Higher-order perturbation theory for highly-improved actions

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Lattice 2003

Widespread use of highly-improved actions

- *perturbative* and non-perturbative

To demonstrate reliability of perturbation theory &
to achieve few % precision required by *e.g.*
b-physics experiments,
we must calculate through second-order in PT

$$\alpha_V(1/a) \approx a\Lambda_{\mathrm{QCD}} \approx 0.2 - 0.3$$

Requires an ambitious perturbative program

- coupling constants
- action parameters
- matrix elements

Our eventual goal:

Two-loop perturbation theory for most highly-improved action

$$S_{\rm Glue}^{\rm Symanzik} = \beta_{\rm pl} \sum_{x;\mu<\nu} (1 - P_{\mu\nu})$$

$$+\beta_{\rm rt} \sum_{x;\mu\neq\nu} (1 - R_{\mu\nu}) + \beta_{\rm pg} \sum_{x;\mu<\nu<\sigma} (1 - C_{\hat{\mu},\pm\hat{\nu},\pm\hat{\sigma}})$$

$$\beta_{\rm pl} = \frac{10}{g^2}, \quad \beta_{\rm rt} = -\frac{\beta_{\rm pl}}{20 u_{0,P}^2}(1 + 0.4805\alpha_s), \quad \beta_{\rm pg} = -\frac{\beta_{\rm pl}}{u_{0,P}^2} 0.03325\alpha_s$$

$$u_{0,P} = (W_{1,1})^{1/4} \quad \alpha_s \equiv -4\ln(u_{0,P})/3.0684.$$

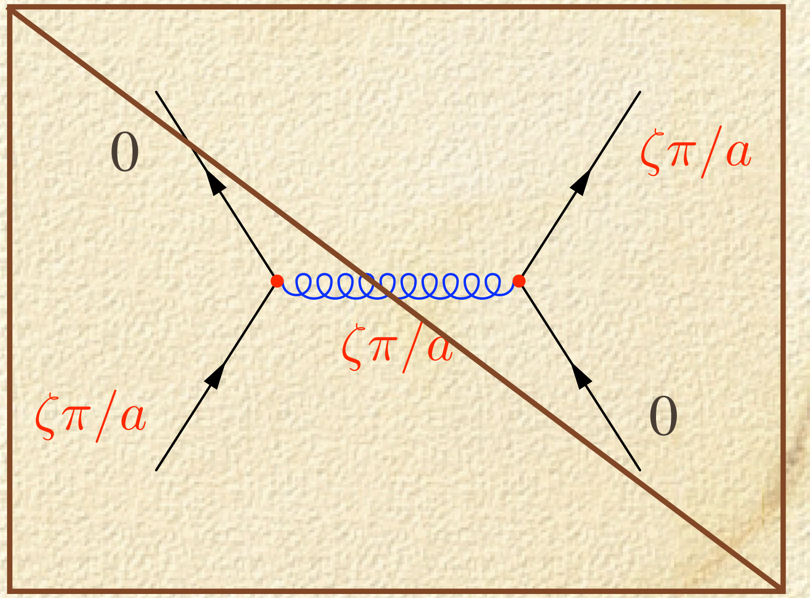
Used by MILC unquenched

$$S^{\mathrm{Imp}}_{\mathrm{Stagg}} = \sum_x \bar{\psi}(x) \left(\gamma \cdot \Delta' - \frac{a^2}{6} \gamma \cdot \Delta^3 + m \right) \psi(x)$$

$$\Delta'_\mu \psi(x) = \frac{1}{2a} \Big[V'_\mu(x) \psi(x + a\hat{\mu}) - V'_\mu(x - a\hat{\mu}) \psi(x - a\hat{\mu}) \Big],$$

$$\vdots$$

$$V_\mu = \prod_{\rho \neq \mu} \left(\frac{1 + a^2 \Delta^{(2)}_\rho}{4} \right) \Bigg|_{\mathrm{symm.}} U_\mu .$$



Tree-level taste-changing interactions

G.P. Lepage, Phys. Rev. D 59, 074502 (1999)

"Cures" bad perturbation theory of unimproved staggered fermions (quark tadpoles)

Sharpe & Lee - one-loop currents and four-quark operators

- also for HYP smearing (A. Hasenfratz)

Outline of Talk

- Automated methods for higher-order PT
- Application to two-loop $\alpha_V(q^*)$,
  three-loop Wilson loops
- Other quantities calculated / underway
- Numerical perturbation theory
- Preliminary investigation third-order $\alpha_{\overline{MS}}(M_Z)$

HPQCD Perturbation Theory "Subgroup"

- Quentin Mason
- G.P. Lepage

- C. Davies, A. Gray

- M. Nobes, K. Wong, R. Woloshyn

- A. El-Khadra, A. Kronfeld, P. Mackenzie, B. Oktay

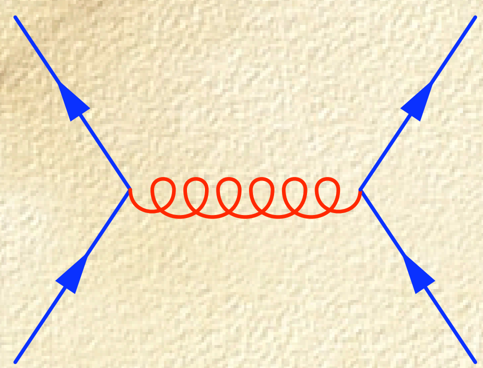
- J. Shigemitsu, E. Gulez, M. Wingate

- I.T. Drummond, A. Hart, R.R. Horgan, L.C. Storoni

Perturbation Theory: We need the Feynman rules

$$-4\pi C_F \frac{\alpha_V(q^2)}{q^2}$$

=



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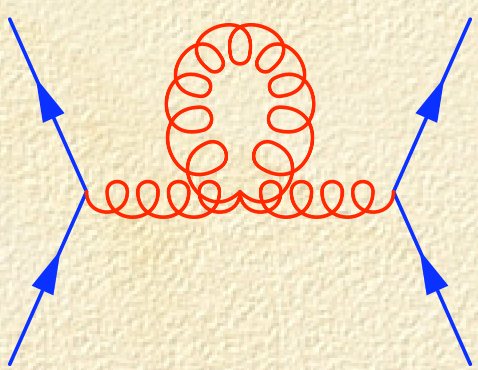

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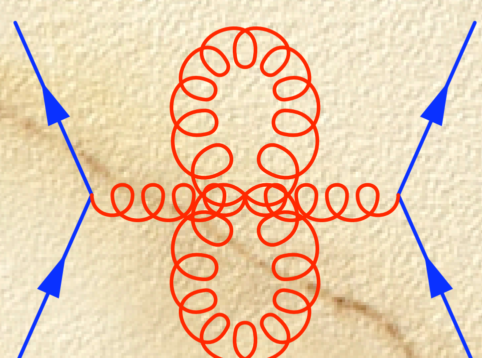

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Needless to say, the calculation of the four gluon vertex (see fig. on page 212) from the fourth order contribution in θ_i^A to the effective action is quite tedious and we shall not present it here. The expression is very lengthy and has been given in the appendix of the paper by Kawai et al. (1981):*

$$
\begin{aligned}
\Gamma^{ABCD}_{\mu\nu\lambda\rho}(p,q,r,s) = \\
& - g^2 f_{ABE} f_{CDE} \Big\{ \delta_{\mu\lambda}\delta_{\nu\rho} \left[\cos\frac{a(q-s)_\mu}{2} \cos\frac{a(p-r)_\nu}{2} - \frac{a^4}{12}\hat{p}_\nu \hat{q}_\mu \hat{r}_\nu \hat{s}_\mu \right] \\
& - \delta_{\mu\rho}\delta_{\nu\lambda} \left[\cos\frac{a(q-r)_\mu}{2} \cos\frac{a(p-s)_\nu}{2} - \frac{a^4}{12}\hat{p}_\nu \hat{q}_\mu \hat{r}_\mu \hat{s}_\nu \right] \\
& + \frac{a^2}{12}\delta_{\mu\nu}\delta_{\mu\lambda}\delta_{\mu\rho} \sum_\sigma (\hat{q}_\sigma e^{-i\frac{a}{2}p_\sigma} - \hat{p}_\sigma e^{-i\frac{a}{2}q_\sigma})(\hat{s}_\sigma e^{-i\frac{a}{2}r_\sigma} - \hat{r}_\sigma e^{-i\frac{a}{2}s_\sigma}) \\
& - \frac{a^2}{6}\delta_{\mu\nu}\delta_{\mu\lambda}(\hat{q}_\rho e^{-i\frac{a}{2}p_\rho} - \hat{p}_\rho e^{-i\frac{a}{2}q_\rho})\hat{s}_\mu \cos\frac{a r_\rho}{2} \\
& + \frac{a^2}{6}\delta_{\mu\nu}\delta_{\mu\rho}(\hat{q}_\lambda e^{-i\frac{a}{2}p_\lambda} - \hat{p}_\lambda e^{-i\frac{a}{2}q_\lambda})\hat{r}_\mu \cos\frac{a s_\lambda}{2} \\
& - \frac{a^2}{6}\delta_{\mu\lambda}\delta_{\mu\rho}(\hat{s}_\nu e^{-i\frac{a}{2}r_\nu} - \hat{r}_\nu e^{-i\frac{a}{2}s_\nu})\hat{q}_\mu \cos\frac{a p_\nu}{2} \\
& + \frac{a^2}{6}\delta_{\nu\lambda}\delta_{\nu\rho}(\hat{s}_\mu e^{-i\frac{a}{2}r_\mu} - \hat{r}_\mu e^{-i\frac{a}{2}s_\mu})\hat{p}_\nu \cos\frac{a q_\mu}{2} \Big\} \\
& + (B \leftrightarrow C, \nu \leftrightarrow \lambda, q \leftrightarrow r) + (B \leftrightarrow D, \nu \leftrightarrow \rho, q \leftrightarrow s) \\
& + g^2 \frac{a^4}{12} \Big\{ \frac{2}{3}(\delta_{AB}\delta_{CD} + \delta_{AC}\delta_{BD} + \delta_{AD}\delta_{BC}) \\
& + (d_{ABE}d_{CDE} + d_{ACE}d_{BDE} + d_{ADE}d_{BCE}) \Big\} \Big\{ \delta_{\mu\nu}\delta_{\mu\lambda}\delta_{\mu\rho} \sum_\sigma \hat{p}_\sigma \hat{q}_\sigma \hat{r}_\sigma \hat{s}_\sigma \\
& - \delta_{\mu\nu}\delta_{\mu\lambda}\hat{p}_\rho \hat{q}_\rho \hat{r}_\rho \hat{s}_\mu - \delta_{\mu\nu}\delta_{\mu\rho}\hat{p}_\lambda \hat{q}_\lambda \hat{s}_\lambda \hat{r}_\mu \\
& - \delta_{\mu\lambda}\delta_{\mu\rho}\hat{p}_\nu \hat{r}_\nu \hat{s}_\nu \hat{q}_\mu - \delta_{\nu\lambda}\delta_{\nu\rho}\hat{q}_\mu \hat{r}_\mu \hat{s}_\mu \hat{p}_\nu \\
& + \delta_{\mu\nu}\delta_{\lambda\rho}\hat{p}_\lambda \hat{q}_\lambda \hat{r}_\mu \hat{s}_\mu + \delta_{\mu\lambda}\delta_{\nu\rho}\hat{p}_\nu \hat{r}_\nu \hat{q}_\mu \hat{s}_\mu + \delta_{\mu\rho}\delta_{\nu\lambda}\hat{p}_\nu \hat{s}_\nu \hat{q}_\mu \hat{r}_\mu \Big\}
\end{aligned}
\tag{14.44}
$$

* The expression given in the above reference is however not completely correct. We give here the corrected form which was provided to us by W. Wetzel.

H. Rothe, LGT: An Introduction

There exists a class of remarkably simple automated algorithms:

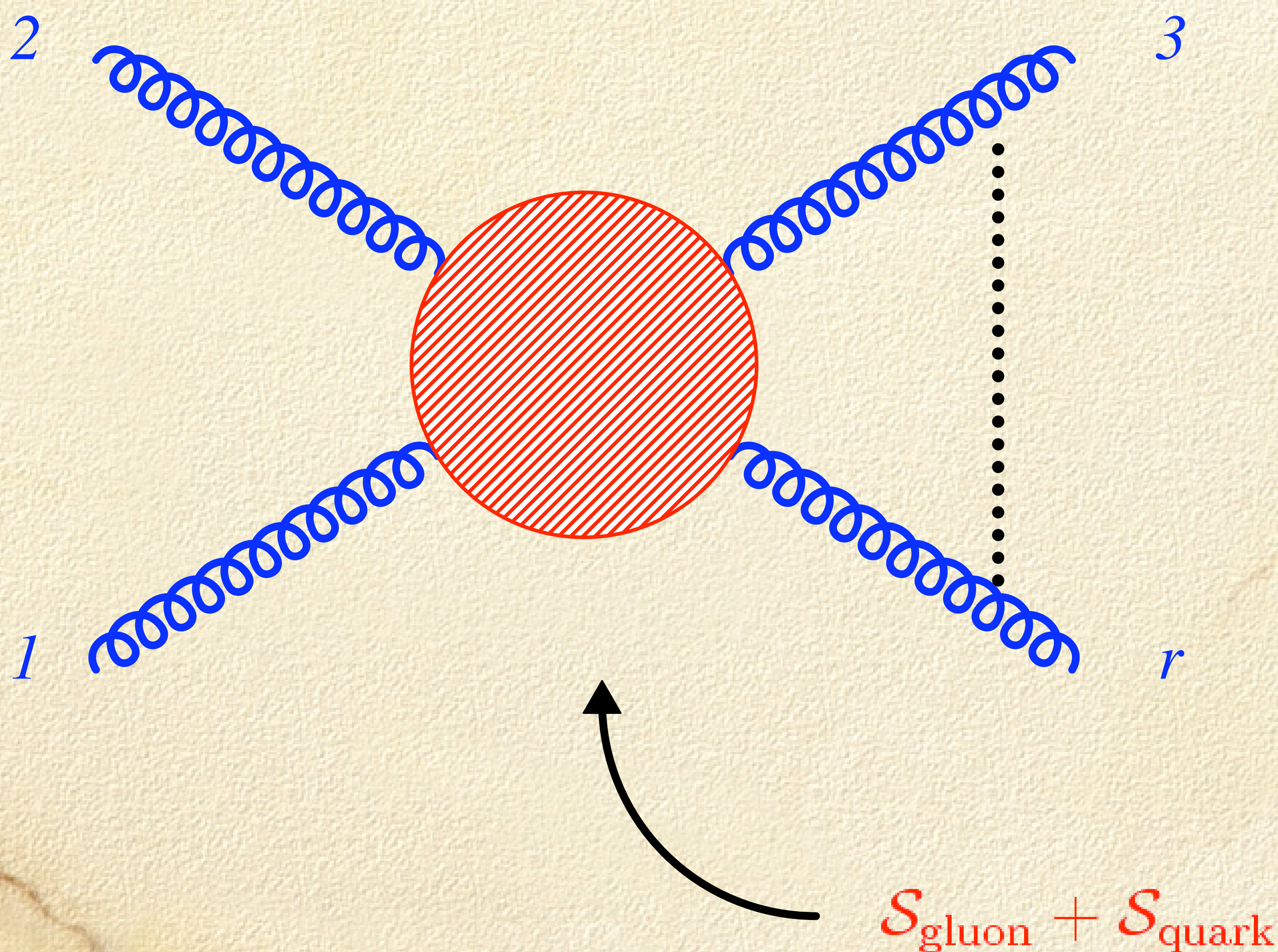
- entirely symbolic/numeric manipulation
- generate Feynman rules for essentially arbitrary lattice actions

- M. Lüscher and P. Weisz, Nucl. Phys. B266, 309 (1986).

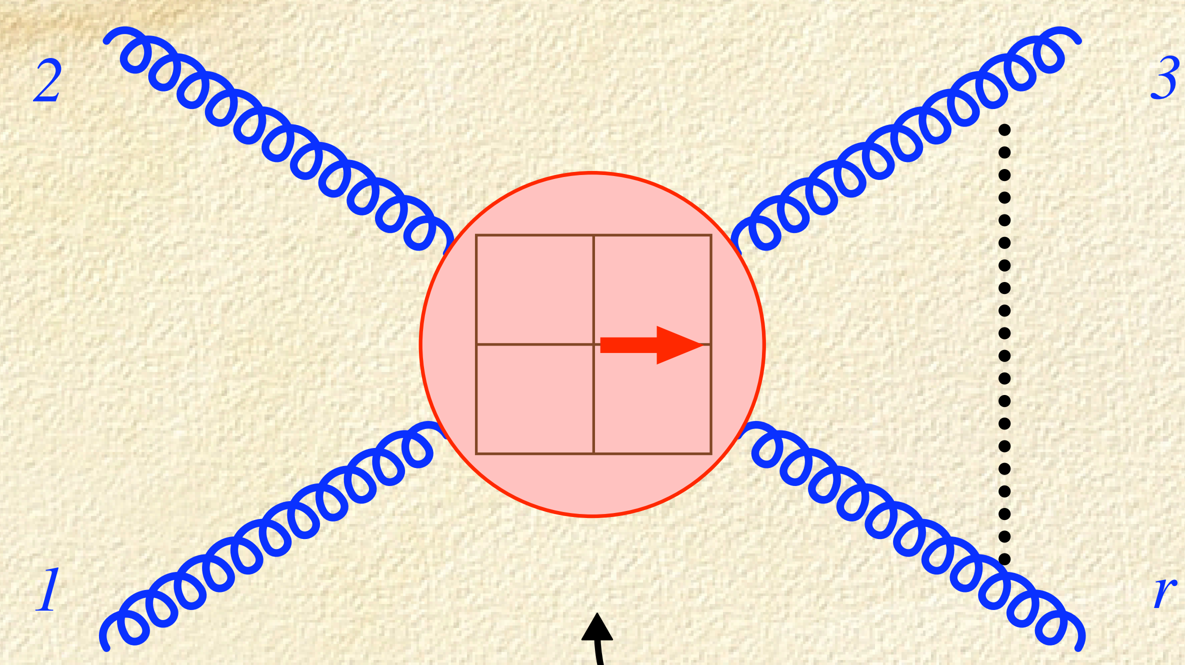
- C. Morningstar, Phys. Rev. D48, 2265 (1993).
- B. Allés, M. Campostrini, A. Feo, H. Panagopoulos, Nucl. Phys. B413, 553 (1994).
- S. Capitani, G. Rossi, hep-lat/9504014; hep-lat/0211036.

$$V\left(\begin{Bmatrix} k_1 \\ \mu_1 \\ a_1 \end{Bmatrix}, \begin{Bmatrix} k_2 \\ \mu_2 \\ a_2 \end{Bmatrix}, \ldots, \begin{Bmatrix} k_r \\ \mu_r \\ a_r \end{Bmatrix}\right)$$

$$\equiv$$



Simplest Case:
Gluon "action" with links in a ***single*** direction



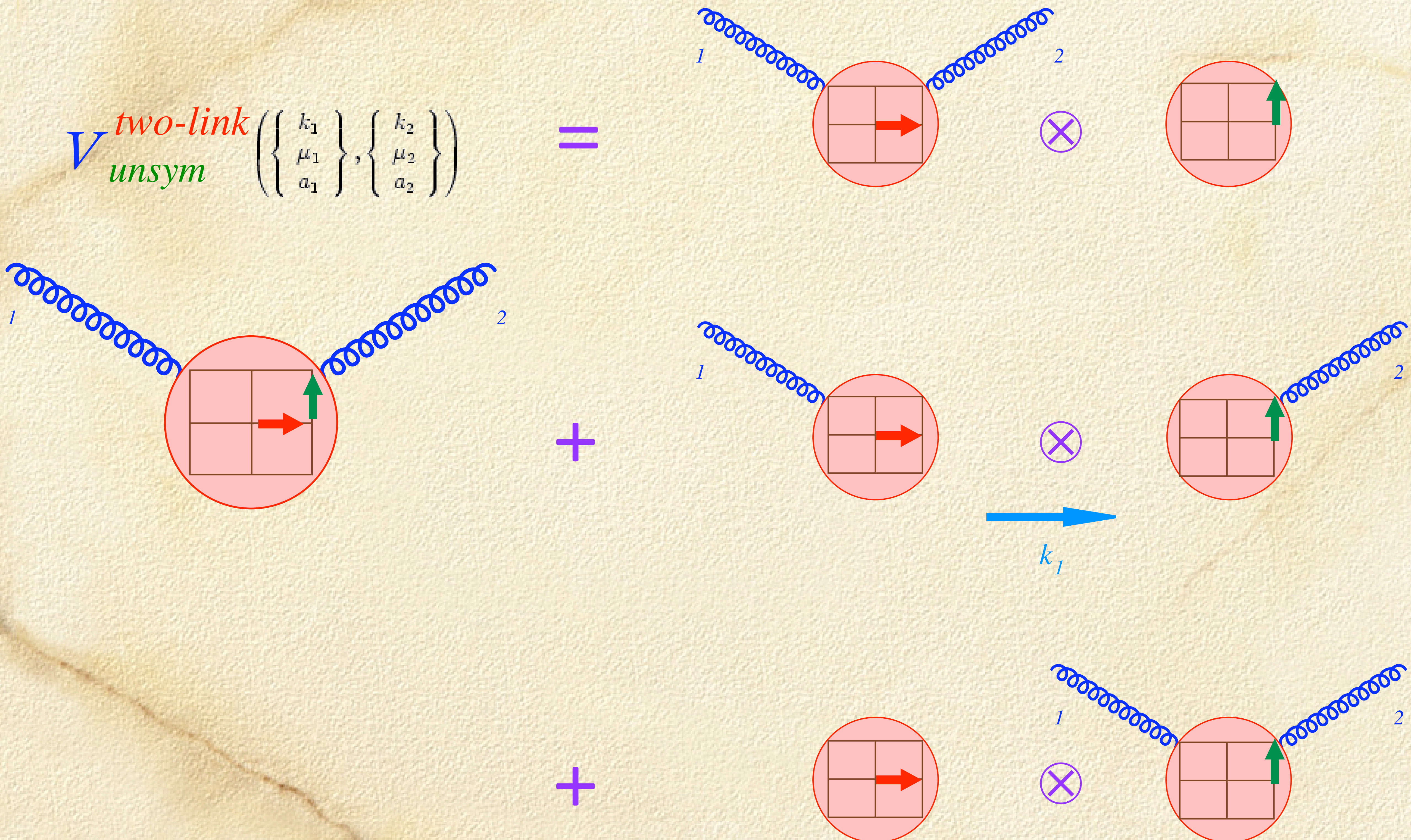
$$\mathcal{S} = \sum_x U_\ell(x) = \sum_x e^{A_{\hat{\ell}}(x+\frac{1}{2}a_\ell)}$$

$$V^{link}_{unsym}\left(\left\{\begin{array}{c}k_1\\ \mu_1\\ a_1\end{array}\right\},\left\{\begin{array}{c}k_2\\ \mu_2\\ a_2\end{array}\right\},\ldots,\left\{\begin{array}{c}k_r\\ \mu_r\\ a_r\end{array}\right\}\right) = (2\pi)^4\delta(k_1+k_2+\ldots+k_r=k_{\rm tot})\times$$

$$\frac{1}{r!}\delta_{\hat{\mu}_1=\hat{\mu}_2=\ldots=\hat{\mu}_r=\hat{\ell}} \times \quad e^{i(k_1\cdot\frac{a_\ell}{2}+k_2\cdot\frac{a_\ell}{2}+\ldots+k_r\cdot\frac{a_\ell}{2})} \times \quad (T^{a_1}T^{a_2}\ldots T^{a_r})$$

Handle arbitrarily complicated actions by convolution

$$V^{\textit{two-link}}_{\textit{unsym}}\left(\begin{Bmatrix} k_1 \\ \mu_1 \\ a_1 \end{Bmatrix}, \begin{Bmatrix} k_2 \\ \mu_2 \\ a_2 \end{Bmatrix}\right) =$$



Apply same algorithm to any gluon / quark action

$$\mathcal{L}_{\text{Heavy Q}} = \bar{\psi}\left(1 + \frac{\Delta^{(2)}}{4nM_Q^0}\right)^n U_4^\dagger \left(1 + \frac{\Delta^{(2)}}{4nM_Q^0}\right)^n (1 - \delta H)\,\psi$$

$$\delta H = \ldots - c_3 \frac{g}{8(M_Q^0)^2}\boldsymbol{\sigma}\cdot(\boldsymbol{\Delta}\times\mathbf{E} - \mathbf{E}\times\boldsymbol{\Delta}) - c_4\frac{g}{2M_Q^0}\boldsymbol{\sigma}\cdot\mathbf{B} + \ldots$$

- Convolute U_μ's to get Feynman rules for $\boldsymbol{\Delta}$, $\mathbf{E}, \ldots$
- Convolute $\boldsymbol{\Delta}$, $\mathbf{E}, \ldots$ to get rules for $\mathcal{L}_{\text{Heavy Q}}$

Major effort over past year:

Third-order determination of $\alpha_{\overline{\mathrm{MS}}}(M_Z)$ from unquenched simulations (MILC)

A calculation in four parts

$\alpha_{\overline{MS}} \leftrightarrow \alpha_{\mathrm{lat}}$ through two-loops

- Gluonic loops
- Fermionic loops

Wilson loops through *three*-loops

- Gluonic loops
- Fermionic loops

Background-field method

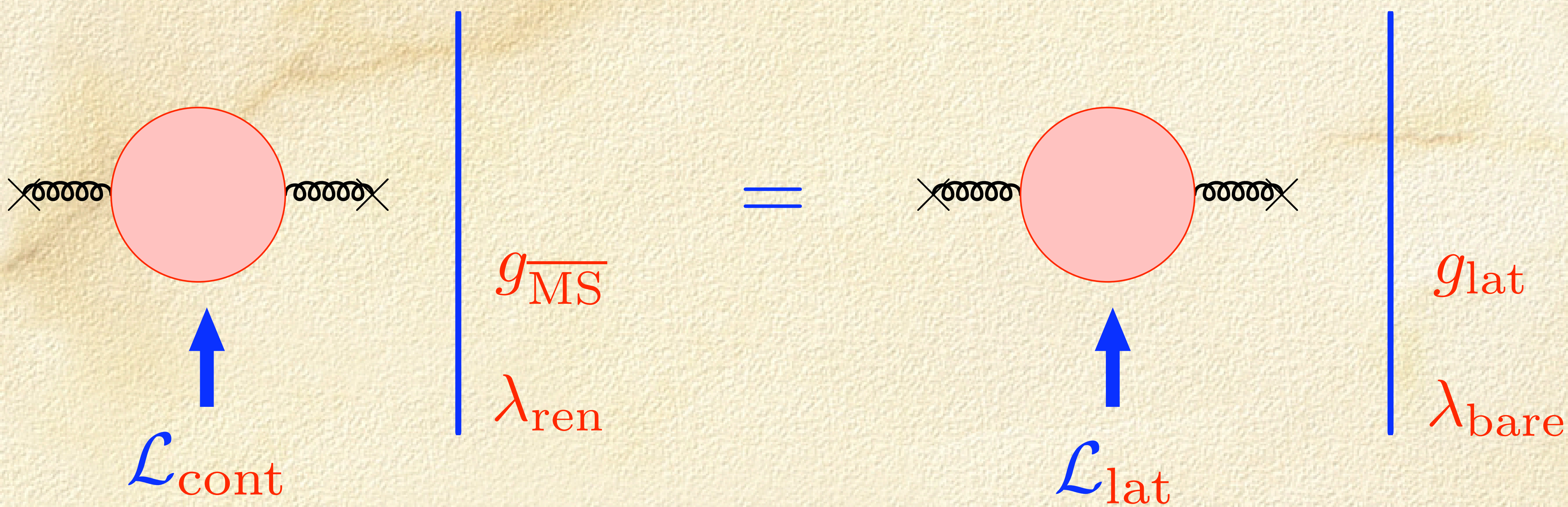
$\alpha_{\overline{MS}} \leftrightarrow \alpha_{\rm lat}$ through two-loops

- M. Lüscher and P. Weisz, Nucl. Phys. B 452, 234 (1995);
  Unimproved Wilson gluon action - gluon loops

- C. Christou, A. Feo, H. Panagopoulos, E. Vicari;
  A. Bode and H. Panagopoulos,
  Nucl. Phys. B 525, 387 (1998); B 625, 198 (2002);
  Unimproved Wilson & Clover Fermion quark actions - quark loops

- Trottier, Mason and Lepage, Lattice 2003
  Improved gluon action and improved staggered;
  Also unimproved staggered & checked unimproved Wilson.

Background-Field Matching



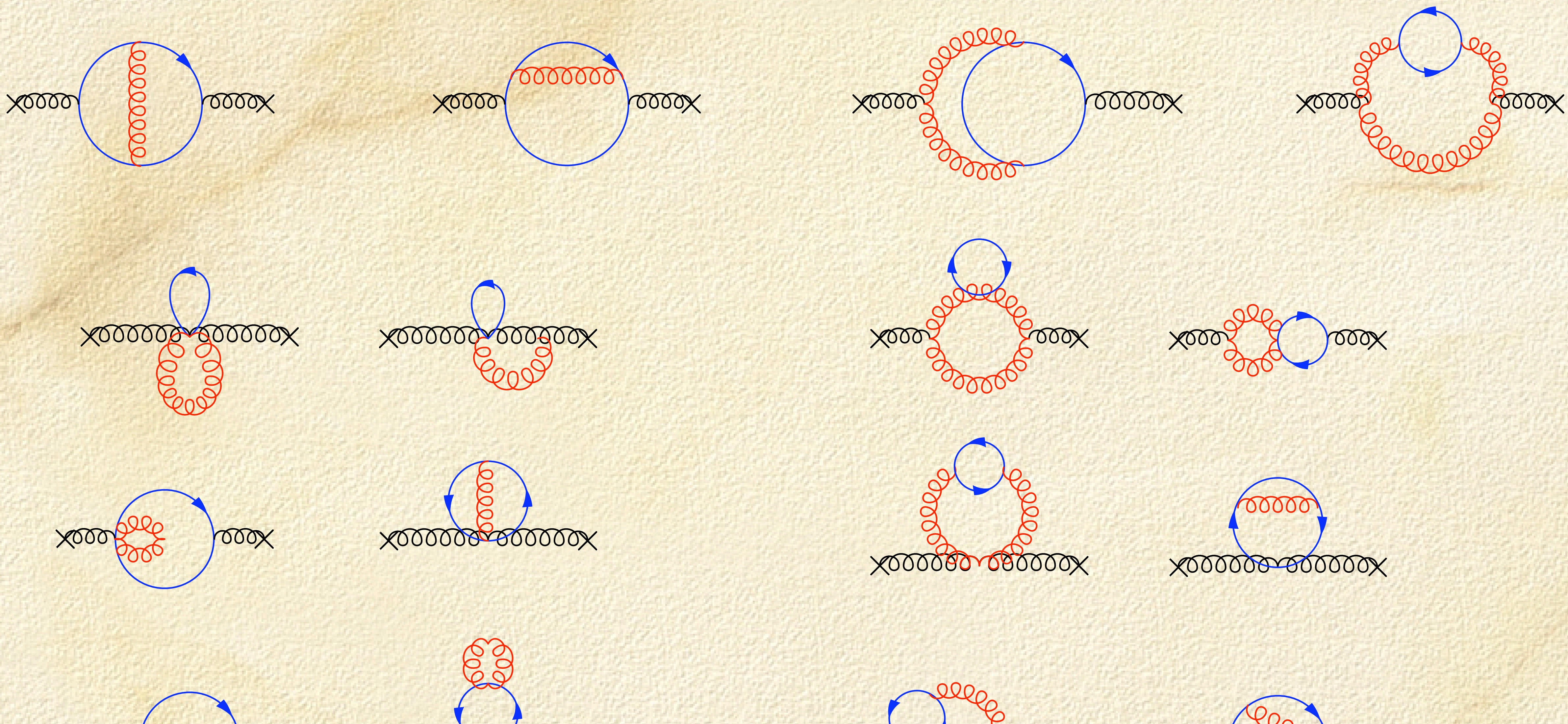
$$g_{\overline{\mathrm{MS}}} = \mathcal{Z}_g \, g_{\mathrm{lat}}$$

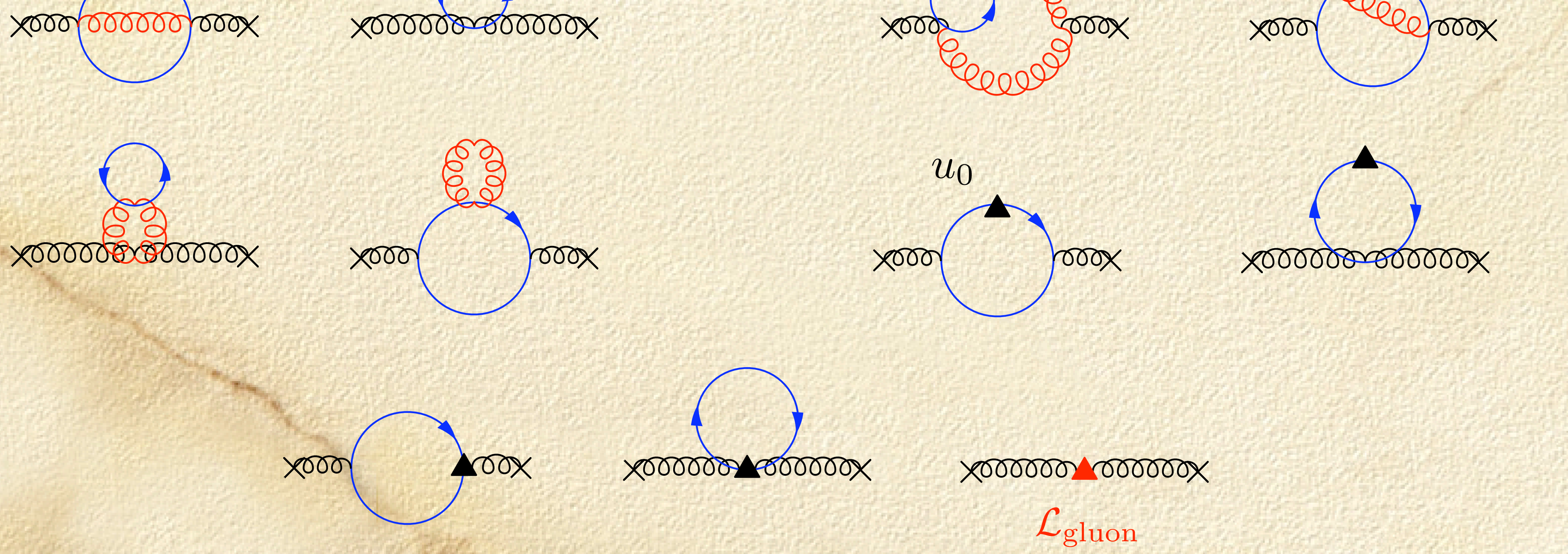
(gauge parameter) $$\lambda_{\mathrm{ren}} = \mathcal{Z}_3 \, \lambda_{\mathrm{bare}}$$

Continuum - Gluonic: K. Ellis (1984); L&W, van de Ven (1995)

- Fermionic: Panagopoulos et al. (1998); HDT et al. (current)

Lattice Two Loop N_f portion of $\alpha_V(q^*) = \alpha_{\text{lat}} + \ldots$





&
31
gluonic
two-loop
diagrams
&
various
one-loop
diagrams

Results

$$\alpha_{\text{lat}} = \alpha_V(q)\left[1 - v_1(q)\alpha_V(q) - v_2(q)\alpha_V^2(q)\right] + O\left(\alpha_V^4\right)$$

$$v_1(q) = \frac{\beta_0}{4\pi}\ln\left(\frac{\pi}{aq}\right)^2 + v_{1,0} \qquad v_2(q) = \frac{\beta_1}{16\pi^2}\ln\left(\frac{\pi}{aq}\right)^2 - \left[v_1(q^2)\right]^2 + v_{2,0}$$

Unimproved Staggered Quarks

$$v_{1,0} = 4.70181 + 4.81(64)\times 10^{-5} \qquad v_{2,0} = 9.52806 - 0.5440(65)N_f$$

Unimproved Wilson Quarks

$$v_{1,0} = 4.70181 - 0.05110423(6)\,N_f \qquad v_{2,0} = 9.52806 - 0.67697(81)N_f$$

MILC - Improved Glue & Improved Staggered

$$v_{1,0} = 3.57123(17) - 1.196(53)\times 10^{-4}\,N_f \qquad v_{2,0} = 5.382(39) - 1.0511(51)N_f$$

Short-distance Wilson loops

- Third order expansion means we must calculate through three loops!
- We extend a clever trick of Panagopoulos et al. to reduce # diagrams:

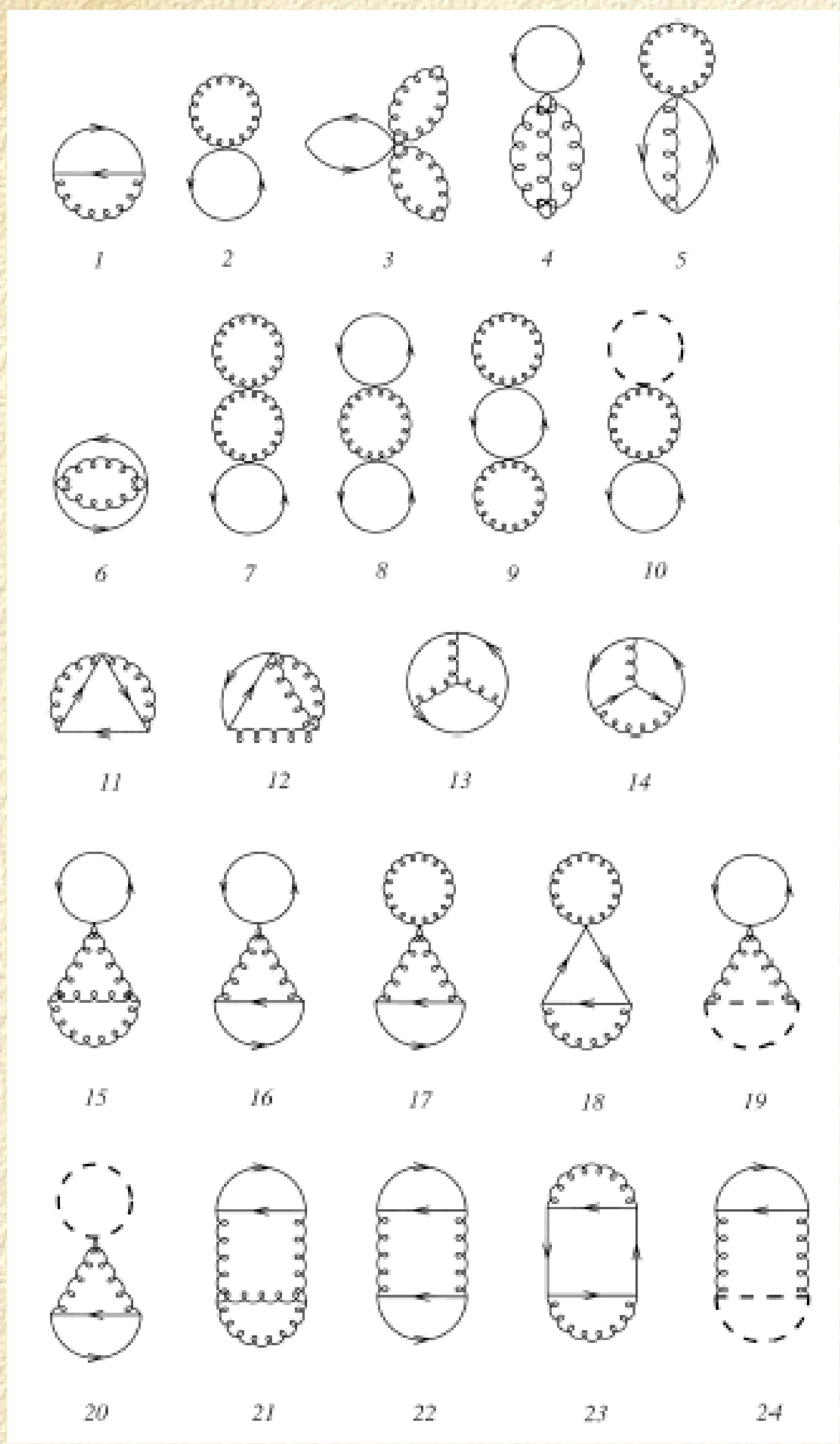
Feynman Rules

$$\langle W_{R,T} \rangle \Big|_{S_{\rm lat}} = \frac{\partial}{\partial \rho} \langle 1 \rangle \Big|_{S_{\rm lat} + \rho W_{R,T}}$$

  - vacuum-to-vacuum susceptibility has < 1/2 diagrams

- Panagopoulos et al.: average plaquette for unimproved glue and unimproved Wilson quarks
- We've done several Wilson loops - improved action (also did: unimproved naive & checked Wilson quarks)

Three-loop fermionic diagrams



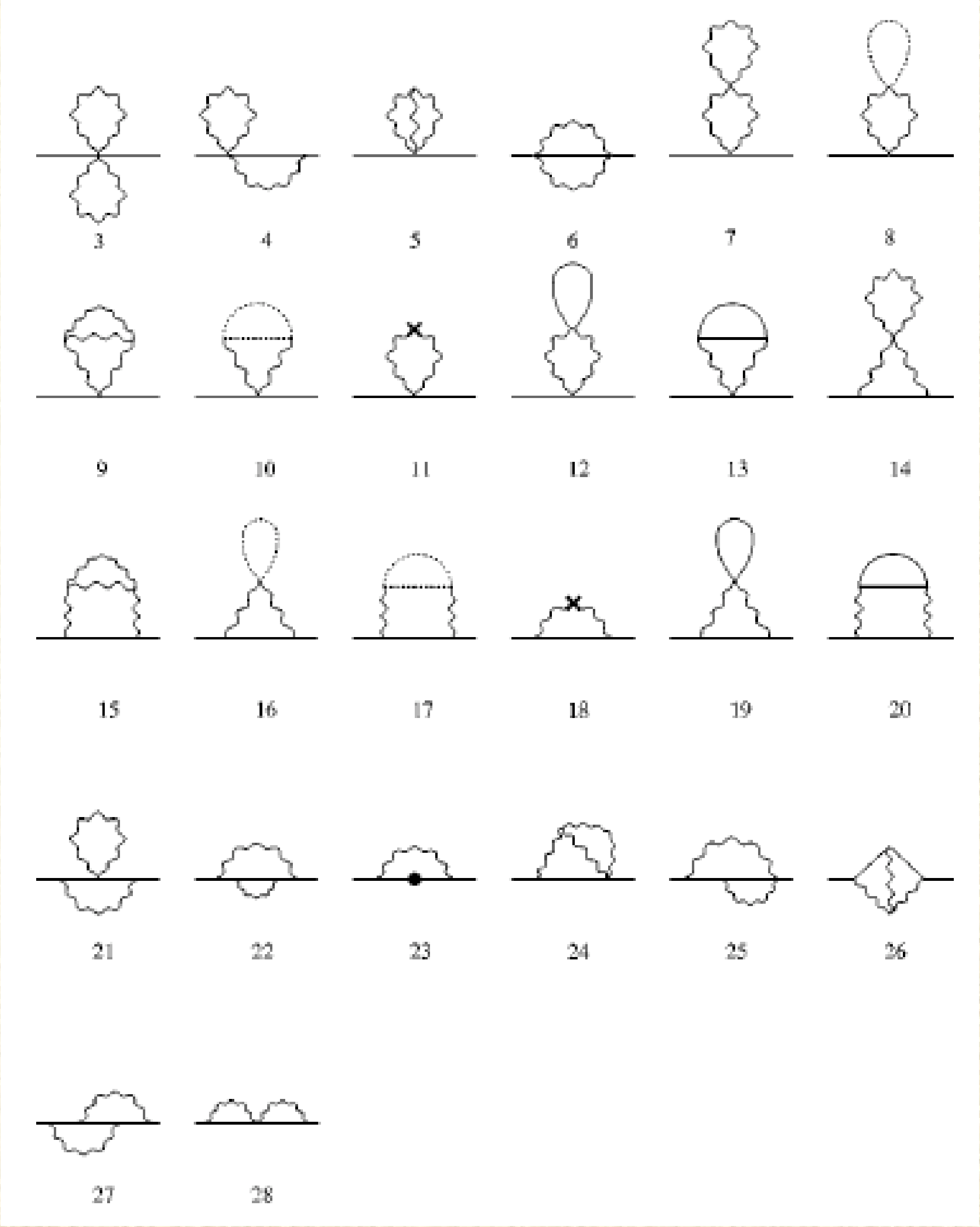
+ 11 two-loop including tadpole counterterms

Plus comparable number of three-loop gluonic diagrams

Some other work in progress

Two-loop kinetic mass renormalization

Q. Mason & HDT



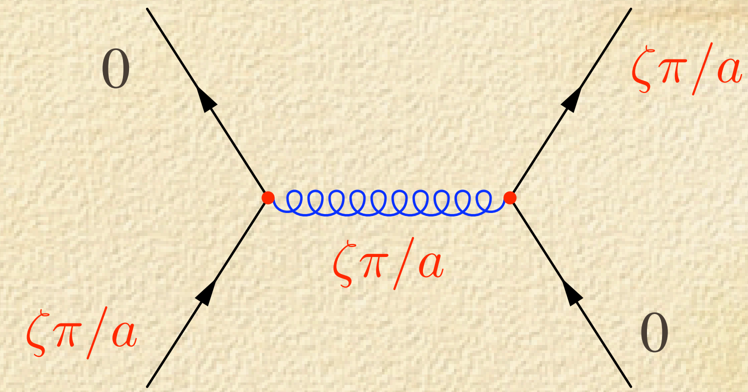
+ tadpole counterterms

Additive mass renormalization for Wilson quarks
E. Follana and H. Panagopoulos, PRD 63 (2000)

Taste-changing Interactions @ One-Loop

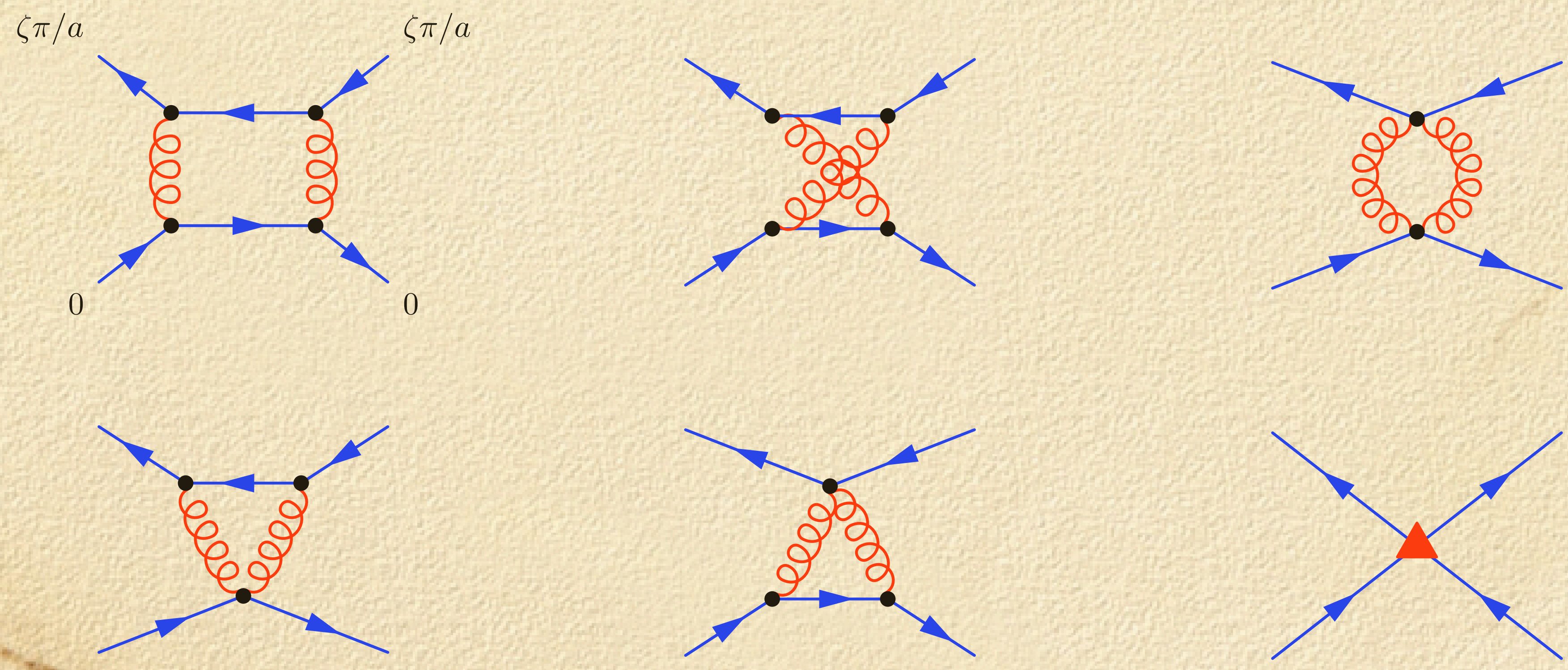
Perturbation Theory: Q. Mason (poster), G. P. Lepage & HDT

Improved staggered action (Lepage)
eliminates
tree-level taste-changing interactions



$= \; 0$

One-Loop $=$

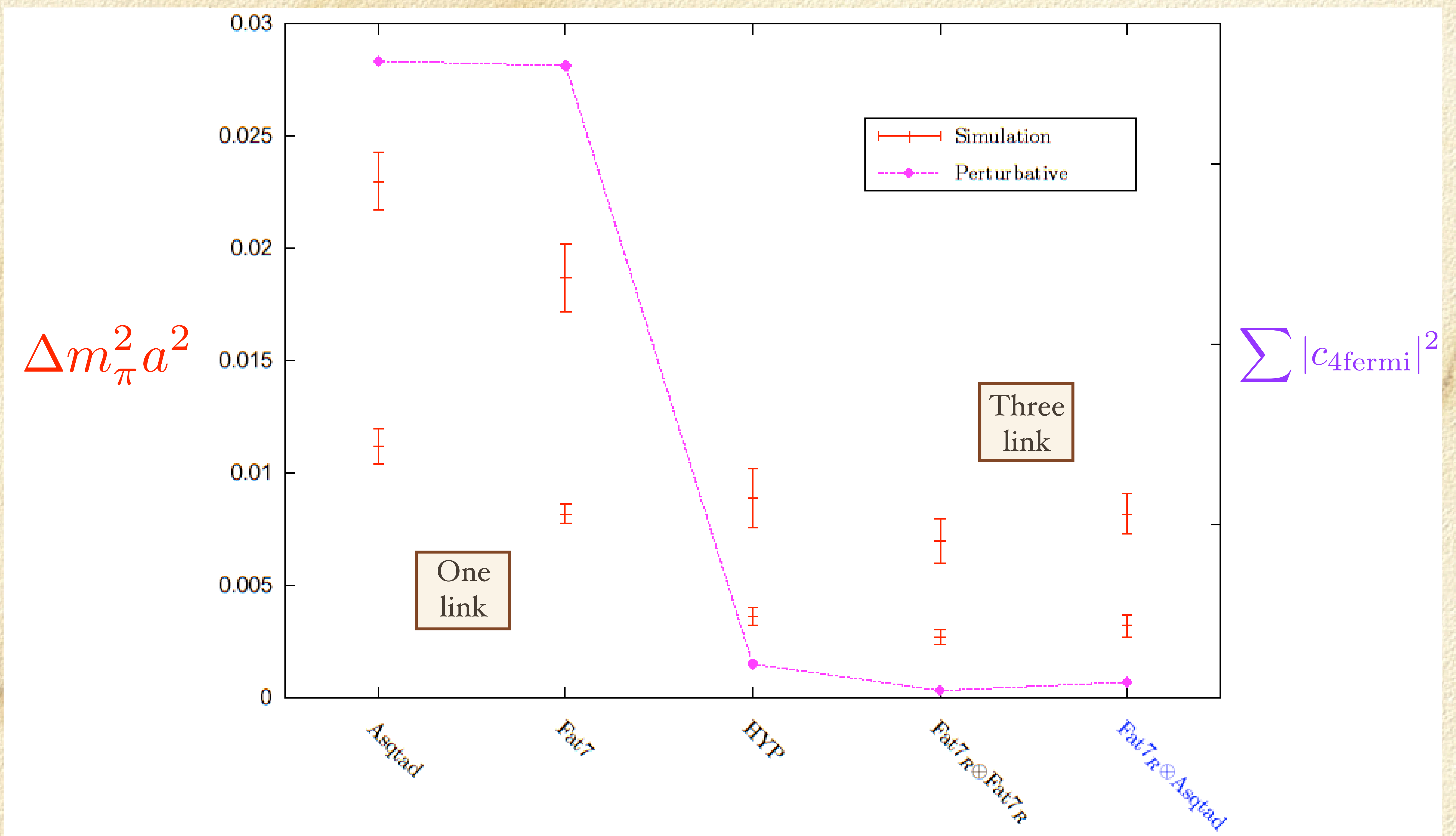


$\mathcal{L}_{\rm 4fermi}$

$\equiv \; 0$

Quenched Simulations vs. Perturbation Theory

E. Follana (talk), C. Davies, A. Gray, P. Lepage, & K. Hornbostel:

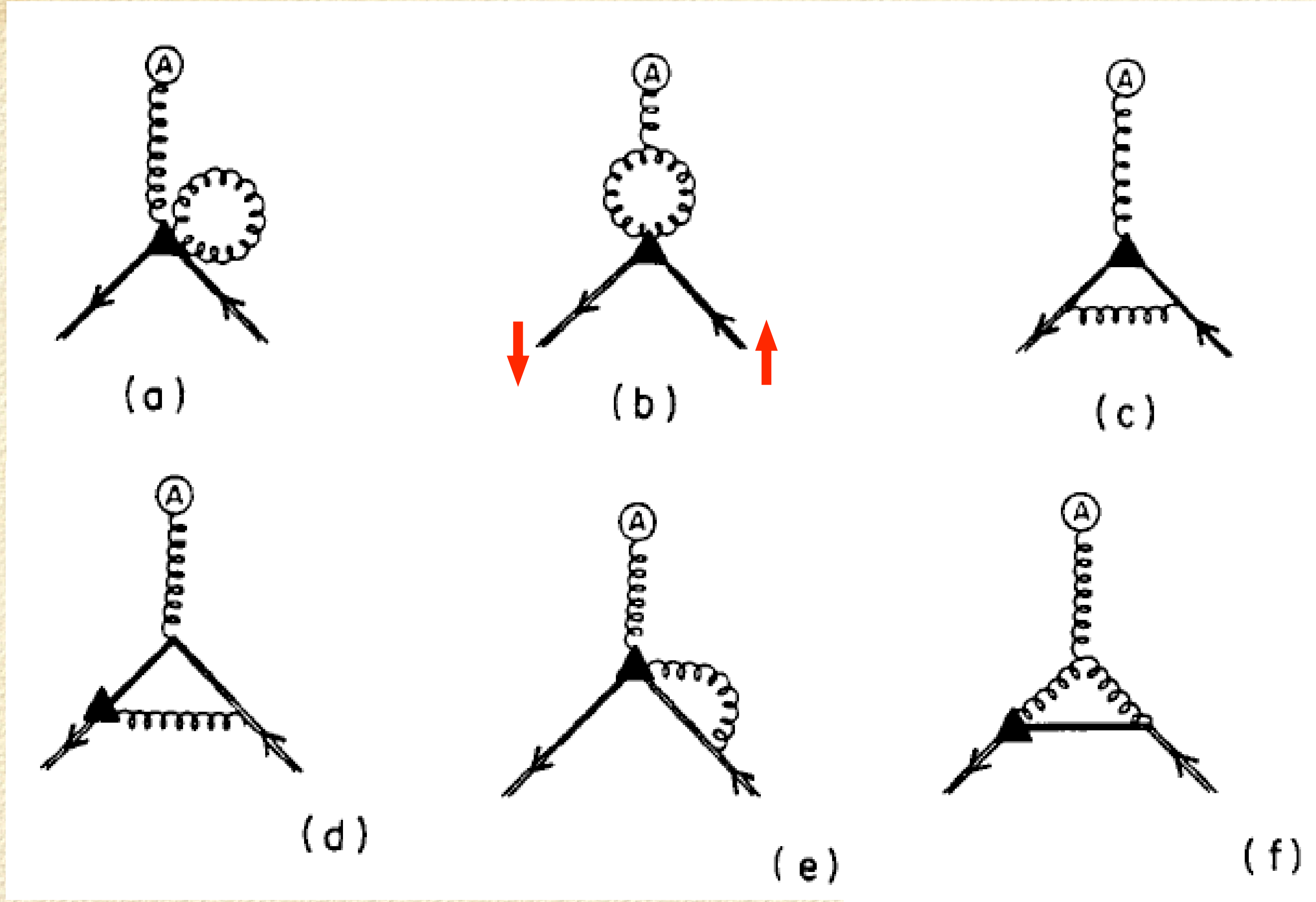


Algorithm for unquenched simulations:

Hornbostel & Lepage; Woloshyn & Wong

One-loop heavy quark action parameters

M. Nobes & HDT - see parallel talk by Nobes



One-loop $\dfrac{\sigma \cdot B}{2M}$ renormalization for HQET quarks

J. Flynn & B. Hill, Phys. Lett. B 264, 173 (1991)!

Other recent perturbative calculations
completed and under development
(HPQCD Collaboration and related)

Harada, Hashimoto, Kronfeld, Onogi et al.

- perturbative renormalization of clover fermion action
- heavy-light and light-light currents

Hornbostel, Morningstar & Lepage

- scale setting for α_s beyond leading-order

Shigemitsu, Gulez & Wingate

- heavy-light currents:
  NRQCD & Improved Staggered actions

Davies & Gray

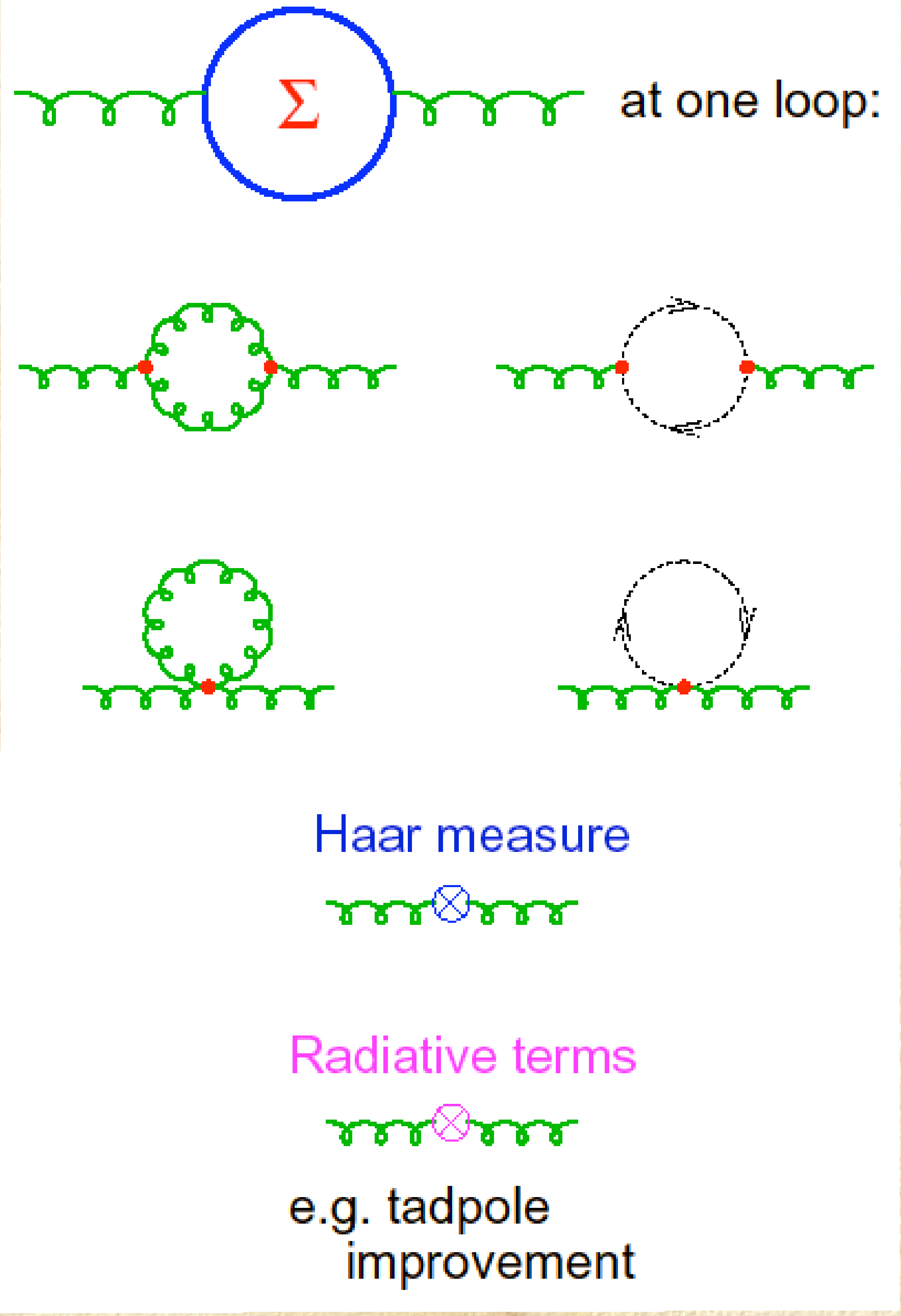
- heavy-heavy currents in NRQCD:
  velocity-dependent corrections

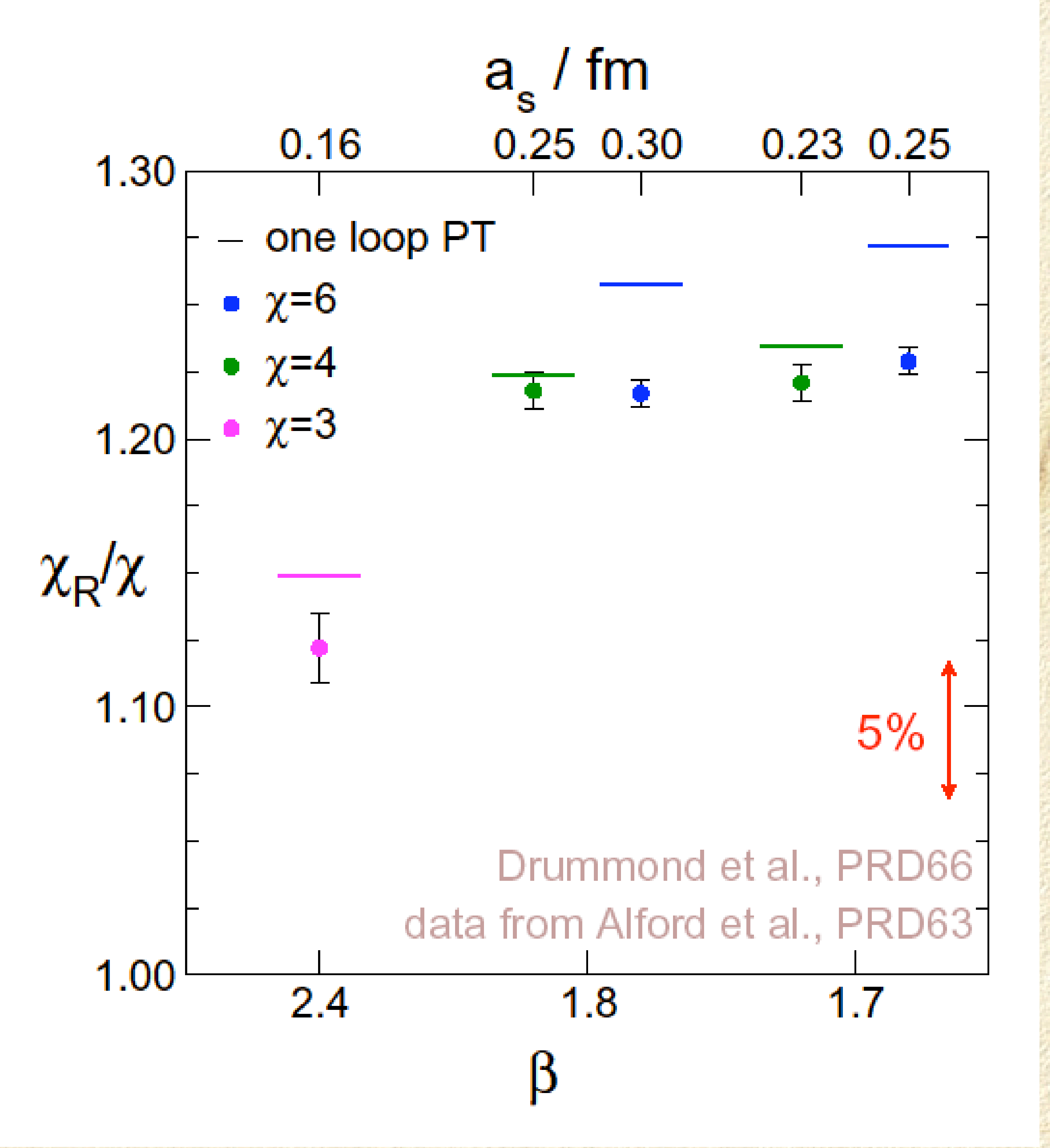
El-Khadra, Kronfeld, Mackenzie, Oktay, Nobes & HDT

- heavy-light & light-light currents:
  Fermilab action including dimension-7 improvement terms

One-loop renormalization of lattice anisotropy

I.T. Drummond, A. Hart, R.R. Horgan, L.C. Storoni





Also have done two-loop Landau-gauge mean-link

Other recent perturbative calculations

Sharpe & Lee

- improved staggered, HYP etc. currents & 4-quark ops.
- how to handle reunitarization perturbatively

Becher & Melnikov

- continuum-like methods for analytical evaluation of one-loop integrals (IR) for highly-improved actions

Burgio, Feo, Peardon & Ryan

- one-loop anisotropic gluon actions

Göckeler, Horsley, Perlt, Rakow, Schierholz & Schiller

- one-loop quark currents & DIS first-moments for overlap fermions

Aoki & Kuramashi

- perturbative improvement clover fermions with improved glue

Bali & Boyle - recent review perturbative α_s

Capitani - review of techniques & results in lattice PT

A Radically Different Technique

"Do" perturbation theory by numerical integration of the lattice path integral

(Lepage, Mackenzie & Dimm, Lattice 1995)

$$\langle W_{R,T} \rangle = \int [dU_\mu(x)[d\bar{\psi}d\psi] \, W_{R,T} \, e^{-\beta(S_{\text{gluon}}+S_{\text{quark}})}$$

=> enters perturbative phase on finite volume @ high β

- Run at several values of $\beta \approx 9 - 60$ (Planck-box!)

$$= c_0 \alpha_V(q^*) + c_1 \alpha_V^2(q^*) + \ldots$$

(We assume PT known for 1x1 loop => Λ_V at each β)

Also: Stochastic Perturbation Theory

(Di Renzo, Onofri and Marchesini 1994)

Third-order Wilson Loops

Quenched High-β

$$-\ln W_{R\times T} = w_0 \alpha_V(q^*) \left[1 + r_1 \alpha_V(q^*) + r_2 \alpha_V^2(q^*)\right] + \ldots$$

Wilson Loop	High-β Monte Carlo	Perturbation Theory
1x2	-1.34(8)	-1.31(1)
1x3	-1.17(9)	-1.13(5)
1x4	-1.04(9)	-1.00(8)
1x5	-0.98(11)	-1.05(12)
2x2	-0.71(8)	-0.71(5)
2x3	-0.44(9)	-0.48(9)
3x3	-0.12(9)	0.15(17)

Trottier, Shakespeare, Lepage, Mackenzie, Phys. Rev. D65 (2002)

Trottier & Mason, Lattice 2003

Second-order Unquenched Wilson Loops

(K. Wong, R. Woloshyn & HDT, Lattice 2003)

$$-\ln W_{R\times T} = w_0 \alpha_V(q^*)\left[1 + (r_{1,g} + r_{1,f} N_f)\,\alpha_V(q^*)\right] + \ldots$$

Wilson Loop	High-β Monte Carlo	Perturbation Theory
1x2	-0.071(7)	-0.0665(1)
1x3	-0.073(5)	-0.0681(2)
2x2	-0.075(13)	-0.0579(2)
2x3	-0.076(13)	-0.0584(7)
3x3	-0.072(13)	-0.0588(10)

Unimproved Staggered

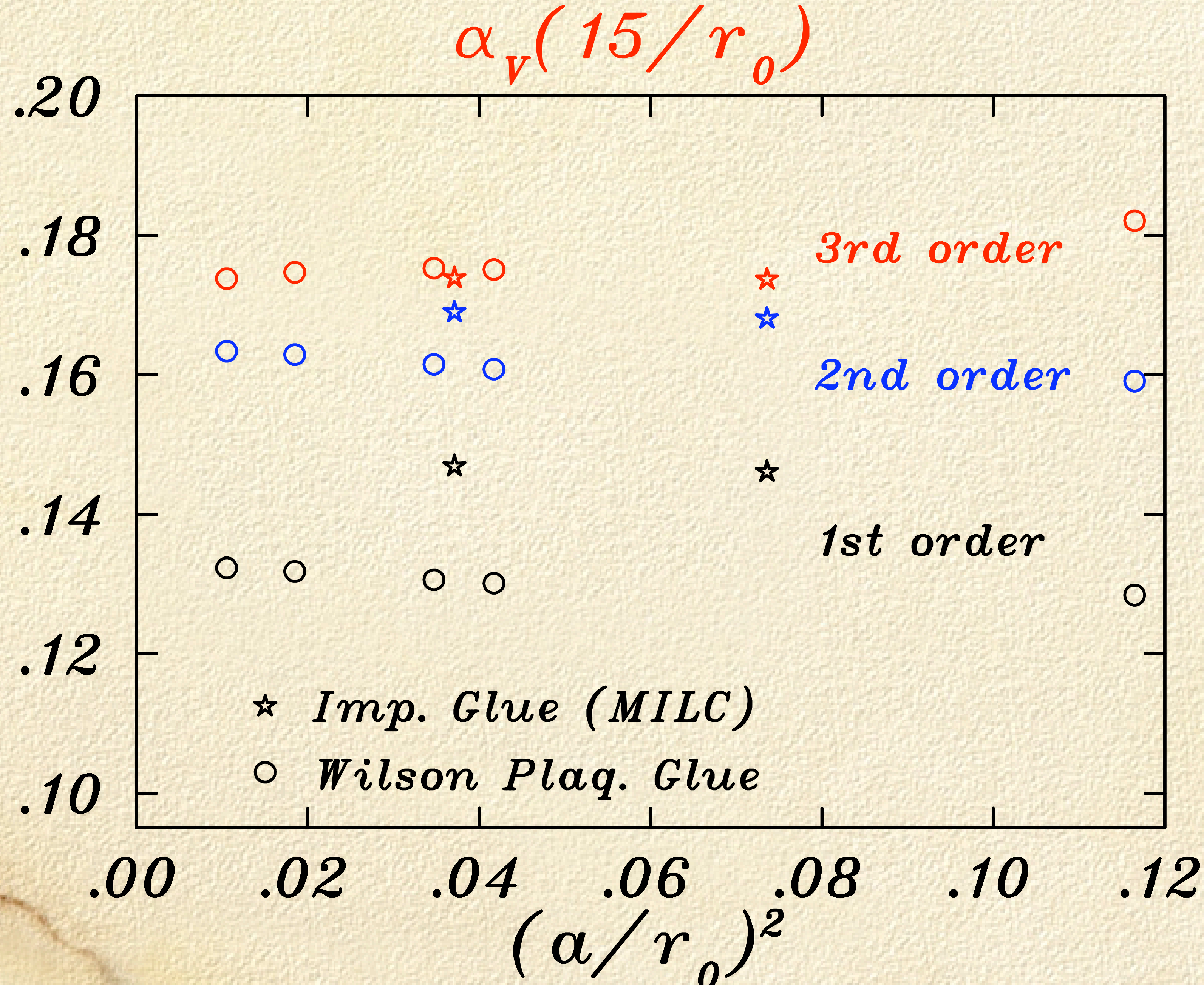
Unquenched Stochastic Perturbation Theory

See talk by Di Renzo, Lattice 2003

Preliminary Third-Order Strong Coupling

Here quenched only: Unquenched under analysis

Convergence with increasing order



(C. Davies)

Summary

- Automated methods allow one to routinely go to higher-orders for highly-improved lattice actions
- Numerical perturbation theory may offer powerful alternative to conventional perturbative methods
  - we've only scratched the surface here
- We have already done some important two- and three-loop calculations for unquenched MILC
  - anticipate two-loop results soon for $\alpha_{\overline{\mathrm{MS}}}(M_z),\ m_s,\ m_c,\ m_b$
- Collaboration committed to complete two-loop program: couplings, action parameters, currents ...