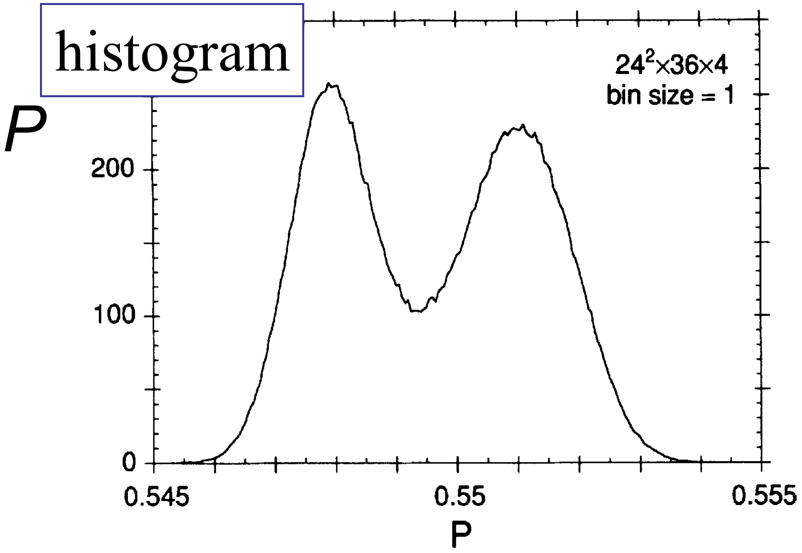
- Partition function

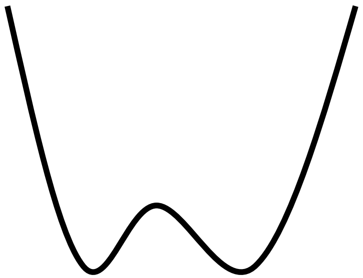
$$Z(T) = \int dP \, \underbrace{W(P,T)}_{\text{histogram}}$$

- Effective potential

$$V_{\text{eff}}(P) \equiv -\ln\left(W(P)\right)$$

SU(3) Pure gauge theory
QCDPAX, PRD46, 4657 (1992)

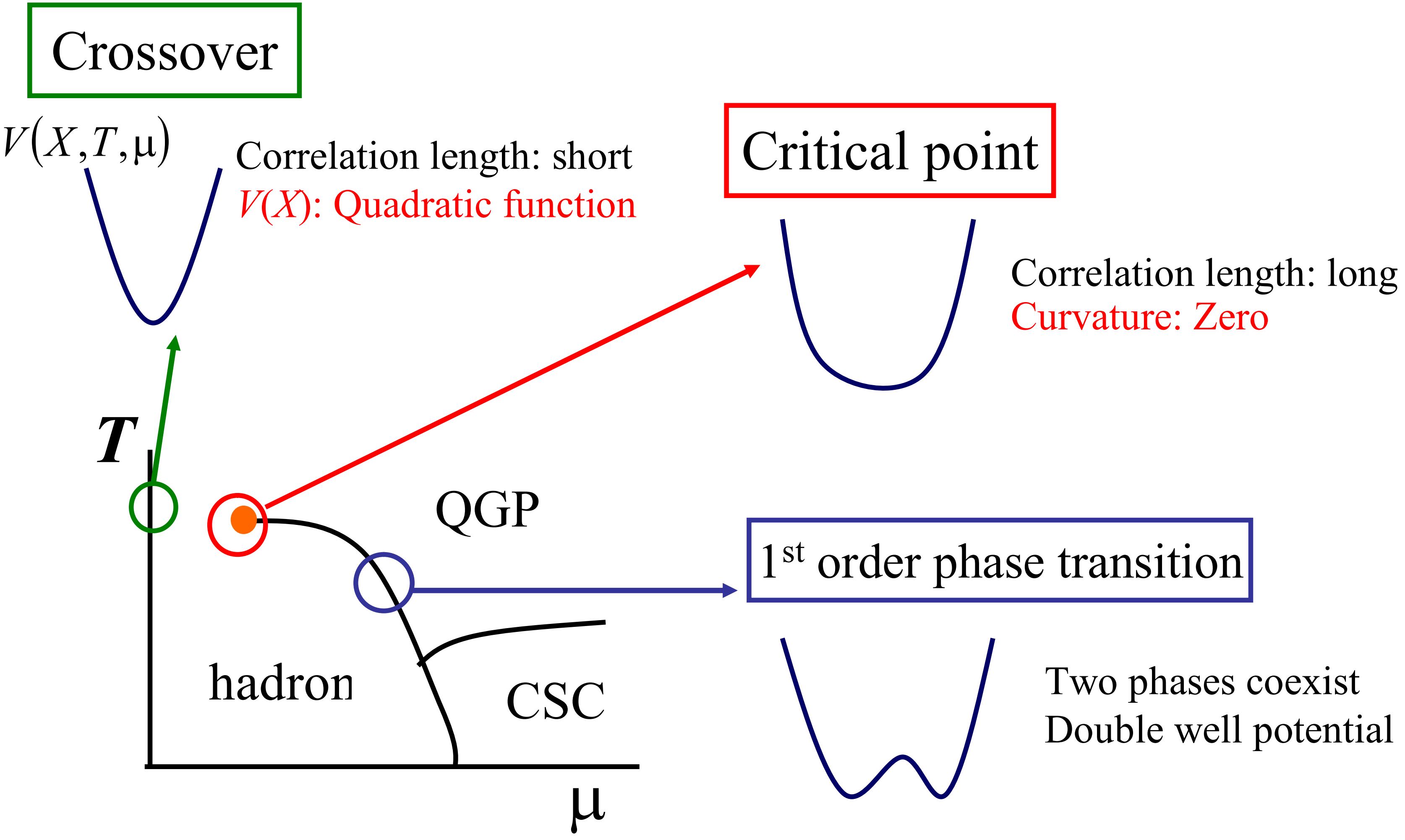




μ-dependence of the effective potential

$$Z(T,\mu)=\int dX\,W(X,T,\mu), \qquad V_{\rm eff}(X)=-\ln W(X)$$

X: order parameters, total quark number, average plaquette etc.



Crossover

$V(X,T,\mu)$

Correlation length: short

$V(X)$: Quadratic function

Critical point

Correlation length: long

Curvature: Zero

1st order phase transition

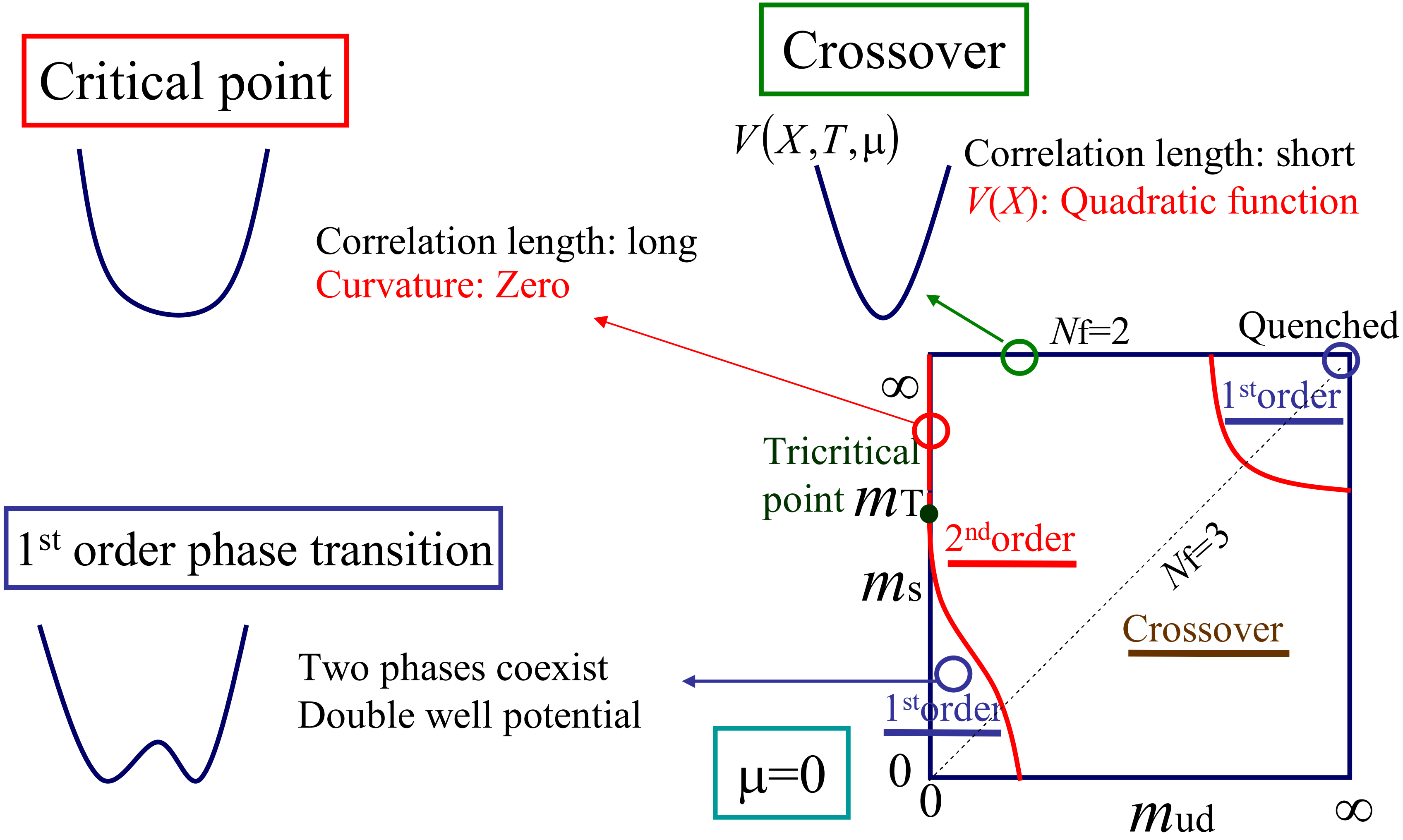
Two phases coexist

Double well potential

mass-dependence of the effective potential

$$Z(T,\mu)=\int dX\, W(X,T,\mu), \qquad V_{\mathrm{eff}}(X) = -\ln W(X)$$

X: order parameters, total quark number, average plaquette etc.



Critical point

Correlation length: long

Curvature: Zero

Crossover

$V(X,T,\mu)$

Correlation length: short

$V(X)$: Quadratic function

1st order phase transition

Two phases coexist

Double well potential

μ=0

Reweighting method at $\mu\neq 0$ (S.E., Phys.Rev.D77, 014508(2008))

- Distributions of plaquette *P* (1x1 Wilson loop for the standard action)

$$Z(\beta,m,\mu)=\int dP\, \underline{\underline{R(P,m,m_0,\mu)W(P,m_0,\beta)}} \qquad S_g=-6N_{site}\beta P \qquad \left(\beta=6/g^2\right)$$

$$W(\bar{P},\beta)\equiv\int DU\delta(P\text{-}\bar{P})\left(\det M(m_0,0)\right)^{N_f}\,\mathrm{e}^{-S_g}$$

(Weight factor at $\mu=0$)

(Reweight factor)

$$R(\bar{P})\equiv\frac{\int DU\,\delta(P\text{-}\bar{P})\left(\det M(m,\mu)\right)^{N_f}}{\int DU\,\delta(P\text{-}\bar{P})\left(\det M(m_0,0)\right)^{N_f}}=\frac{\left\langle \delta(P\text{-}\bar{P})\left(\dfrac{\det M(m,\mu)}{\det M(m_0,0)}\right)^{N_f}\right\rangle_{(\beta,\mu=0)}}{\left\langle\delta(P\text{-}\bar{P})\right\rangle_{(\beta,\mu=0)}}\equiv\left\langle\left(\frac{\det M(m,\mu)}{\det M(m_0,0)}\right)^{N_f}\right\rangle_P$$

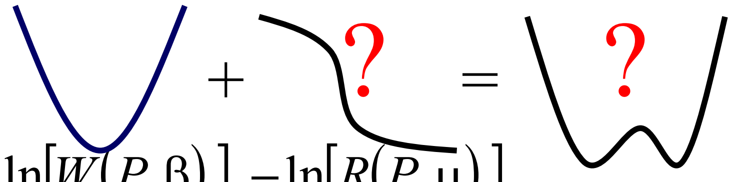
$R(P,m,\mu)$: independent of β, → $R(P,m,\mu)$ can be measured at any β.

Effective potential:

crossover
non-singular

1st order phase transition?

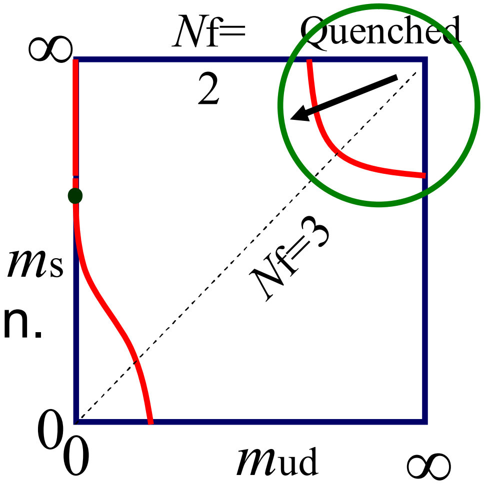
$$V_{\rm eff}(P)=-\ln\left[R(P,\mu)W(P,\beta)\right]=$$



$-\ln[W(P,\beta)]$ $-\ln[R(P,\mu)]$

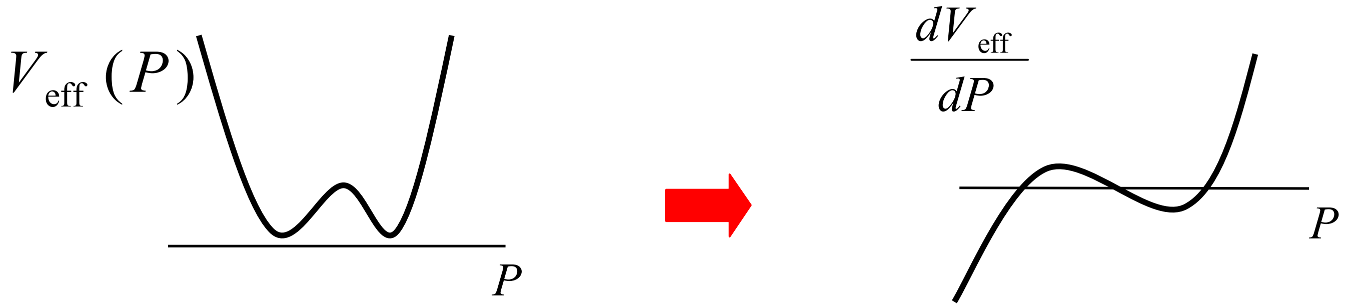
Reweighting from quenched simulations

WHOT-QCD Collab.



- Order of phase transition near the quenched limit
  - First order ($m_q \to \infty$) → crossover (small m_q)

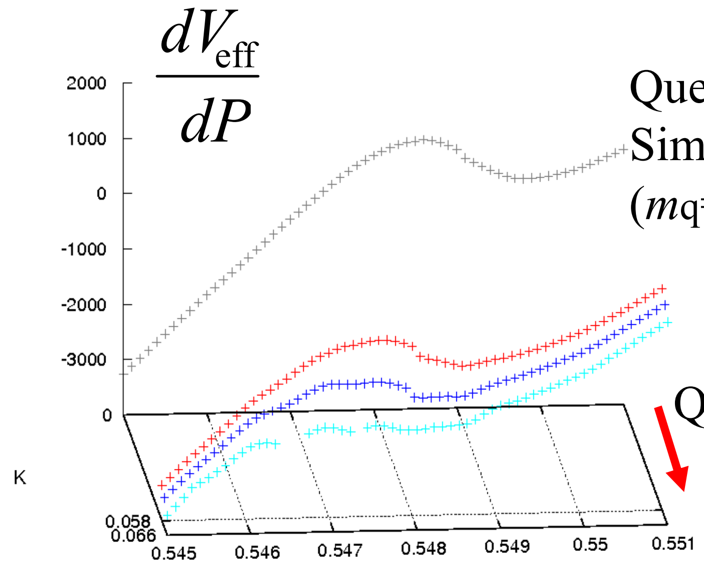
- Quenched simulations + Reweighting method
  - det*M* is estimated by a hopping parameter expansion.
- Effective potential in a wide range of *P*: required.
  - Derivative of the effective potential



- One can combine the data of $dV_{\rm eff}/dP$ obtained at different β using

$$V_{\rm eff}(\beta_1) - 6N_{\rm site}(\beta_2-\beta_1)P = V_{\rm eff}(\beta_2) \Rightarrow \frac{dV_{\rm eff}}{dP}(\beta_1) - 6N_{\rm site}(\beta_2-\beta_1) = \frac{dV_{\rm eff}}{dP}(\beta_2)$$

Effective potential near the quenched limit

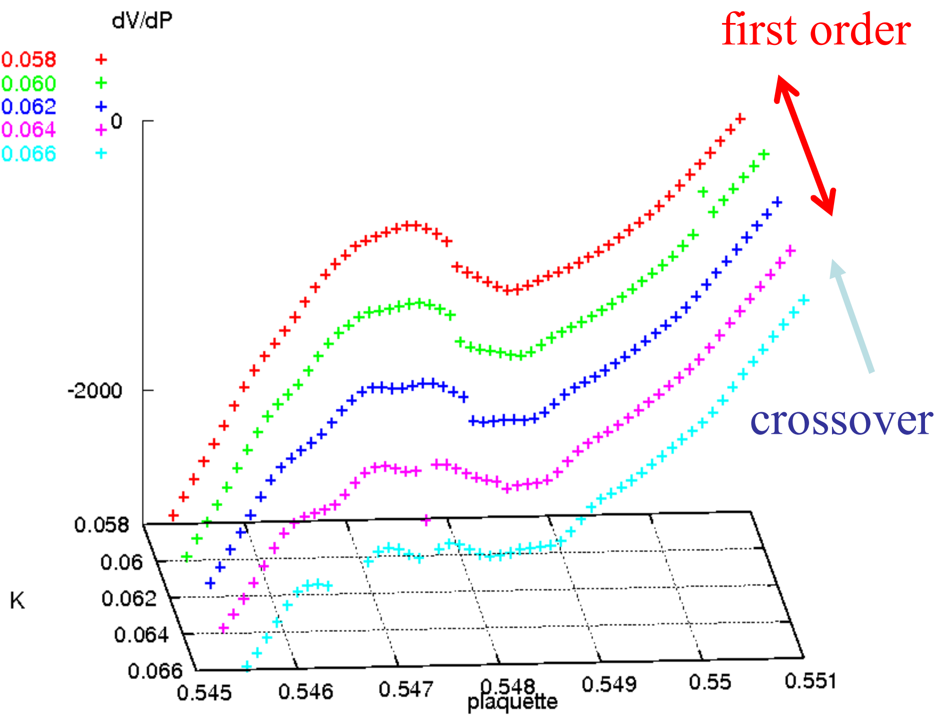


Quenched Simulation ($m_q=\infty$)

Quark mass smaller

$K \sim 1/m_q$ for large m_q

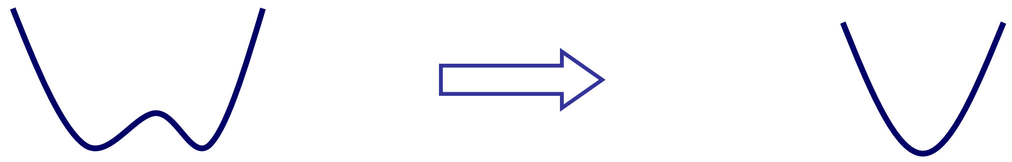
$24^3 \times 4$ lattice, 5 β points



first order

crossover

- Quenched simulations are performed and m_q is changed by the reweighting.
- First order transition at $m_q=\infty$ changes to crossover at $m_q< \infty$.



Application to finite density QCD

- Solving the sign problem

Avoiding the sign problem

SE, Phys.Rev.D77,014508(2008),
WHOT-QCD, arXiv0909.2121

θ: complex phase

$$\boxed{F = \left| \frac{\det M(\mu)}{\det M(0)} \right|^{N_f}}$$

- Sign problem: If $e^{i\theta}$ changes its sign,

$$R(P,\mu) = \left\langle \left(\frac{\det M(\mu)}{\det M(0)} \right)^{N_f} \right\rangle_{P\,\text{fixed}} \equiv \left\langle e^{i\theta} F \right\rangle_P \ll (\text{statistical error})$$

- Cumulant expansion

$<..>_{F,P}$: expectation values fixed F and P.

$$\left\langle e^{i\theta} F \right\rangle_P = \int F \left\langle e^{i\theta} \right\rangle_{F,P} dF$$

$$\approx \int F \exp\left[\underset{\to 0}{\cancel{i\langle\theta\rangle_C}} - \frac{1}{2}\left\langle\theta^2\right\rangle_C - \underset{\to 0}{\cancel{\frac{i}{3!}\left\langle\theta^3\right\rangle_C}} + \frac{1}{4!}\left\langle\theta^4\right\rangle_C + \cancel{\cdots} \right] dF$$

cumulants

$$\langle\theta\rangle_C = \langle\theta\rangle_{F,P},\quad \left\langle\theta^2\right\rangle_C = \left\langle\theta^2\right\rangle_{F,P} - \langle\theta\rangle_{F,P}^2,\quad \left\langle\theta^3\right\rangle_C = \left\langle\theta^3\right\rangle_{F,P} - 3\left\langle\theta^2\right\rangle_{F,P}\langle\theta\rangle_{F,P} + 2\langle\theta\rangle_{F,P}^3,\quad \left\langle\theta^4\right\rangle_C = \cdots$$

  - Odd terms vanish from a symmetry under $\mu \leftrightarrow -\mu$ $(\theta \leftrightarrow -\theta)$
    Source of the complex phase

If the cumulant expansion converges, No sign problem.

Convergence of the cumulant expansion

- Because θ ~ $O(\mu)$, $\left\langle \theta^n \right\rangle_C \sim O(\mu^n)$
  - The cumulant expansion is a power expansion of μ.
    - Applicable at low density.
  - If one takes into account $\left\langle \theta^n \right\rangle_C$, the truncation error does not affect up to $O(\mu^n)$.

- Gaussian distribution function
  - The cumulants vanish except for $\left\langle \theta^2 \right\rangle_C$.

$$\left\langle F e^{i\theta} \right\rangle = \int dF \int d\theta \; F e^{i\theta} W\left(F,\theta\right) \approx \int dF \, F e^{-1/(4\alpha)} W'(F)$$

$$\boxed{W\left(F,\theta\right) \approx \sqrt{\frac{\alpha(F)}{\pi}} e^{-\alpha(F)\theta^2} W'(F)} \quad \Rightarrow \quad \boxed{\left\langle F e^{i\theta} \right\rangle \approx \left\langle F e^{-\left\langle \theta^2 \right\rangle_F / 2} \right\rangle}$$

$$\frac{1}{2\alpha(F')} = \frac{\int \theta^2 W(F',\theta) d\theta}{\int W(F',\theta) d\theta} \equiv \left\langle \theta^2 \right\rangle_F$$

Gaussian distribution of the complex phase

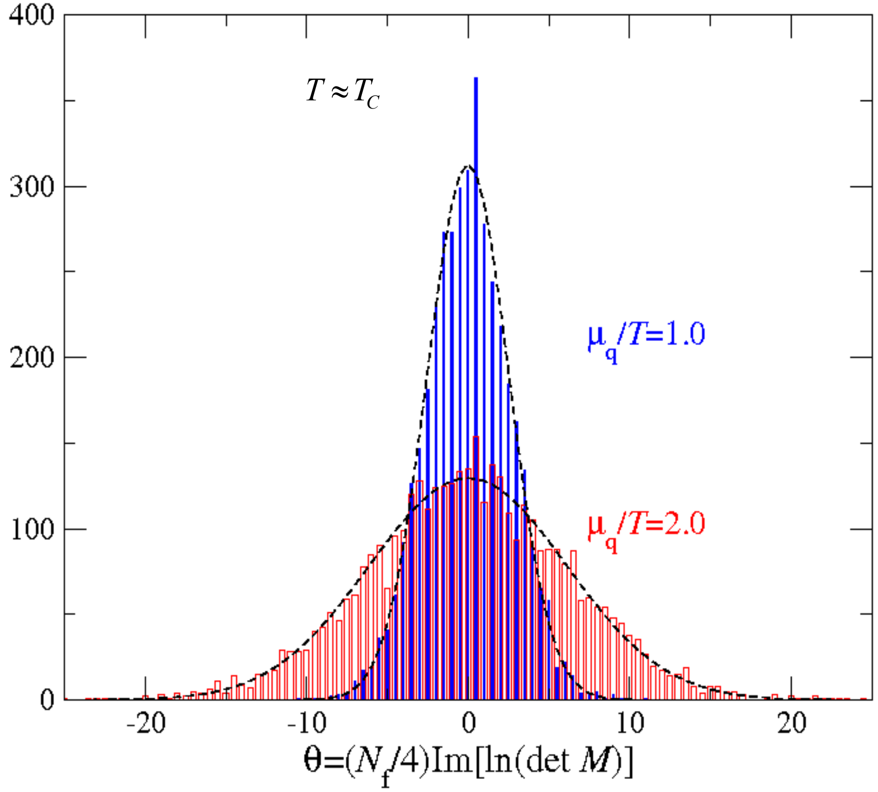
- Complex phase of det*M* $\theta = N_f \, \mathrm{Im}[\ln \det M(\mu)]$
  - Taylor expansion: odd terms of ln det *M* (Bielefeld-Swansea, PRD66, 014507 (2002))

$$\theta = N_f \mathrm{Im}\left[\frac{\mu}{T}\frac{\mathrm{d}\ln\det M}{\mathrm{d}(\mu/T)} + \frac{1}{3!}\left(\frac{\mu}{T}\right)^3 \frac{\mathrm{d}^3 \ln\det M}{\mathrm{d}^3(\mu/T)} + \frac{1}{5!}\left(\frac{\mu}{T}\right)^5 \frac{\mathrm{d}^5 \ln\det M}{\mathrm{d}^5(\mu/T)} + \cdots\right]$$

- Gaussian distribution
  - Results for p4-improved staggered
  - Taylor expansion up to $O(\mu^5)$
  - Dashed line: fit by a Gaussian function
  - SE, Phys.Rev.D77,014508(2008)

Well approximated ⬅

histogram of θ



Convergence in the large volume(V) limit

- Worst case: Because $\theta \sim O(V)$, Naïve expectation: $\left\langle \theta^n \right\rangle_C \sim O(V^n)?$
  - If so, the cumulant expansion does not converge.

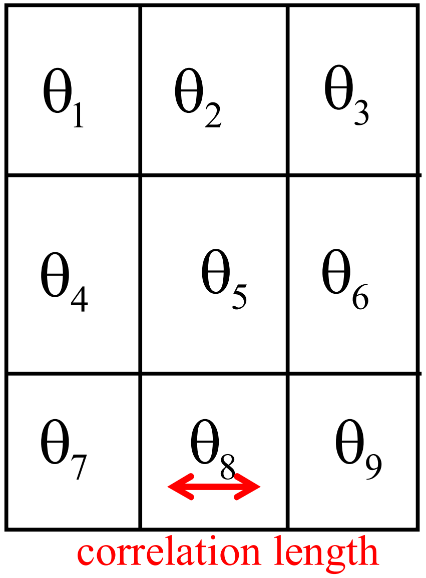
However, this problem is solved in the following situation.

- The phase is given by $\theta = \sum_x \theta_x$
  - No correlation between θx.
  - This situation is realized if we define the phase as

$$\theta = N_f \mathrm{Im}\left[\frac{\mu}{T} \frac{\mathrm{d} \ln \det M}{\mathrm{d}(\mu/T)} + \frac{1}{3!}\left(\frac{\mu}{T}\right)^3 \frac{\mathrm{d}^3 \ln \det M}{\mathrm{d}^3(\mu/T)} + \frac{1}{5!}\left(\frac{\mu}{T}\right)^5 \frac{\mathrm{d}^5 \ln \det M}{\mathrm{d}^5(\mu/T)} + \cdots \right]$$

or

$$\theta = N_f \int_0^{\mu/T} \mathrm{Im}\left[\frac{\mathrm{d} \ln \det M}{\mathrm{d}(\mu/T)} \right]_{\mu'/T} d\left(\frac{\mu'}{T}\right)$$



  - The first derivative is the sum of the local density operator.

The spatial density correlation length is short at a non-singular point.

$$\mathrm{Im}\left[\frac{d \ln \det M}{d(\mu/T)} \right] = \mathrm{Im}\left[Tr\left(M^{-1} \frac{\partial M}{\partial(\mu/T)} \right) \right]$$

Diagonal element:
local density operator

Convergence in the large volume (*V*) limit

This problem is solved in the following situation.

- The phase is given by $\theta = \sum_x \theta_x$
  - No correlation between θ_x.

$$\langle e^{i\theta}\rangle_{F,P} = \left\langle e^{i\sum_x \theta_x}\right\rangle_{F,P} \approx \prod_x \langle e^{i\theta_x}\rangle_{F,P} = \exp\left[\sum_x\sum_n \frac{i^n}{n!}\langle\theta_x^n\rangle_C\right]$$

$$\langle e^{i\theta}\rangle_{F,P} = \exp\left[\sum_n \frac{i^n}{n!}\langle\theta^n\rangle_C\right] \quad \Rightarrow \quad \langle\theta^n\rangle_C \approx \sum_x \langle\theta_x^n\rangle_C \sim O(V)$$

  - Ratios of cumulants do not change in the large *V* limit.
  - Convergence property is independent of *V* although the phase fluctuation becomes larger as *V* increases
  - The application range of μ can be measured on a small lattice.

Gaussian approximation (S.E., Phys.Rev.D77, 014508(2008))

If Gaussian distribution,

➡ Higher order cumulants vanish.

- If the second term is dominated, the calculation: much easier.

$$\boxed{\left\langle e^F e^{i\theta} \right\rangle_P \approx \left\langle e^F e^{-\left\langle \theta^2 \right\rangle_{F,P}/2} \right\rangle_P}$$

No sign problem

- Configurations with large fluctuations of θ: not important.
  - Such a configuration is suppressed as

$$W(P) \sim \exp\left[-\frac{1}{2} \left\langle \theta^2 \right\rangle_{F,P} \right]$$

  - If configurations with small θ are important,
    the cumulant expansion must be good.

Canonical approach

- Effective potential of total quark number
- Provability distribution function of quark number
  - Related to event by event analyses in heavy-ion collisions.

Canonical approach

- Canonical partition function: *Zc* (Fugacity expansion)

$$Z_{GC}(T,\mu)=\sum_N \underline{Z_C(T,N)}\exp(N\mu/T)\equiv\sum_N W(N)$$

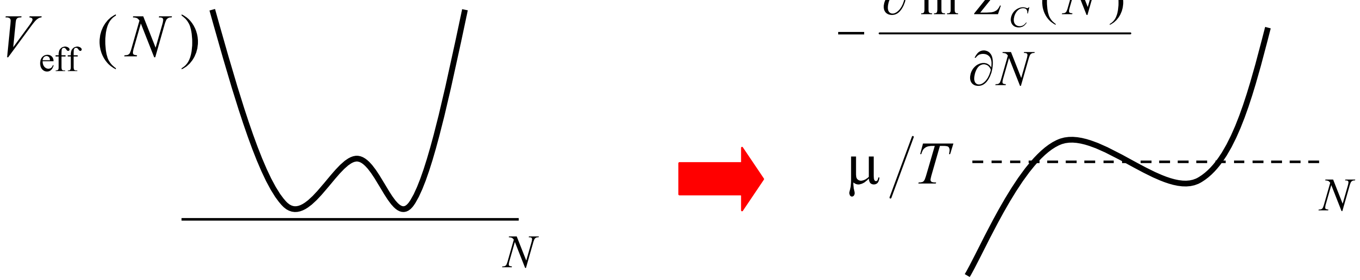
- Effective potential as a function of the quark number *N*.

$$V_{\text{eff}}(N)=-\ln W(N)=-\ln Z_C(T,N)-N\mu/T$$

- At the minimum,

$$\frac{\partial V_{\text{eff}}(N)}{\partial N}=-\frac{\partial\ln W(N)}{\partial N}=\underline{\underline{-\frac{\partial\ln Z_C(T,N)}{\partial N}}}-\frac{\mu}{T}=0$$

- First order phase transition: Two phases coexist.

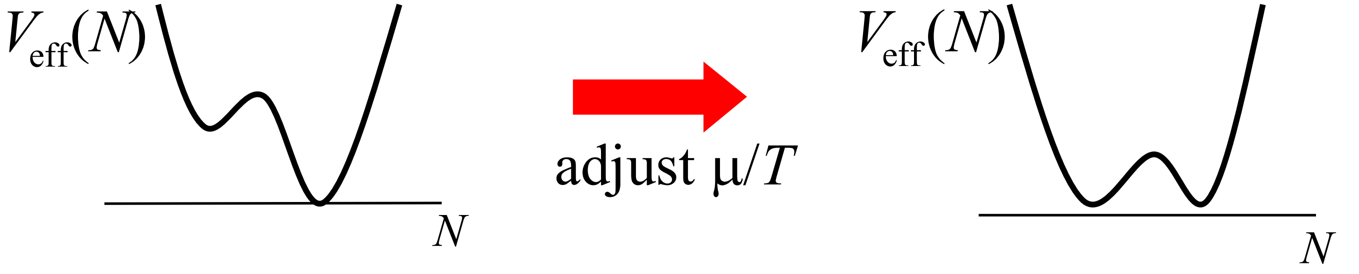


First order phase transition

- Critical μ/T

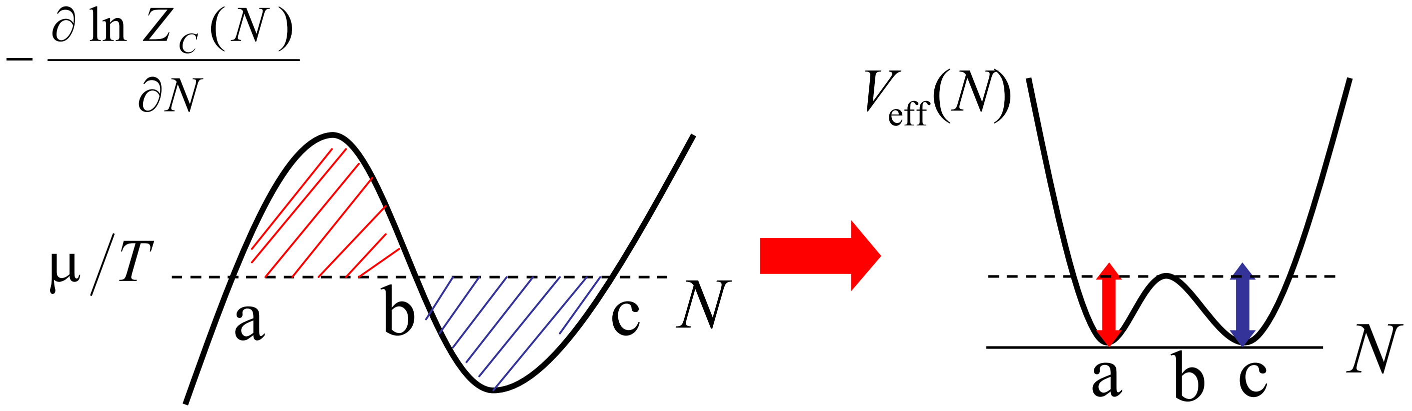
$$V_{\rm eff}(N) = -\ln Z_C(T,N) - \underline{N\mu/T}$$

liner function



- Derivative of lnZ

$$\frac{\partial V_{\rm eff}(N)}{\partial N} = -\frac{\partial \ln Z_C(T,N)}{\partial N} - \frac{\mu}{T}$$



- Area of ▨ and ▨ are the same at the critical μ/T.

Chemical potential in finite density lattice QCD

- Grand partition function

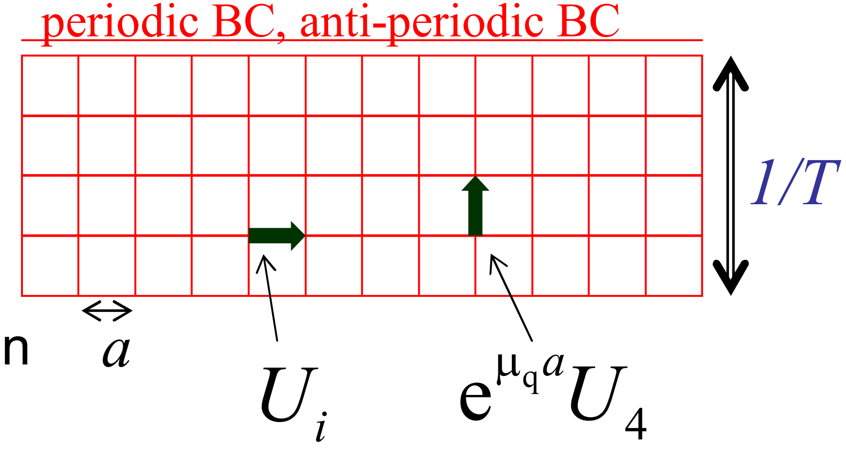
$$Z = \int \prod_{x,\mu} dU_\mu(x) \left(\det M\right)^{N_f} e^{-S_g}$$

- Lattice formulation: to obtain correct continuum limit

(Hasenfratz,Karsch,Phys.Lett.,125B,308(1983))

$$U_\mu \approx e^{igaA_\mu} \in SU(N_C)$$

(Gauge connection)



  - Modify the link field for time direction in the quark action.

$$\boxed{\begin{aligned} U_4(x) &\Rightarrow e^{\mu_q a} U_4(x) \\ U_4^+(x) &\Rightarrow e^{-\mu_q a} U_4^+(x) \end{aligned}}$$

$N_t a = 1/T$

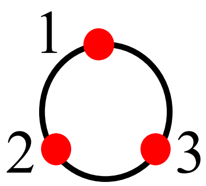
Simulation parameter: μ_q/T

Fermion determinant

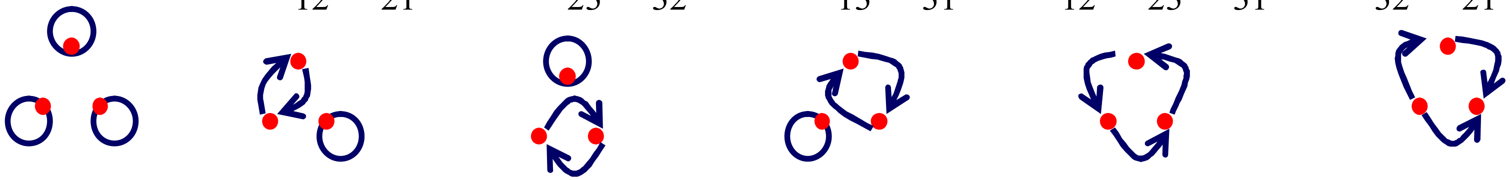
(see e.g.,Montvay's book 4.1.4)

For example,

$$M = \begin{pmatrix} m & -K_{12} & -K_{13} \\ -K_{21} & m & -K_{23} \\ -K_{31} & -K_{32} & m \end{pmatrix}$$

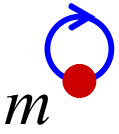


$$\det M = mmm - mK_{12}K_{21} - mK_{23}K_{32} - mK_{13}K_{31} - K_{12}K_{23}K_{31} - K_{32}K_{21}K_{13}$$

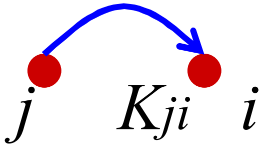


In general,

$$M_{ji} = m\delta_{ji} - K_{ji} \quad (K_{ii} = 0)$$



monomer

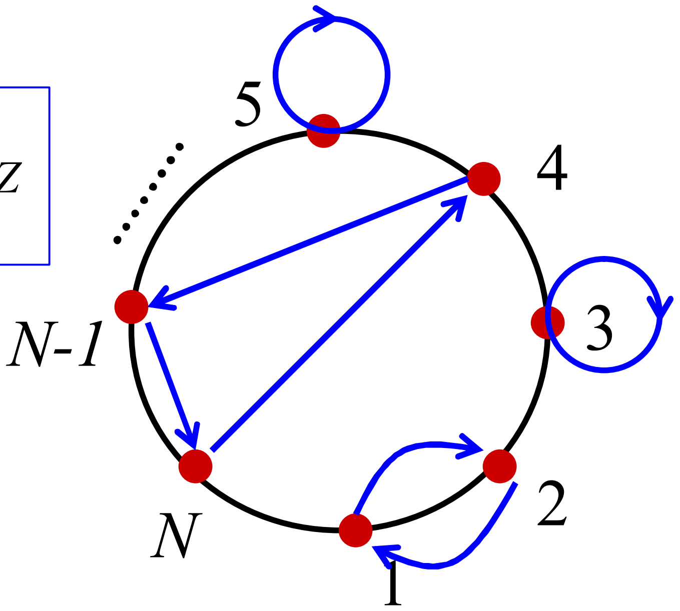


polymer

$$\boxed{\det M = \sum_{Z \in C} R_Z}$$

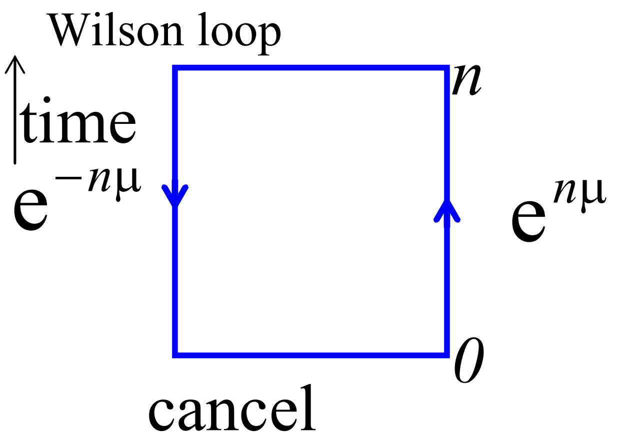
$$R_Z = (-1)^{(\#\text{ closed loop})} m^{(\#\text{ monomer})} \prod_{j,k} K_{kj}$$

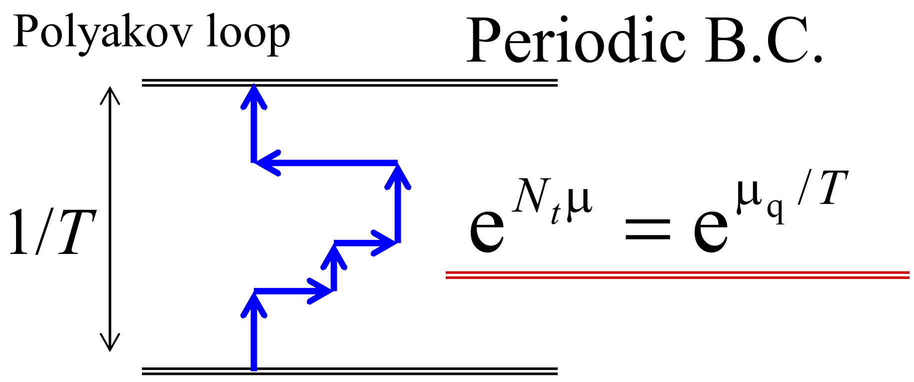
$$R_{\text{Fig}} = (-1)^2 m^2 K_{12} K_{21} K_{4N-1} K_{N-1,N} K_{N,4} \cdots$$



Non-zero chemical potential; Fugacity expansion

$$\boxed{K_{x,x+\hat{4}} = \mathrm{e}^{\mu}\eta_{x,4}U_4(x)}$$





- Chemical potential enhances winding current loops (static currents).
- Classify by the number of static currents.
- Fugacity expansion; expansion with the winding number *N*.

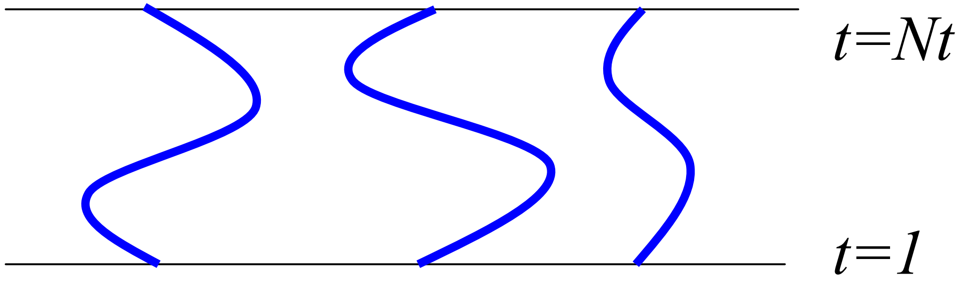
$$Z_{GC}(T,\mu) = \sum_{N=-3V}^{3V} Z_C(N,T)\exp(N\mu/T)$$

Heavy quark limit (Engels et al., Nucl.Phys.B558(1999)307)

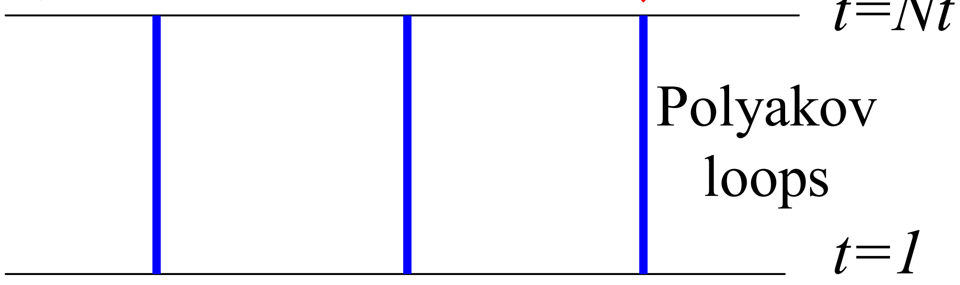
- Canonical Partition function for fixed quark number *N*

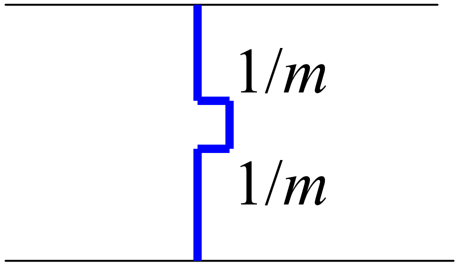
$$Z_C(N) = \int DU\ f_N\ \mathrm{e}^{-S_g}$$

- f_N is a product of *N* quark propagators with all possible combination



Quark mass ∞ limit





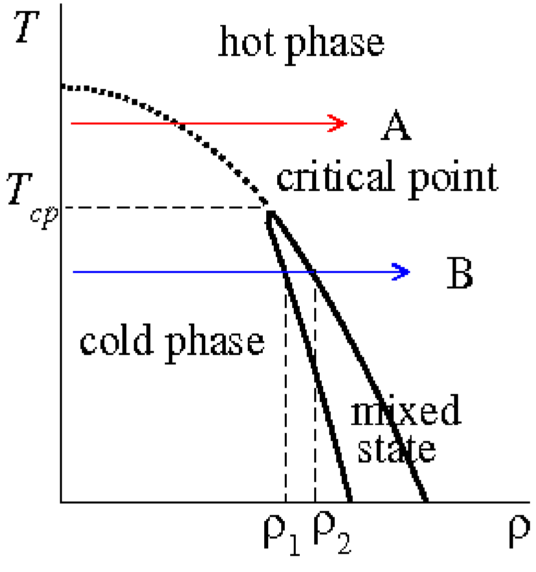
$1/m^2$ expansion

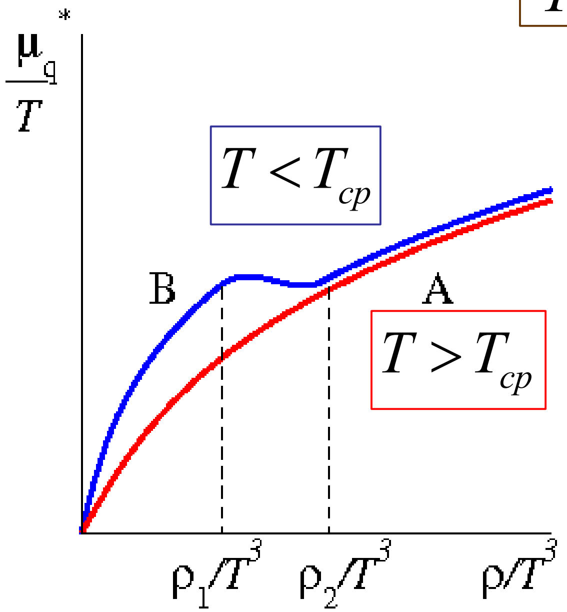
f_N is a sum of products of Polyakov loops

calculable analytically for each *N*

First order phase transition line

In the thermodynamic limit, $\frac{\partial V_{\rm eff}(N)}{\partial N} = 0,$ ⇨ $\boxed{\frac{\mu^*}{T} \equiv -\frac{\partial \ln Z_C(T,N)}{\partial N}}$





$$\frac{\mu^*}{T} \to \frac{\mu}{T} \quad \left(N_s^3 \to \infty\right)$$

Mixed state

$\Downarrow$

First order transition

- Inverse Laplace transformation using Glasgow method

  Kratochvila, de Forcrand, PoS (LAT2005) 167 (2005)

  *N*f=4 staggered fermions, $6^3 \times 4$ lattice, (*N*f=4: First order for all ρ.)
- Simulations with canonical ensemble

  Kentucky group (arXiv:0810.2349, Anyi Li's poster in QM09)
- With a saddle point approximation (S.E., Phys.Rev.D78, 074507 (2008)) ⬅

  this talk

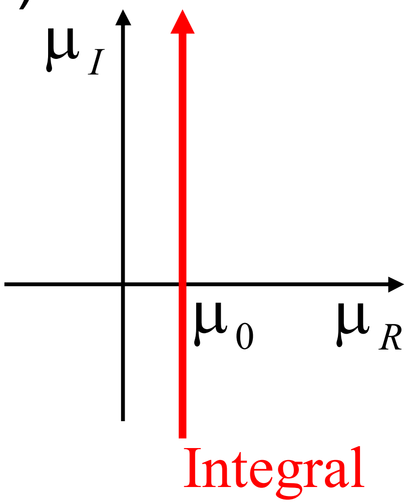
Numerical calculation of the canonical partition function on a lattice

Phys. Rev. D 78, 074507 (2008) [arXiv:0804.3227]

- Effective potential for total quark number
- Simulations:
  - Bielefeld-Swansea Collab., PRD71,054508(2005).
  - 2-flavor p4-improved staggered quarks with $m_\pi \approx 770$MeV
  - 16^3x4 lattice
  - In det M: Taylor expansion up to $O(\mu^6)$

Canonical partition function

- Fugacity expansion (Laplace transformation)



$$Z_{GC}(T,\mu)=\sum_N \underline{\underline{Z_C(T,N)}}\exp(N\mu/T) \qquad \rho = N/V$$

canonical partition function

- Inverse Laplace transformation

$$Z_C(T,N)=\frac{3}{2\pi}\int_{-\pi/3}^{\pi/3} d(\mu_I/T)\,e^{-N(\mu_0/T+i\mu_I/T)}Z_{GC}(T,\mu_0+i\mu_I)$$

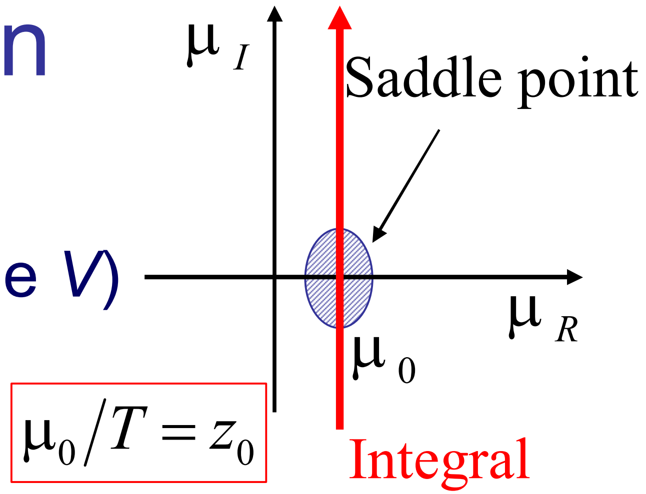
Arbitrary μ_0

$$\frac{Z_{GC}(\mu)}{Z_{GC}(0)}=\frac{1}{Z_{GC}(0)}\int DU(\det M(\mu))^{N_f}e^{-S_g}=\left\langle\left(\frac{\det M(\mu)}{\det M(0)}\right)^{N_f}\right\rangle_{\mu=0}$$

Integral path, e.g.
1, imaginary μ axis
2, Saddle point

  - Note: periodicity $Z_{GC}(T,\mu+2\pi iT/3)=Z_{GC}(T,\mu)$
  - det*M*(μ) : Quark determinant

Saddle point approximation



- Inverse Laplace transformation with
  Saddle point approximation (valid for large *V*)
  - Taylor expansion at the saddle point.

Saddle point: z_0 $$\left[\frac{N_f}{V}\frac{\partial(\ln\det M)}{\partial(\mu/T)} - \rho\right]_{\frac{\mu}{T}=z_0} = 0$$

$$\rho = N/V \qquad V \equiv N_s^3$$

- At low density: The saddle point and the quark determinant can be estimated from data of Taylor expansion around μ=0.

$$N_f \ln\det M(\mu) = N_f \sum_{n=0}^{\infty}\left[\frac{1}{n!}\left(\frac{\mu}{T}\right)^n \frac{d^n \ln\det M}{d(\mu/T)^n}\right] \equiv V N_f N_t \sum_{n=0}^{\infty}\left[D_n\left(\frac{\mu}{T}\right)^n\right]$$

- The calculation by the Taylor expansion is much easier than the exact calculation. ⟶ Large lattices: applicable.

Saddle point approximation

- Canonical partition function in a saddle point approximation

$$\frac{Z_C(T,\rho)}{Z_{GC}(T,0)} = \frac{3}{\sqrt{2\pi}} \left\langle \exp\left[N_f \ln\left(\frac{\det M(z_0)}{\det M(0)} \right) - V\rho z_0 \right] e^{-i\alpha/2} \sqrt{\frac{1}{V\left|R''(z_0)\right|}} \right\rangle_{(T,\mu=0)}$$

$$\equiv \frac{3}{\sqrt{2\pi}} \left\langle \exp\left(F + i\theta\right) \right\rangle_{(T,\mu=0)}$$

Saddle point: z_0

$$R''\left(\frac{\mu}{T}\right) = \frac{N_f}{V} \frac{\partial^2 (\ln \det M)}{\partial (\mu/T)^2} \equiv \left|R''\right| e^{i\alpha}$$

- Chemical potential

$$\frac{\mu^*(\rho)}{T} \equiv \frac{-1}{V} \frac{\partial \ln Z_C(T,\rho)}{\partial \rho} \approx \frac{\left\langle \underline{z_0}\ \underline{\exp\left(F + i\theta\right)} \right\rangle_{(T,\mu=0)}}{\left\langle \underline{\exp\left(F + i\theta\right)} \right\rangle_{(T,\mu=0)}}$$

saddle point

reweighting factor

⇨ Similar to the reweighting method

(sign problem & overlap problem)

Technical problem 1: Sign problem

- Complex phase of det*M*

$$(\text{phase}) = N_{\mathrm{f}}\, \mathrm{Im}\left[\ln \det M(\mu)\right]$$

  - Taylor expansion (Bielefeld-Swansea, PRD66, 014507 (2002))

$$\theta = \mathrm{Im}\left[V\left(N_{\mathrm{f}} N_{\mathrm{t}} \sum_{n=1}^{\infty} D_n z_0 - \rho z_0 \right)\right] - \frac{\alpha}{2}$$

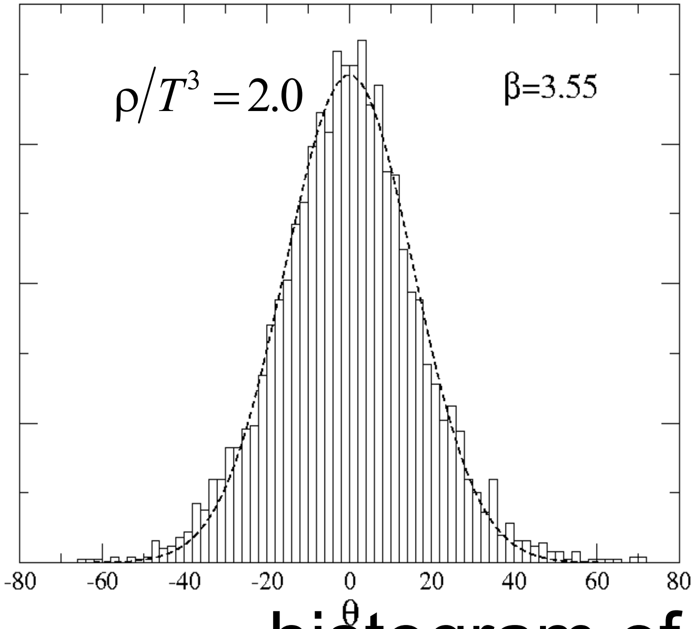
θ: NOT in the range [-π, π]

- |θ| > π/2: Sign problem happens.

  $e^{i\theta}$ changes its sign.

- Gaussian distribution
  - Taylor expansion up to $O(\mu^5)$
  - Dashed line: fit by a Gaussian function

Well approximated



histogram of θ

$$\boxed{W(\theta) \approx \sqrt{\frac{a}{\pi}}\, e^{-a\theta^2}} \Rightarrow \boxed{\left\langle e^{i\theta} e^{F} \right\rangle \approx \left\langle e^{-\langle \theta^2 \rangle_F / 2} e^{F} \right\rangle}$$

real and positive

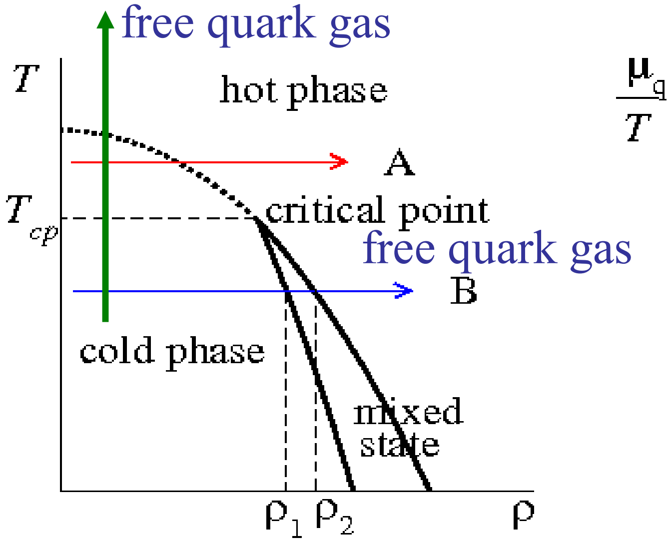
No sign problem

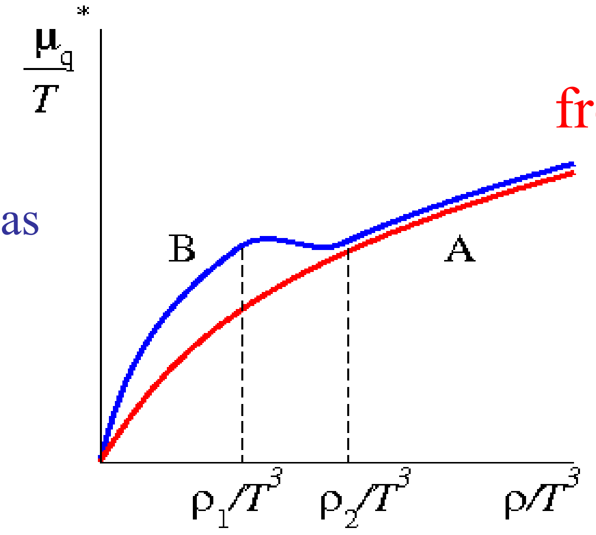
Technical problem 2: Overlap problem

Role of the weight factor $\exp(F+i\theta) \sim \exp(F-\langle\theta^2\rangle/2)$

- When the density increases, the position of the importance sampling changes.
- We combine all data generated at many T by multi-β reweighting

  (Ferrenberg-Swendsen, Phys.Rev.Lett.63,1195(1989))
  - Important configurations are selected automatically.
- Configurations at each T (μ=0)
  - $T<Tc$: Phase fluctuations: large, weight: small ➡ <u>suppressed at large ρ.</u>
  - $T>Tc$: Phase fluctuations: small, weight: large ➡ <u>enhanced at large ρ.</u>
- μ^*/T approaches the free quark gas value in the high density limit for all T.





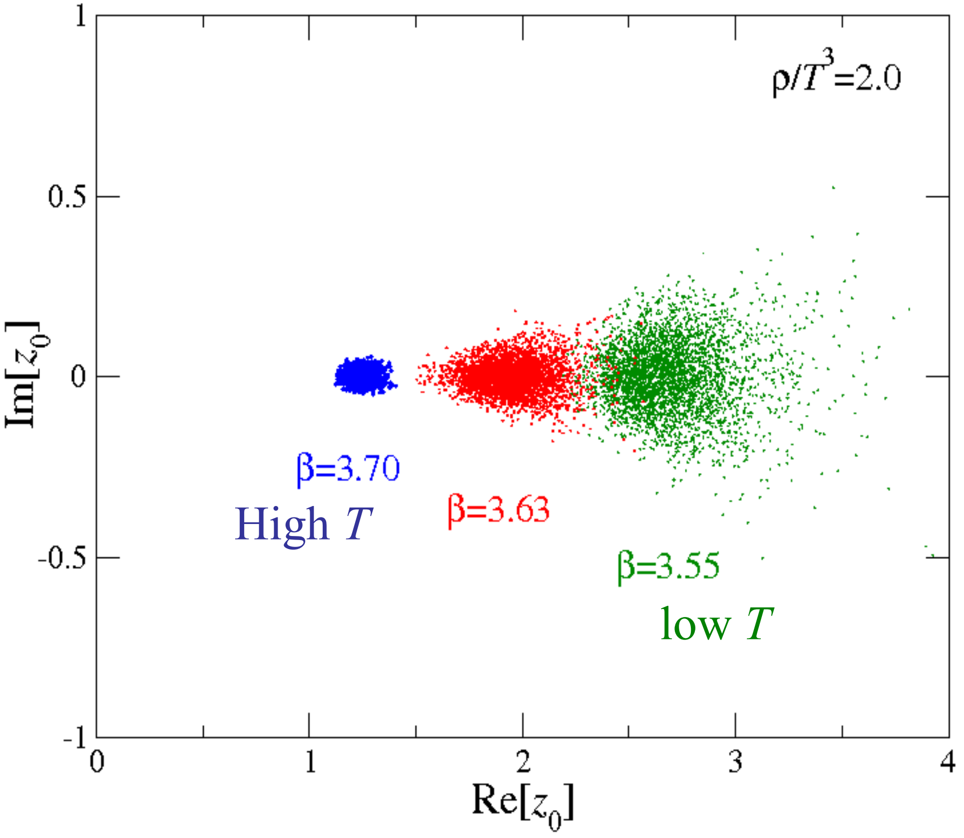
free quark gas

$$\frac{\rho}{T^3} = N_f\left[\frac{\mu}{T} + \frac{1}{\pi^2}\left(\frac{\mu}{T}\right)^3\right]$$

Saddle point in complex μ/T plane

- Find a saddle point z_0 numerically for each conf.

$$\left[\frac{N_f}{V}\frac{\partial(\ln\det M)}{\partial(\mu/T)}-\rho\right]_{\frac{\mu}{T}=z_0}=0$$



Chemical potential vs density

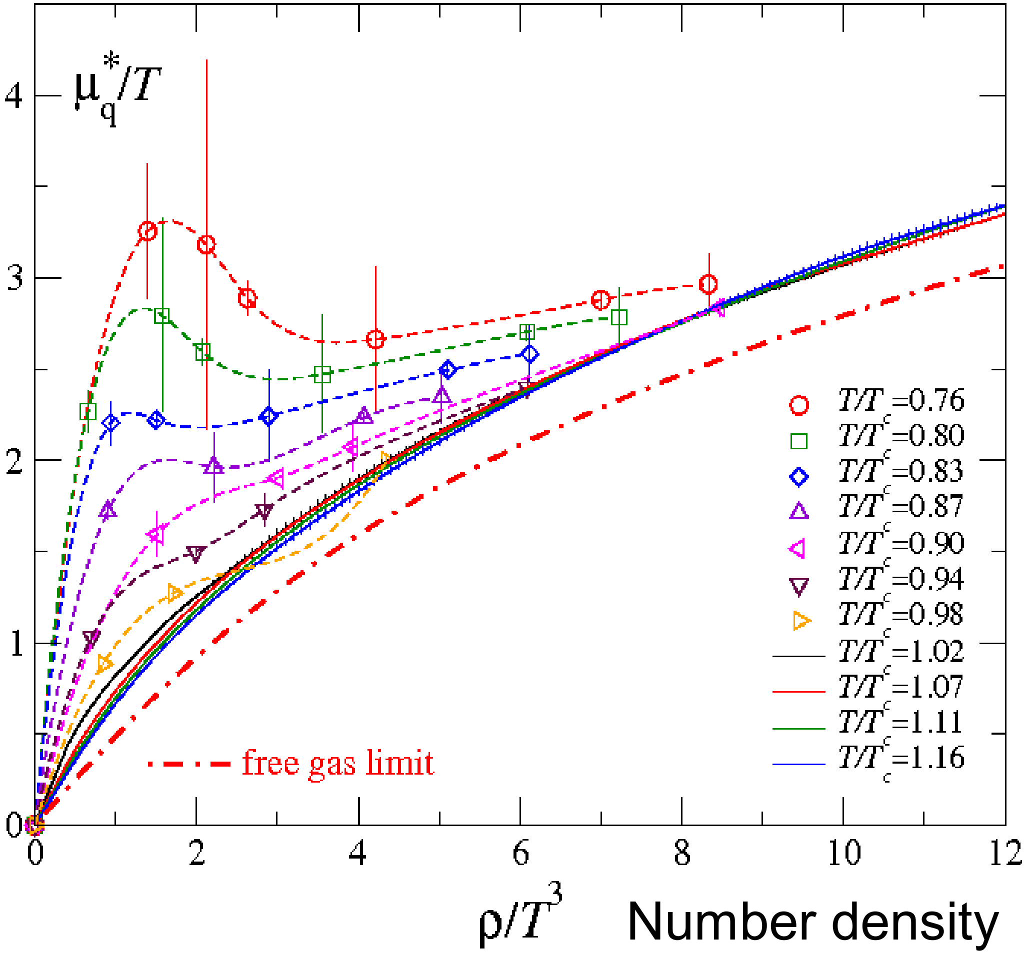
- Approximations:
  - Taylor expansion: ln det *M*
  - Gaussian distribution: θ
  - Saddle point approximation

- Two states at the same μ_q/T
  - First order transition at $T/T_c < 0.83$, $\mu_q/T > 2.3$

- μ^*/T approaches the free quark gas value in the high density limit for all *T*.

- Solid line: multi-b reweighting
- Dashed line: spline interpolation
- Dot-dashed line: the free gas limit

$N_f=2$ p4-staggered, $m_\pi \approx 770$MeV, $16^3\times 4$ lattice



Number density

•Studies near physical quark mass: important.

まとめ

- 有限温度・密度格子QCDの最近の成果についてまとめた。
- ボルツマンの重みが複素数であることの問題を避けるために、確率分布関数のようなものを定義して、議論した。
- 符号問題を避けるために、複素位相の部分をキュムラント展開して、符号問題を展開の収束の問題に置き換えた。体積を大きくしても、その展開の収束性が悪くならないことが期待できる。
- クォーク数（バリオン数）を固定した計算法を議論した。そこで得られるバリオン数に関する確率分布関数は、重イオン衝突実験のイベントごとの解析で得られる確率分布関数と直接関係つくかもしれない。