Localization in Lattice QCD

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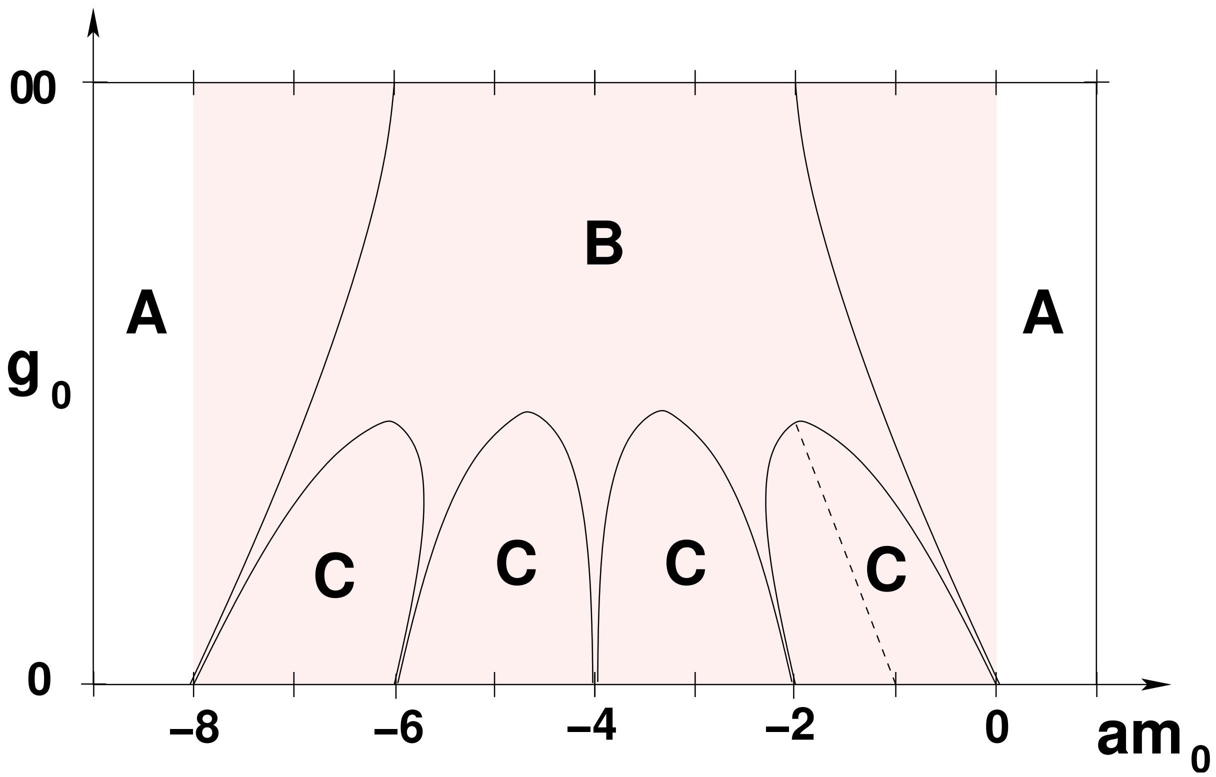
Two-flavor Wilson QCD



• Aoki phase (B) entered when

$$m_\pi^2 \propto (m_0 - m_c(g_0))$$ goes negative.

• SSB of flavor (and parity): $\langle \pi_3 \rangle \neq 0.$

• Goldstone bosons: $\pi_\pm$.

• Phases C and A: no Goldstone bosons*
in quenched numerical simulations;
but exceptional configurations discarded!
 $*$ defines (complement of) the Aoki phase

The story unfolds ...

• Extensive evidence[1,2,3] for small-size near-zero modes outside the Aoki phase, in the super-critical region $-8 \leq m_0 \leq 0$ of the quenched theory.

Quenched Puzzle

• By definition: Aoki phase $\Leftrightarrow$ Goldstone bosons

• Outside the Aoki phase:

Density of (localized) near-zero modes

$\longrightarrow \rho(0) \neq 0 \longrightarrow \langle \pi_3 \rangle \neq 0$

$\longrightarrow$ SSB $\longrightarrow$ Goldstone bosons !!??

[1] SCRI: Edwards, Heller, Narayanan, Nucl. Phys. B522 (1998) 285; B535 (1998) 403; Phys. Rev. D60 (1999) 034502

[2] Domain-wall and overlap simulations; more...

[3] Berruto, Narayanan, Neuberger, Phys. Lett. B489 (2000) 243

In a nut-shell:

- The quenched theory can have SSB of a continuous symmetry (even in finite volume), without accompanying Goldstone bosons.

- Quenched Wilson QCD is ill-defined without a "twisted mass." [4]

- The super-critical quenched theory always has $\rho(0) > 0$. Nevertheless, the phase diagram is qualitatively similar to the unquenched case.

- MC simulations using domain-wall fermions or overlap fermions (with a Wilson kernel) should only be done well outside the Aoki phase. Otherwise, one looses locality and/or chirality.

[4] Frezzotti, Grassi, Sint, Weisz, JHEP 0108 (2001) 058

Banks-Casher relation

Wilson fermions with "twisted mass"

$$D(m_0) = \not{D}^{\text{naive}} + (W + m_0)$$

Lagrangian: $\overline{\psi}' = \overline{\psi}\gamma_5, \;\; H(m_0) = \gamma_5 D(m_0),$

$$\begin{aligned} \mathcal{L} &= \overline{\psi}(D(m_0) - i m_1 \gamma_5 \tau_3)\psi \\ &= \overline{\psi}'(H(m_0) - i m_1 \tau_3)\psi \end{aligned}$$

hermitian

Condensate ($m_1 \to 0$ limit)

$$\begin{aligned} \langle \pi_3 \rangle &= \frac{2}{V}\int \mathcal{D}\mathcal{U}\,\mathcal{B}(\mathcal{U})\sum_n \frac{m_1}{\lambda_n^2 + m_1^2} \\ &= 2\int d\lambda\,\rho(\lambda)\frac{m_1}{\lambda^2 + m_1^2} \\ &\longrightarrow 2\pi\rho(0) \end{aligned}$$

Spectral density

$$\rho(\lambda) = \frac{1}{V}\int \mathcal{D}\mathcal{U}\,\mathcal{B}(\mathcal{U})\sum_n \delta(\lambda - \lambda_n)$$

- Flavor-symmetry Ward identity

$$\sum_{\mu}(1+O(ap))\, p_\mu \tilde{\Gamma}_\mu(p) + 2m_1 \tilde{\Gamma}(p) = \langle \pi_3 \rangle$$

$$\tilde{\Gamma}(p) = F.T.\, \langle \pi_+(x)\, \pi_-(y) \rangle$$

$$\tilde{\Gamma}_\mu(p) = F.T.\, \left\langle J_\mu^+(x)\, \pi_-(y) \right\rangle$$

- Goldstone theorem:

$$\sum_{\mu} p_\mu \tilde{\Gamma}_\mu(p) = \langle \pi_3 \rangle \;, \qquad p \neq 0\,,$$

provided $m_1 \tilde{\Gamma}(p) \to 0$ for $m_1 \to 0$.

Alternative mechanism[5]

No Goldstone bosons needed (in the quenched theory), provided the two-point function of the would-be Goldstone boson diverges:

$$\boxed{\tilde{\Gamma}(p) \propto \frac{1}{m_1}\,, \qquad m_1 \to 0\,.}$$

[5] Mc Kane and Stone, Ann. Phys. 131 (1981) 36

The $1/m_1$ divergence in finite volume

- Spectral representation of the pion two-point function

$$\Gamma(x,y) = \left\langle \sum_{kn} \frac{\Psi_n^\dagger(x)\Psi_k(x)\ \Psi_k^\dagger(y)\Psi_n(y)}{(\lambda_k + im_1)(\lambda_n - im_1)} \right\rangle$$

- Divergence comes from $k = n$ terms only

$$\begin{aligned}\Gamma(x,y) &\approx \frac{1}{m_1} \left\langle \sum_n |\Psi_n(x)|^2 |\Psi_n(y)|^2 \frac{m_1}{\lambda_n^2 + m_1^2} \right\rangle \\ &\approx \boxed{\frac{\pi}{m_1} \mathcal{R}(x,y;0)}\end{aligned}$$

in the limit $m_1 \to 0$. Here

$$\mathcal{R}(x,y;\lambda) = \left\langle \sum_n |\Psi_n(x)|^2 |\Psi_n(y)|^2 \delta(\lambda - \lambda_n) \right\rangle$$

- Super-critical quenched theory in finite volume:

$$\mathcal{R}(x,y;0) > 0$$

Exponential localization

- Definition: Ψ_n is exponentially localized if

$$|\Psi_n(x)|^2 \leq \frac{c_1}{l^4} \exp\left(-\frac{|x - x_n^0|}{l}\right)$$

The localization length is $l_n = \min\{l\}$ for which the above inequality holds.

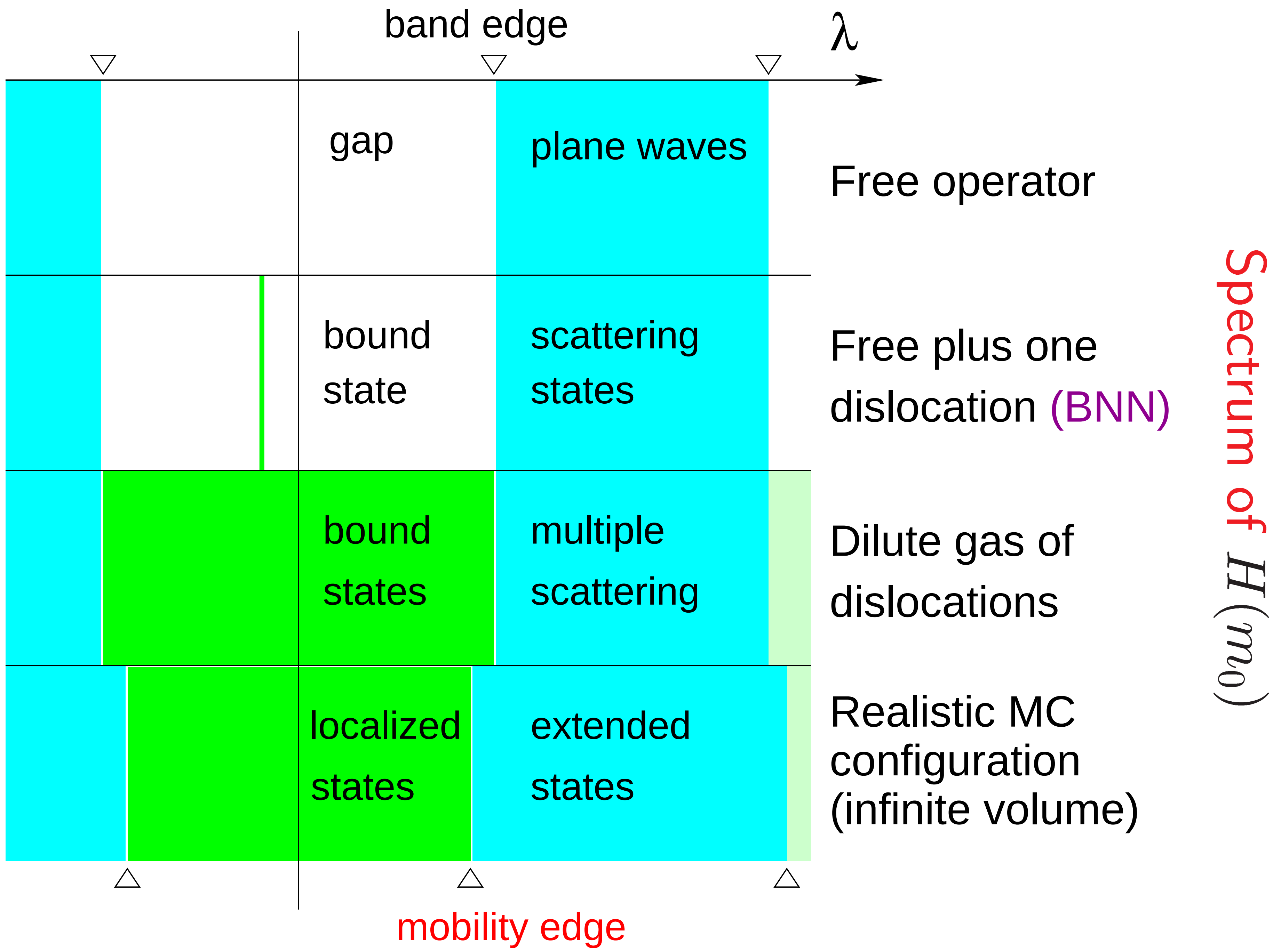
- Make an ansatz

$$|\mathcal{H}_n(p)| = \left| a^4 \sum_x |\Psi_n(x)|^2 e^{-ipx} \right| \sim \frac{1}{1 + p^2 l_n^2}$$

For $m_1 \to 0$ and $p^2 l_n^2 \ll 1$ obtain

$$\begin{aligned} 2m_1 \tilde{\Gamma}(p) &= \frac{2}{V} \left\langle \sum_n |\mathcal{H}_n(p)|^2 \frac{m_1}{\lambda_n^2 + m_1^2} \right\rangle \\ &\rightsquigarrow 2\pi\rho(0) \left(1 + O\left(p^2\, \bar{l}^2\right)\right) \end{aligned}$$

$\Rightarrow$ No Goldstone bosons!
in phase with average localization length $\bar{l} = O(1)$ in lattice units.


band edge
λ
gap
plane waves
Free operator
bound
state
scattering
states
Free plus one
dislocation (BNN)
bound
states
multiple
scattering
Dilute gas of
dislocations
localized
states
extended
states
Realistic MC
configuration
(infinite volume)
mobility edge
Spectrum of $H(m_0)$

Physical picture (a conjecture)

• Quenched and unquenched QCD in the super-critical region for $m_1 \to 0$:

1) Aoki phase $\equiv$ Goldstone bosons (pions)
$\Leftrightarrow$ mobility edge = 0
$\Leftrightarrow$ $\rho(0) > 0$ from extended states only

2) Outside the Aoki phase
$\Leftrightarrow$ mobility edge > 0

Localized modes: $|\lambda_n| \le$ mobility edge,

Extended modes: $|\lambda_n| \ge$ mobility edge.

• Main difference outside the Aoki phase:

Quenched: $\rho(0) > 0$ and $1/m_1$ divergence,

Unquenched: $\rho(0) = 0$ (and $\rho(\lambda) \sim \lambda^2$).

• The simplest physical picture that fits all we know!

Why bother?

Locality of the overlap operator:

$$D_{ov} = \frac{1}{a}(1 - \gamma_5 \hat{\gamma}_5)\,, \qquad \hat{\gamma}_5 \equiv \frac{H(m_0)}{|H(m_0)|}$$

• Outside the Aoki phase: mobility edge $\lambda_c > 0$.
We split $\hat{\gamma}_5 = \hat{\gamma}_5^< + \hat{\gamma}_5^>$ where

$$\hat{\gamma}_5^<(x,y) = a^4 \sum_{|\lambda_n|<\lambda_c/2} \Psi_n(x) \frac{\lambda_n}{|\lambda_n|} \Psi_n^\dagger(y)$$

Assuming that $l_n \le l_{max}$ for $|\lambda_n| < \lambda_c/2$,
We find

$$\left\langle |\hat{\gamma}_5^<(x,y)| \right\rangle \lesssim \exp(-|x-y|/(2l_{max}))$$

• Locality of $\hat{\gamma}_5^>$ proved by Hernández, Jansen, Lüscher[6]

• Loose locality inside or near the Aoki phase because l_{max} is too large!!

[6] Nucl. Phys. B552 (1999) 363

Domain-wall fermions

- Chiral limit $N_s \to \infty$: same as overlap, with

$H(m_0) \to H(m_0, a_5) \equiv -\log(T(m_0, a_5))/a_5$

- Near-zero modes depend weakly on a_5;

mobility edge $\lambda_c(a_5) > 0$ outside the Aoki phase.

- Finite N_s PCAC relation

$$\sum_\mu \partial_\mu A_\mu^a \approx 2(m_q + m_{res})\pi^a$$

- Residual mass

$$m_{res} = \frac{\sum_{\vec{x}} \langle J_{5q}(\vec{x}, \tau)\, \pi(\vec{0}, 0)\rangle_{\text{domain wall}}}{\sum_{\vec{x}} \langle \pi(\vec{x}, \tau)\, \pi(\vec{0}, 0)\rangle_{\text{domain wall}}}$$

For $N_s \gg 1$ and/or $|\tau| \gg a$ we expect

$$m_{res} \sim \frac{\exp(-|\tau|/l_{max})}{f(\tau)} + c\, \exp(-a_5\, \lambda_c(a_5) N_s)$$

Extended modes: suppression with N_s

Localized modes: suppression with space-time separation.