Lattice QCD at finite T and μ

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Outline

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   - Dynamical staggered, Wilson and Overlap results for T_c and the equation of state
   - Heavy quark potential, hadron properties
3. Lattice results at finite μ
   - QCD at $\mu \lesssim T$
   - QCD-like models
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Introduction

Importance of QCD at finite T/μ:

- Early Universe ($T > 0$)
- Neutron stars ($\mu > 0$)
- Heavy-ion collisions ($T, \mu > 0$)
- 31 talks at this conference & 42 contributions to FDQCD, Nara

Very simple formulation through the Euclidean grand-canonical partition function:

$$Z = \int DA_\mu D\bar{\Psi} D\Psi e^{-S_E}$$

Parameters:

gauge coupling g
quark masses m_i ($i = 1..N_f$)
Chemical potentials μ_i
Volume (V) and temperature (T)

Lattice formulation: not unique $\rightarrow$ improved actions

Improved actions

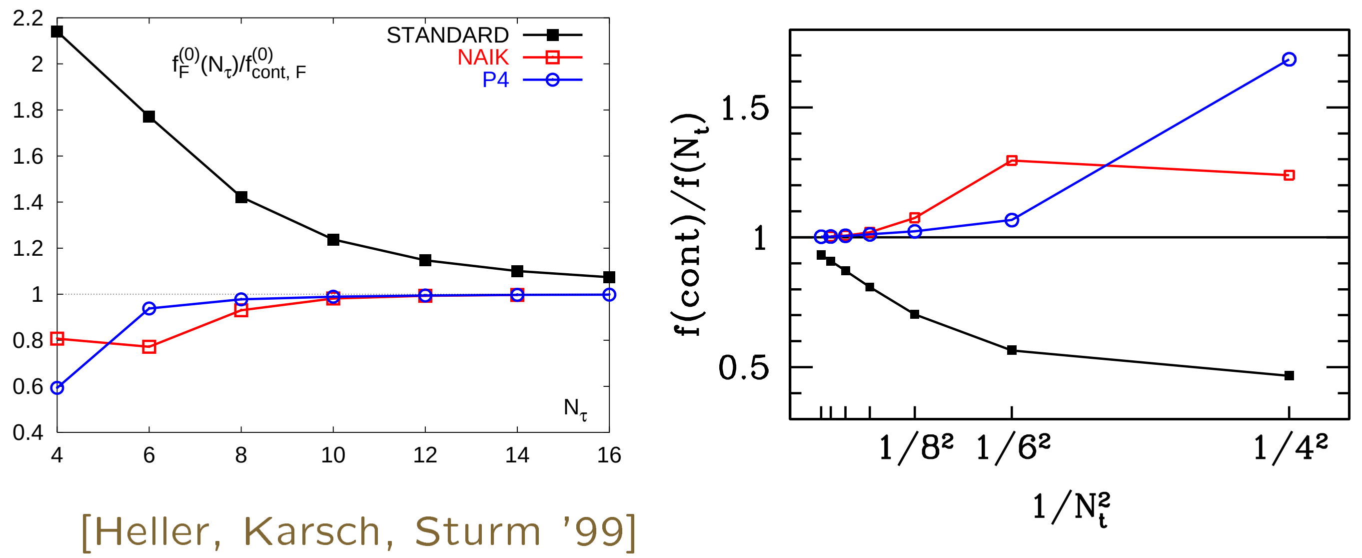
Many possibilities

Gauge action: higher loops, RG improved actions

Fermionic action: $\mathcal{O}(a), \mathcal{O}(a^2)$ improvements,

rotational/flavor symmetry improvements.

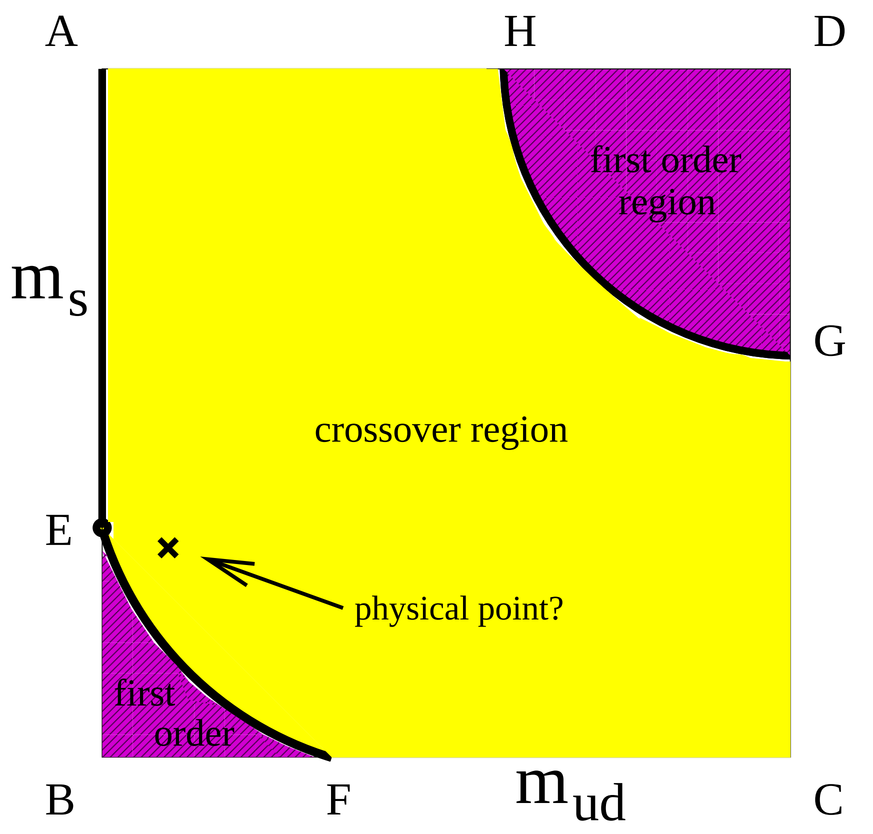
Continuum limit is always important!



[Heller, Karsch, Sturm '99]

Continuum extrapolation from N_t and $N_t + 2$ standard action may be better than using only N_t with improved action

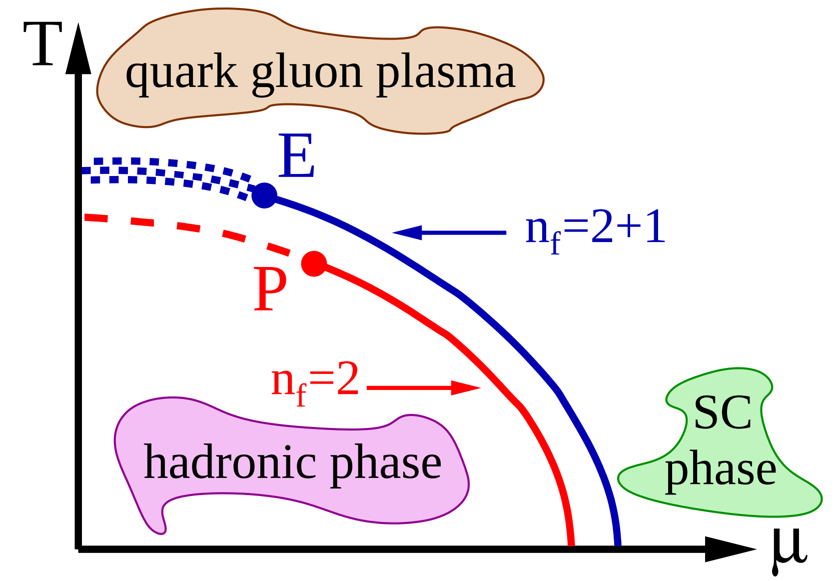
The QCD phase diagram I



$m_{ud}, m_s \to \infty$: pure SU(3) theory, 1st order phase transition
$m_{ud} = 0, m_s \to \infty$: 2 flavor massless QCD, 2nd order transition
$m_{ud} = m_s = 0$: 3 flavor massless QCD, 1st order phase transition

Physical point: probably in the crossover region

The QCD phase diagram II



$n_f = 2$ with $m_q = 0$ at $\mu = 0$ $\Rightarrow$ 2^{nd} order PT
$n_f = 2$ with $m_q = 0$ at $T = 0$ $\Rightarrow$ 1^{st} order PT
$n_f = 2$ with $m_q = 0$ $\Rightarrow$ tricritical point (P) at μ,T$\neq$0

$n_f = 3$ with $m_q = 0$ at $\mu = 0$ $\Rightarrow$ 1^{st} order PT
increasing m_s weakens the 1^{st} order PT $\Rightarrow$ cross-over

$n_f = 2 + 1$ with physical m_q at $\mu = 0$ $\Rightarrow$ cross-over
$n_f = 2 + 1$ with physical m_q at $T = 0$ $\Rightarrow$ 1^{st} order PT
$n_f = 2 + 1$ with physical m_q $\Rightarrow$ critical endpoint (E) at μ,T$\neq$0

The QCD phase transition

Order parameters:

- pure SU(3): Z_3 symmetry, deconfining phase transition
  - Polyakov loop $< L >= 0 (\neq 0)$ in the low (high) temperature phase
- massless QCD: chiral phase transition
  - Chiral condensate $< \bar{\Psi}\Psi >$ as order parameter

Deconfining and chiral transitions at the same temperature

- pure SU(3): $T_c \approx 270$MeV
- full QCD: $T_c \approx 155$ (3 flavors) -175 (2 flavors) MeV

Locating the critical point:

- susceptibility peaks of L or $\bar{\Psi}\Psi$
- Lee-Yang zeroes of the partition function

Finding the order of the phase transition:

Finite size scaling of susceptibility peaks, Lee-Yang zeroes, Binder-cumulants, etc.

The equation of State along the Line of Constant Physics (LCP)

Energy density (ϵ) and pressure (p) from partition function:

$$\epsilon(T) = \frac{T^2}{V}\frac{\partial(\log Z)}{\partial T} \qquad p(T) = T\frac{\partial(\log Z)}{\partial V}.$$

T, V are varied by a, take $\frac{\partial\beta}{\partial a}$ and $\frac{\partial(am_i)}{\partial a} = m_i$ on the LCP

$$\frac{\epsilon - 3p}{T^4} = -\frac{L_t^3}{L_s^3}\left\{a\frac{d(\log Z)}{da} - \mu\frac{d(\log Z)}{d\mu}\right\} =$$

$$-L_t^4\left[a\frac{\partial\beta}{\partial a} < \mathrm{P} > +a\frac{\partial(am_u)}{\partial a} < \bar{u}u > +a\frac{\partial(am_s)}{\partial a} < \bar{s}s >\right]$$

The pressure ($p \propto \log[Z]$) along the LCP by the integral method:

$$\frac{p}{T^4} = L_t^4 \int d(\beta, m\cdot a, \mu a)\left(\frac{\partial(\log Z)}{\partial\beta}, \frac{\partial(\log Z)}{\partial(m\cdot a)}, \frac{\partial(\log Z)}{\partial(\mu a)}\right) =$$

$$L_t^4 \int d\beta\left[< \mathrm{P} > +m_u\frac{\partial a}{\partial\beta} < \bar{u}u > +m_s\frac{\partial a}{\partial\beta} < \bar{s}s > +\mu\frac{\partial a}{\partial\beta}\frac{\partial(\log\det M)}{\partial(\mu a)}\right]$$

Lattice results for the pure SU(3) theory

Standard and improved actions at $N_t = 4, 6, 8$

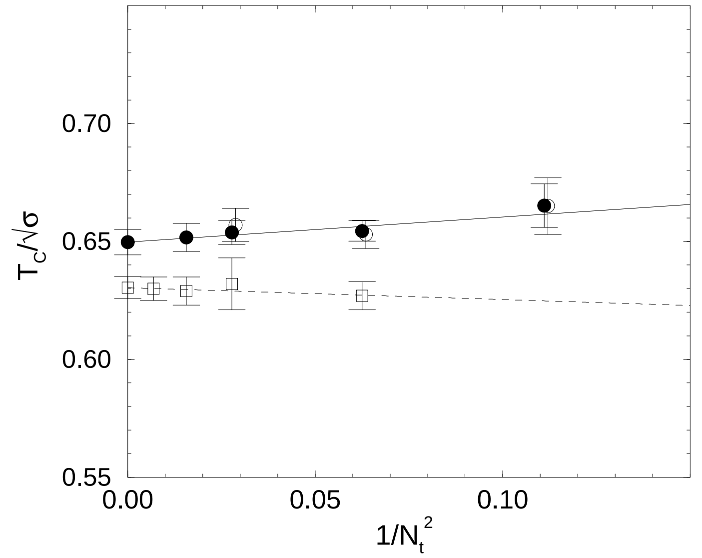
[Boyd et al, '96; Beinlich et al, '98; CP-PACS, '99]

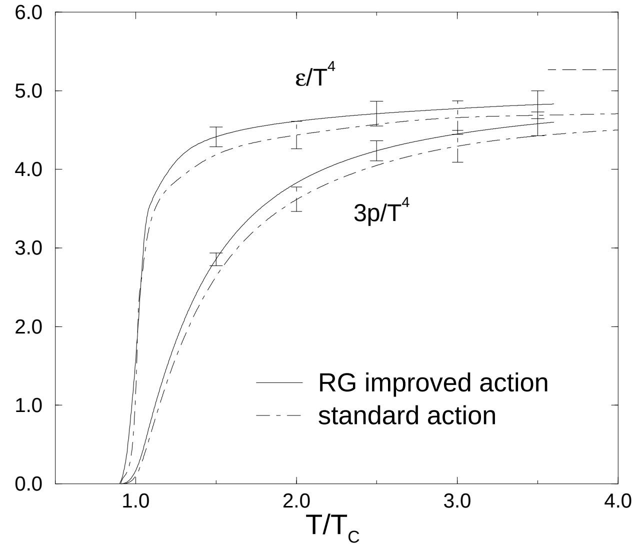
also anisotropic lattices [CP-PACS, '02]

Continuum extrapolation is performed

Critical temperature: $T_c/\sqrt{\sigma} \approx 0.630 - 0.650$

(difference is due to different definitions of σ)





First order phase transition (latent heat)

15% deviation from free gas limit even at $T = 4 - 5 \times T_c$

Full QCD lattice results

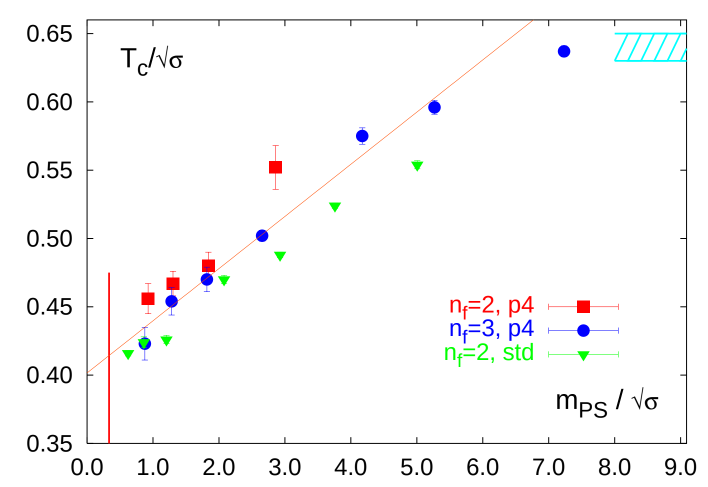
No final results yet

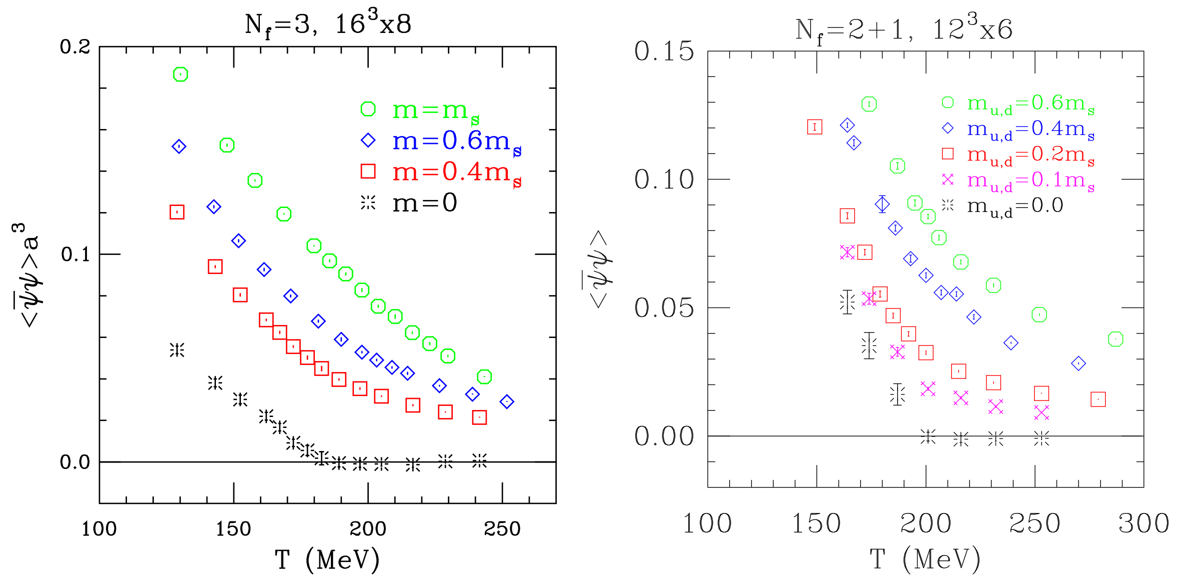
No direct simulations at the physical quark masses $\rightarrow$
chiral extrapolation is done

Continuum limit is still hard, but already results for $N_t = 4, 6, 8$

Staggered results

Results on T_c:



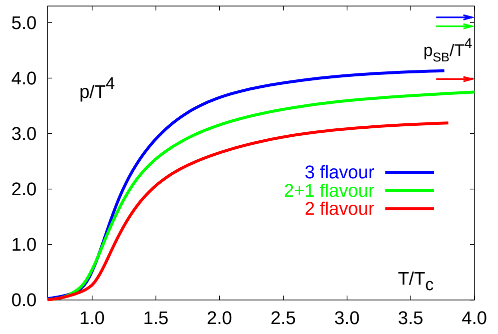


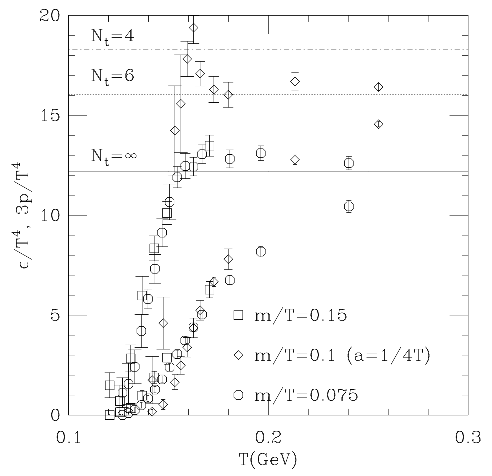
[Karsch, Laermann, Peikert '01] [MILC, '03]

In physical units $T_c \approx 170$MeV

Unchanged with non-degenerate u-d quarks [Gavai, Gupta '02]

Equation of state

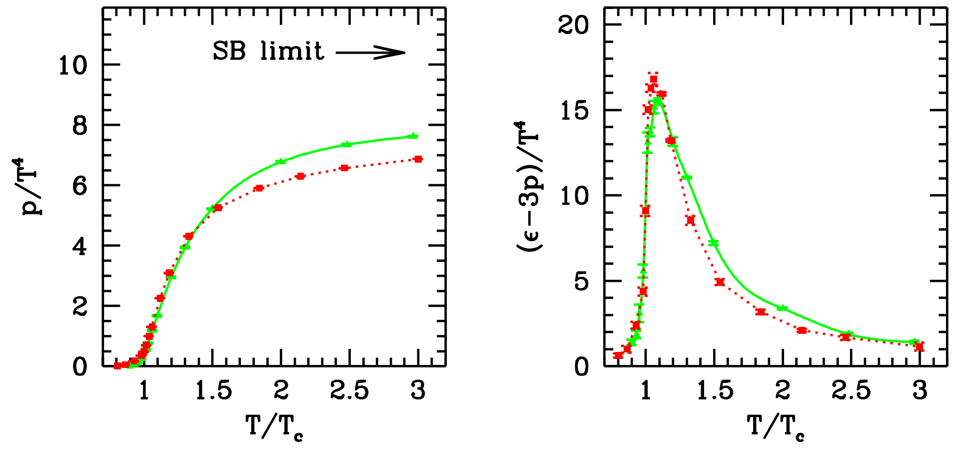




[Karsch, Laermann, Peikert, 2000] [Bernard et al, '96]

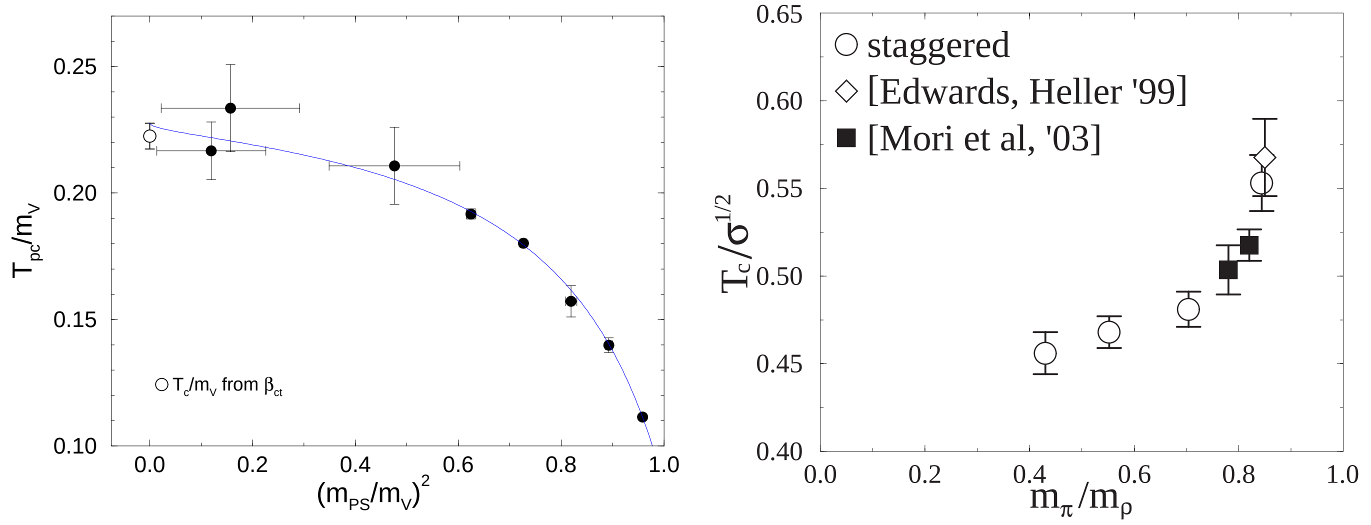
All curves along constant m_q/T, not along the LCP $\rightarrow$ physical quark masses increase with T

Comparison with LCP method



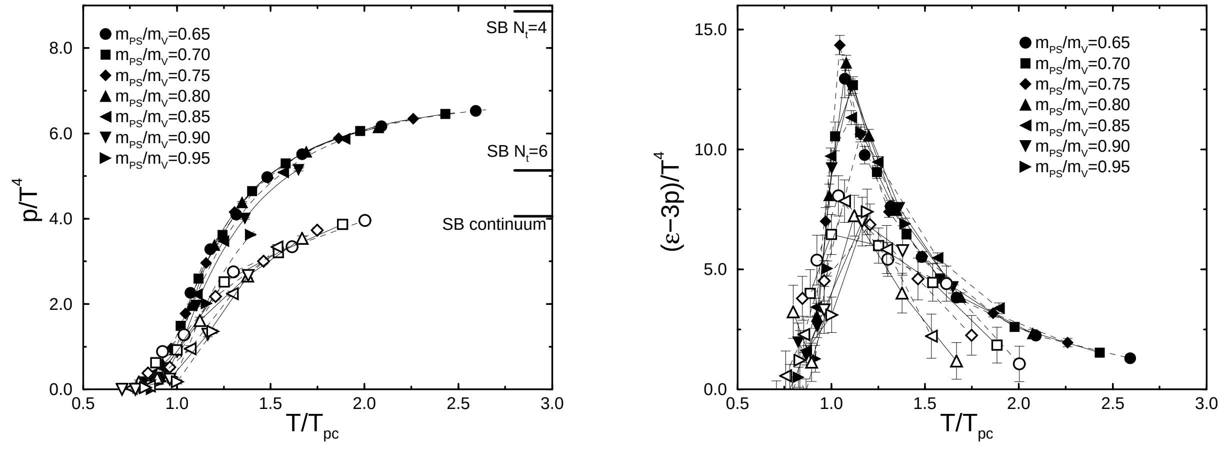
[Fodor, SDK, Szabo '02]

Results with Wilson fermions



[Ali-Khan et al, 2000]

Transition temperature in physical units: $T_c = 171(4)\text{MeV}$
Consistent with staggered results



[Ali-Khan et al, '01]

Preliminary first results using dynamical overlap fermions

[Fodor, SDK, Szabo, preliminary]

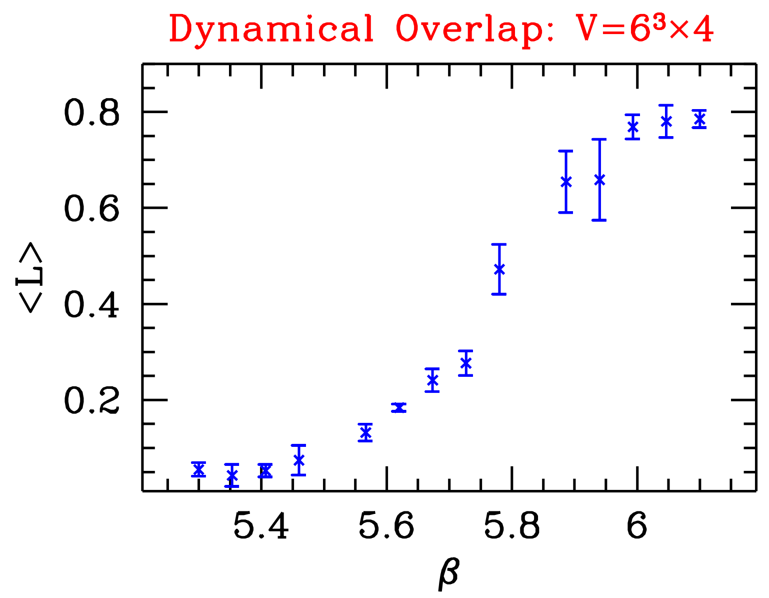
The overlap operator

[Neuberger, '98]

$$D_o = m_0\,(1+\gamma_5 \mathrm{sgn}\,(H_w)) \text{ with } H_w = \gamma_5(D_w - m_0)$$

The sign function is approximated using the optimal Zolotarev rational approximation with $n_z = 20$ [van den Eshof et al., '02; Chiu et al., '02]
Nested inversion for rational polynomials [Neuberger, '98]
Multi-shift solver [Frommer et al., '95, '96]. Extremely expensive (500×)

Standard Hybrid Monte-Carlo, $N_f = 2$; eigenvalue projection
Clear evidence of a transition in β



Nature of the phase transition

2 flavor QCD

In the chiral limit: O(4) second order phase transition expected

lattice results not consistent

- $N_t = 4$ staggered results disagree [JLQCD '98; MILC '99]
- Improved Wilson results agree with O(4) [CP-PACS, 2000]

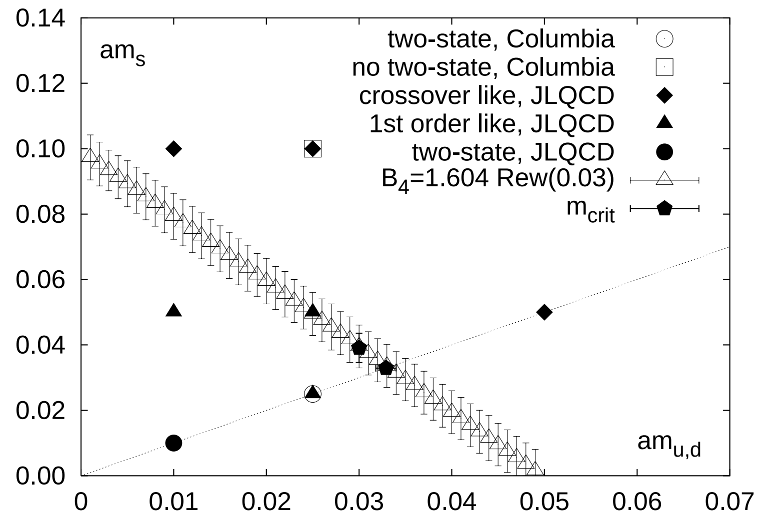
3 flavor QCD

Critical endpoint on $N_t = 4$ lattices at $m_{\text{ps}} \approx 290\text{MeV}$ with standard staggered QCD [Karsch, Laermann, Schmidt '01]

$m_{\text{ps}} \approx 67\text{MeV}$ with p4 action [Allton et al., FDQCD '03]

2+1 flavor QCD

Unclear picture; physical point probably in the crossover region

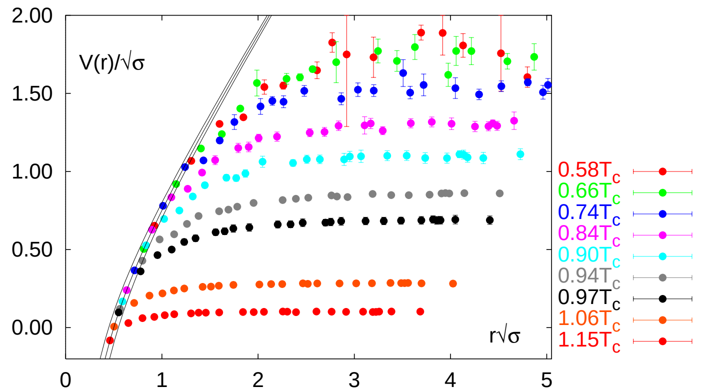


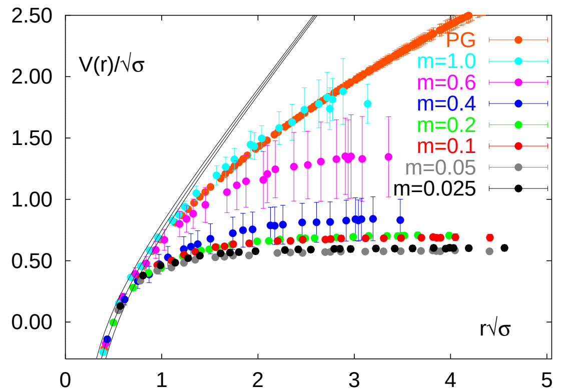
[Schmidt et al., '02]

The heavy quark potential at finite T

Polyakov-loop correlation function:

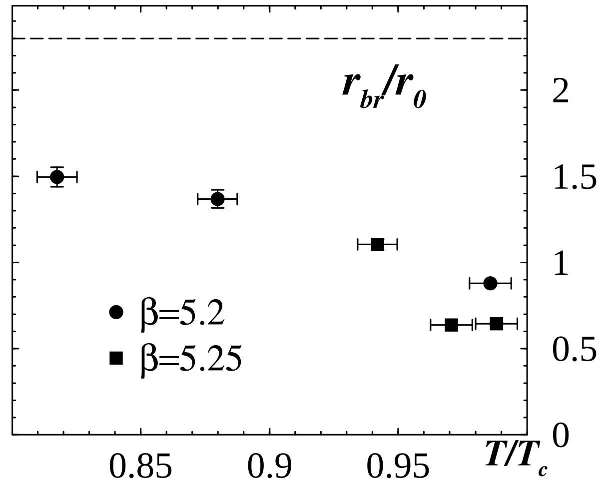
$$< L_x L_y^\dagger > \sim e^{-V(|x-y|,T)/T}$$





[Karsch, Laermann, Peikert '01]

p4 action $16^3 \times 4$ lattices



[Bornyakov et al., '03]

improved Wilson action, $16^3 \times 8$ lattices

Hadron properties at finite T

Correlation functions at finite T: small temporal extension $\rightarrow$

spatial correlators, screening masses

[Laermann, Schmidt, '02; Gavai, Gupta, '02]

temporal correlators also possible $\rightarrow$

anisotropic lattices

Spectral functions $(A(\omega, \underline{k}))$

Related to the Euclidean correlation functions:

$$D(\tau, \underline{k}) = \int_0^\infty K(\tau, \omega) A(\omega, \underline{k}) d\omega$$

The inversion of this equation is difficult $\rightarrow$

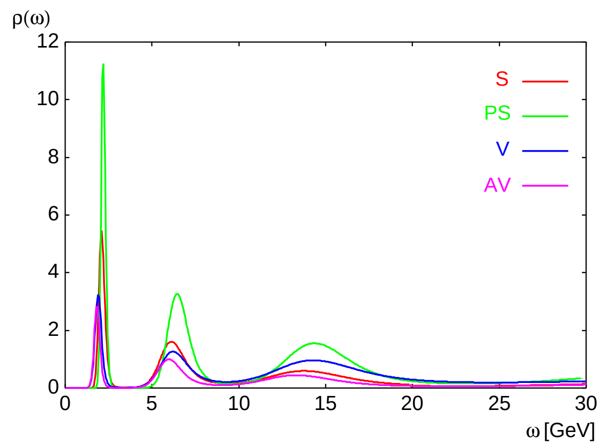
Maximum entropy method (MEM) [Asakawa, Nakahara, Hatsuda '99,'01]

statistical method to recover $A(\omega, \underline{k})$ from noisy data

temporal correlator is required at many points

Lattice results on spectral functions

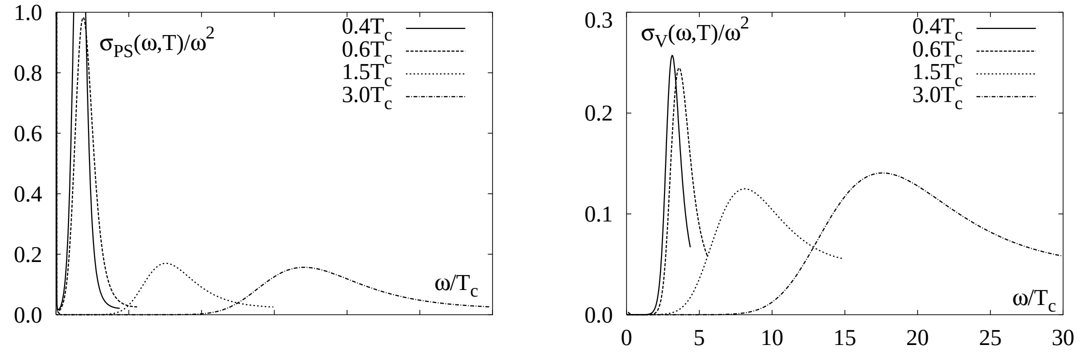
Quenched results:



[Asakawa, Hatsuda, Nakahara, '02]

$T \approx 1.4 T_c, \quad N_t = 54, \quad m_\pi / m_\rho \approx 0.7 (T = 0)$

Anisotropic lattices with $\xi = 4$



[Karsch et al, '02]

$N_t = 16, \quad m_\pi / m_\rho \approx 0.65 (T < T_c)$

Lattice results at finite μ

Main difficulty:

$\det M$ becomes complex

fundamental property of many finite density systems

standard importance sampling does not work

Possible solutions and suggestions:

- Recent new methods for $\mu \lesssim T$
  - multi-parameter reweighting without or with Taylor expansion
  - analytic continuation from imaginary μ
  - factorization method (hard to implement for QCD)
- QCD-like theories with positive action $T = 0; \mu > 0$
  - SU(2) "two-color" QCD
  - QCD with isospin μ
  - four-fermion models

Overlap improving multi-parameter reweighting [Fodor, SDK '01]

$$Z(m,\mu,\beta) = \int \mathcal{D}U \exp[-S_g(\beta,U)] \det M(m,\mu,U) =$$
$$\int \mathcal{D}U \exp[-S_g(\beta_0,U)] \det M(m_0,\mu=0,U)$$
$$\left\{\exp[-S_g(\beta,U)+S_g(\beta_0,U)]\frac{\det M(m,\mu,U)}{\det M(m_0,\mu=0,U)}\right\}$$

first line = measure, field configurations of the Monte-Carlo
curly bracket = can be measured on each configuration, weight

simultaneously changing several parameters: better overlap
e.g. transition configurations are mapped to transition ones

expectation value of an observable O:

$$\langle 0\rangle_{\beta,\mu,m} = \frac{\sum w(\beta,\mu,m)O(\mu,m)}{\sum w(\beta,\mu,m)}$$

observables to get the transition points at $\mu \neq 0$ (susceptibilities)

Goodness of the reweighting

reweighting is accurate only for "very high statistics"

ensured by important but rare fluctuations $\rightarrow$ overlap problem

Overlap measure (α):

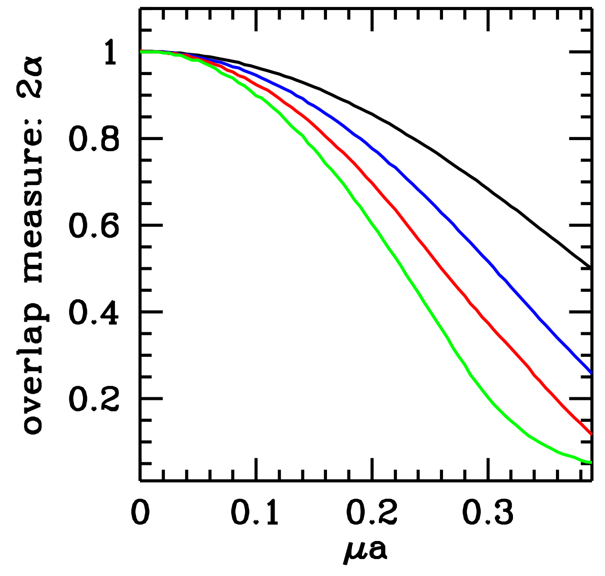
α fraction of the configurations provides $1-\alpha$ weight

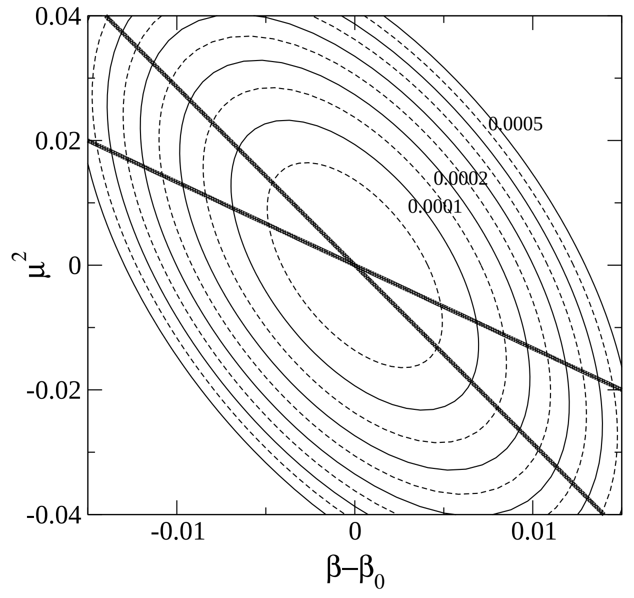
approximate scaling of the half-width:

$\mu_{1/2} \propto V^{-\gamma}$ with $\gamma \approx 0.2 - 0.3$

Fluctuation of the weights

The direction with smallest spread coincides with the critical line





[Ejiri, '02]

Best weight lines: minimizing the spread of $\log w$

also determined diagonalizing covariance matrices [Crompton, '02]

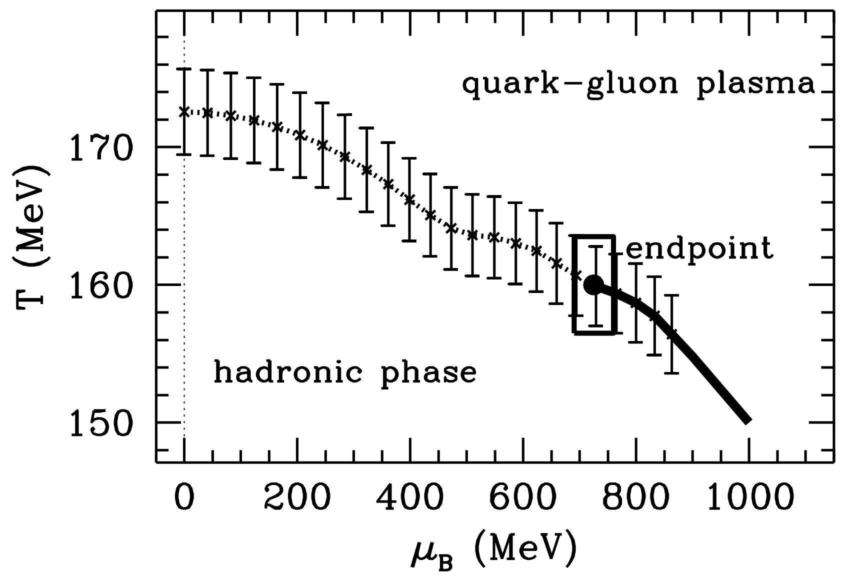
The endpoint of QCD in the $T-\mu$ plane

[Fodor, SDK, '01]

Finding the transition line

- search for the Lee-Yang zeroes of the partition function
- Finite size scaling of the imaginary parts of Lee-Yang zeroes distinguishes between crossover and 1[st] order

$N_f = 2+1$ staggered quarks on $N_t = 4$ lattices with semi-physical masses



endpoint: $T_E = 160 \pm 3.5$ MeV, $\mu_E = 725 \pm 35$ MeV

at μ_B=0 transition temperature: $T_c = 172 \pm 3$ MeV.

$\mu \neq 0$ multi-parameter reweighting with Taylor expansion

[Allton et al., '02]

$$Z(m,\mu,\beta) = \int \mathcal{D}U \, \exp[-S_g(\beta,U)] \, \det M(m,\mu,U) =$$
$$\int \mathcal{D}U \, \exp[-S_g(\beta_0,U)] \, \det M(m_0,\mu=0,U)$$
$$\left\{ \exp[-S_g(\beta,U) + S_g(\beta_0,U)] \frac{\det M(m,\mu,U)}{\det M(m_0,\mu=0,U)} \right\}$$

instead of evaluating determinants expand them in μ or $\exp(\mu)$:

$$\ln\left(\frac{\det M(\mu)}{\det M(0)}\right) = \sum_{n=1}^{\infty} \frac{\mu^n}{n!} \frac{\partial^n \ln \det M(0)}{\partial \mu^n} \equiv \sum_{n=1}^{\infty} R_n \mu^n$$

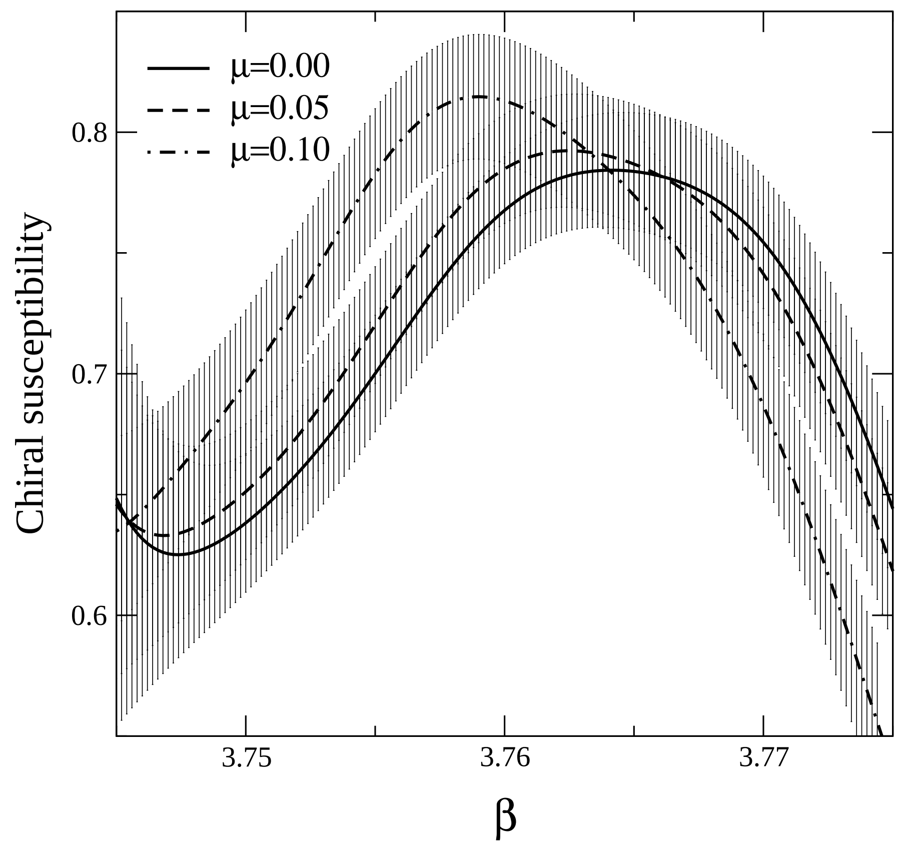
derivatives can be evaluated stochastically $\rightarrow$
faster than the complete evaluation of the determinants
only valid for somewhat smaller μ values than the full technique

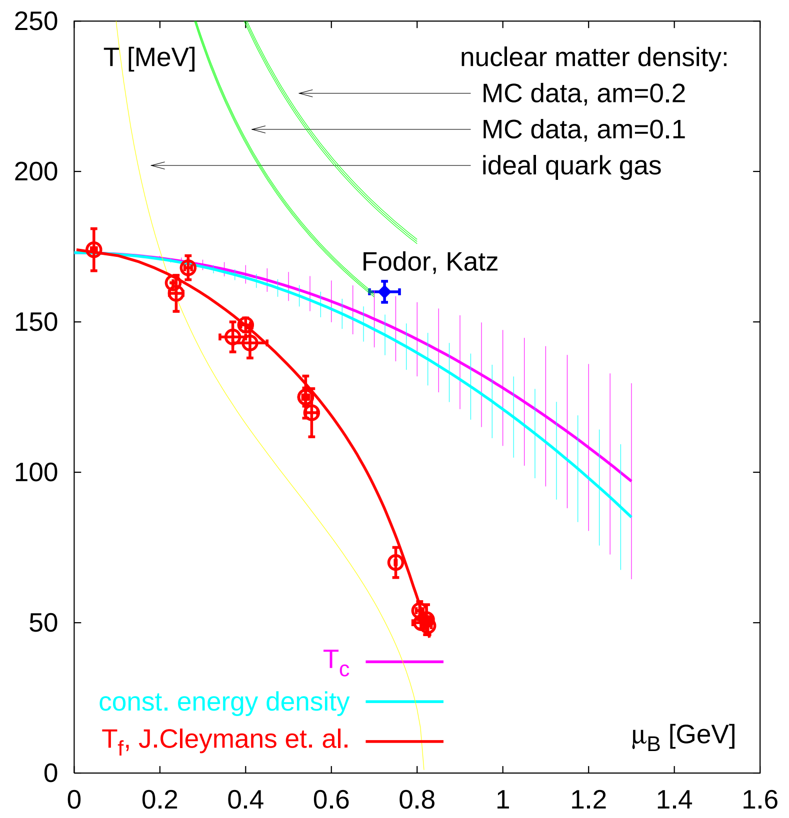
The phase diagram using Taylor expansion

p4 action with 2 flavors

transition points are located by looking for susceptibility peaks

conversion into physical units (T, μ_B in MeV) using $T = 0$ results





[Allton et al., '02]

presence of higher order terms in the Taylor expansion

$\Rightarrow$ uncertainties increase fast with increasing μ

Taylor expansion: determinant's phase lost at $\mu \gtrsim 0.2$ [Takaishi, FDQCD'03]

Mass dependence of the critical endpoint in 3 flavor QCD

[Schmidt et al., '02; Ejiri, FDQCD'03]

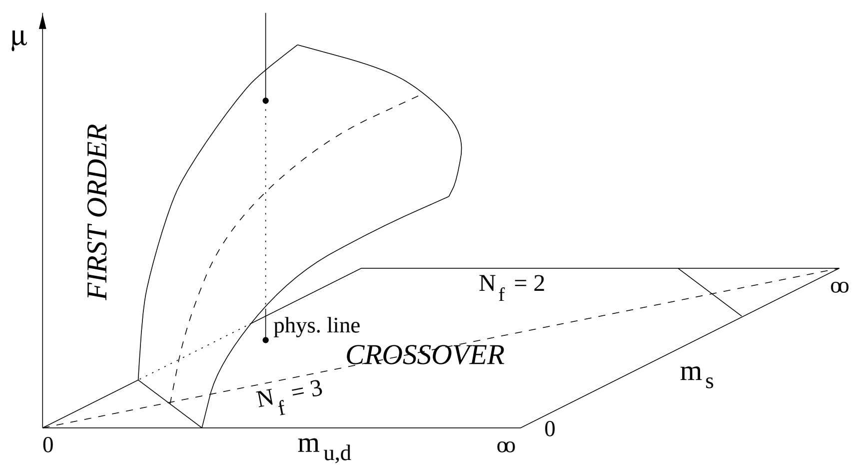
hypothetical theory: pseudoscalar masses $m_\pi = m_K \approx$67 MeV

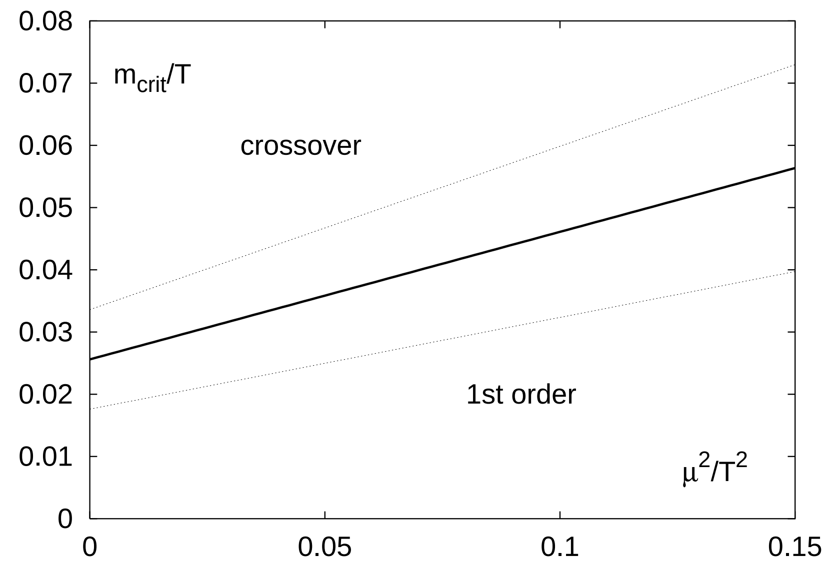
endpoint is at $\mu = 0$, easy to study on the lattice

increasing the mass pushes the endpoint to $\mu \neq 0$

reweighting also in the quark mass

keep the system always on the critical surface





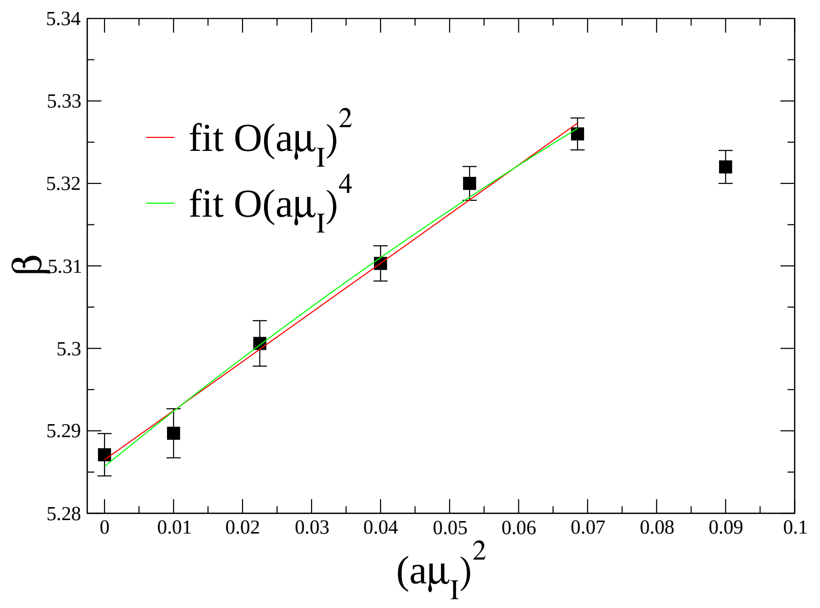
$\Rightarrow$ in principle one can determine the endpoint for physical masses

QCD phase diagram from imaginary chemical potential

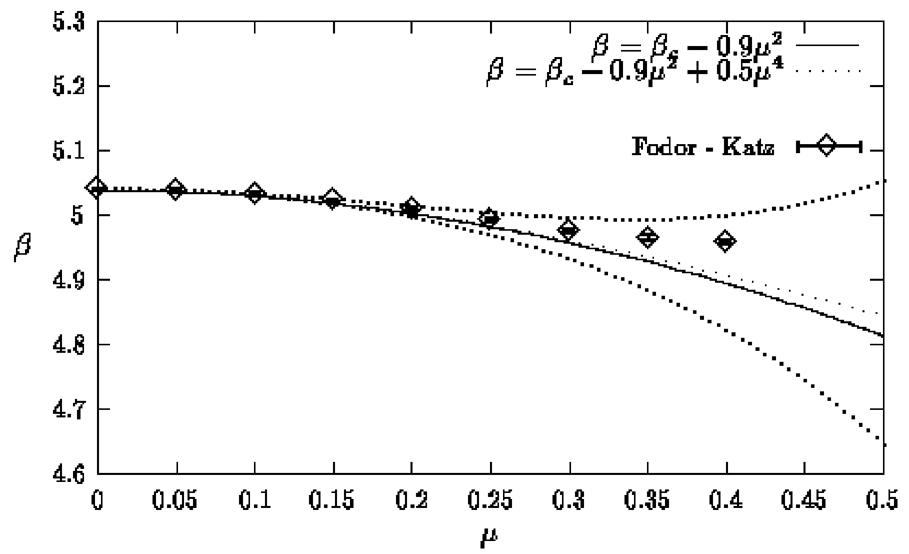
[de Forcrand, Philipsen, '02; D'Elia, Lombardo, '02]

fermion determinant: real for imaginary chemical potential (μ_I)
$\Rightarrow$ no sign problem, no need for reweighting

directly obtain the (β_c,μ_I) transition line
analytically continue it to get the physical (β_c,μ) line
perturbative β function is used to convert to physical values



[de Forcrand, Philipsen, '02]

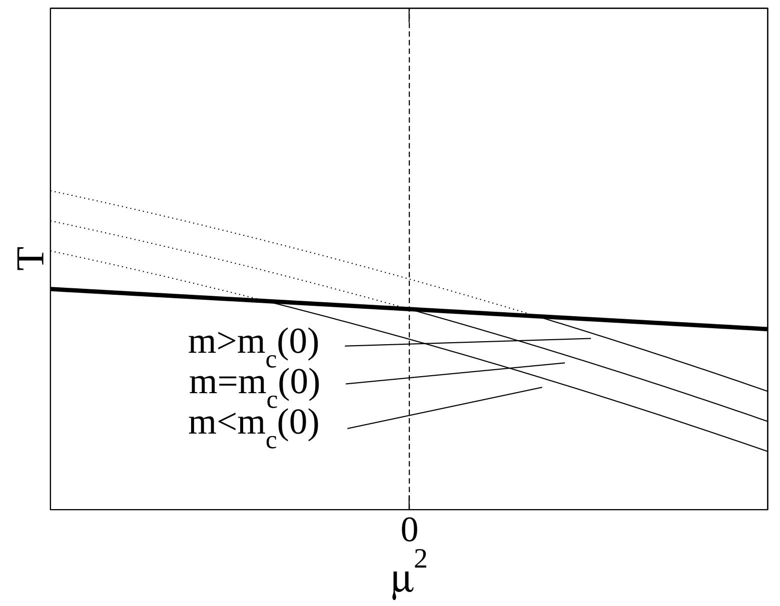


[D'Elia, Lombardo, '02]

$$T_c(\mu) = T_c(0) - 0.0056\mu^2/T$$

mass dependence of the critical point

$N_f = 3$ simulations at imaginary μ and then analytically continue to real μ



[de Forcrand, Philipsen, '03]

$$\frac{m_c(\mu)}{m_c(\mu=0)} = 1 + 0.58(37)\left(\frac{\mu}{\pi T}\right)^2$$

first results for the equation of state

Taylor coefficients of the pressure $p(\mu/T)$ can be determined

Good agreement with reweighting [Lombardo, FDQCD '03]

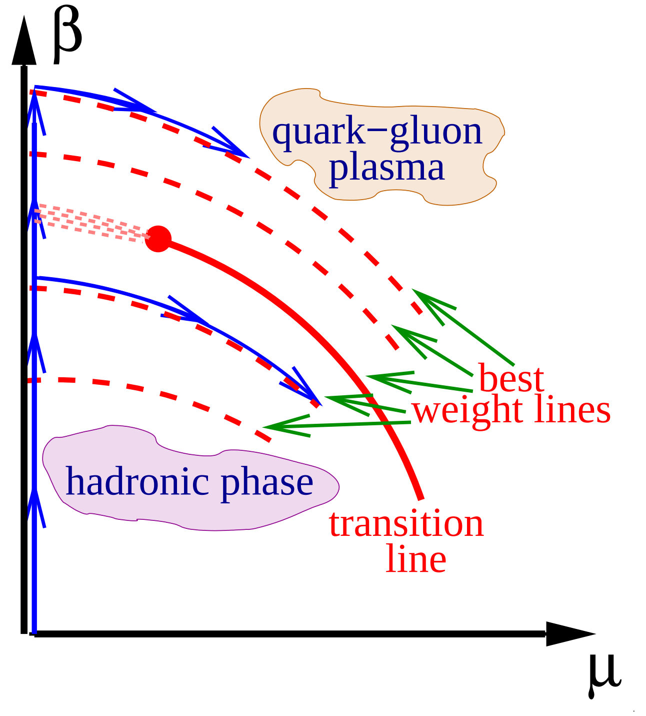
The equation of state at finite μ

Paths used for the pressure integral

integral is performed in two steps (path independence)

a. integral over $\beta = \beta_0 \dots \beta_T$ at $\mu = 0$ (same as for $\mu = 0$)

b. integral over $(\beta_T, \mu = 0) \dots (\beta, \mu \neq 0)$ along the weight line

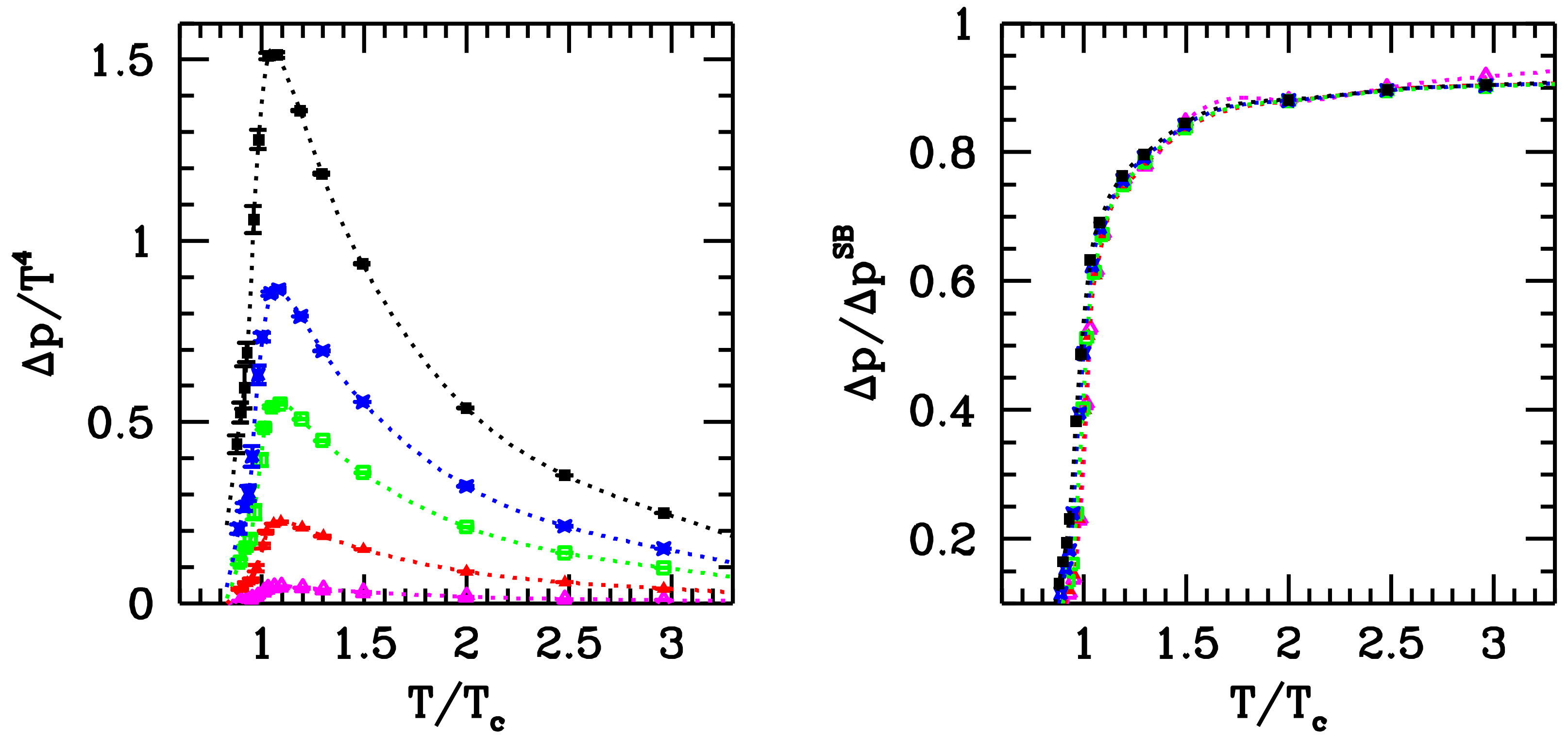


mixing steps a,b one can successfully check path independence

Equation of state at finite chemical potential upto $\mu_q \approx T_c$

difference between the $\mu = 0$ and $\mu \neq 0$ pressure values

chemical potential for $\mu_B \approx$100,210,330,410,530 MeV



[Fodor, SDK, Szabo, '02]

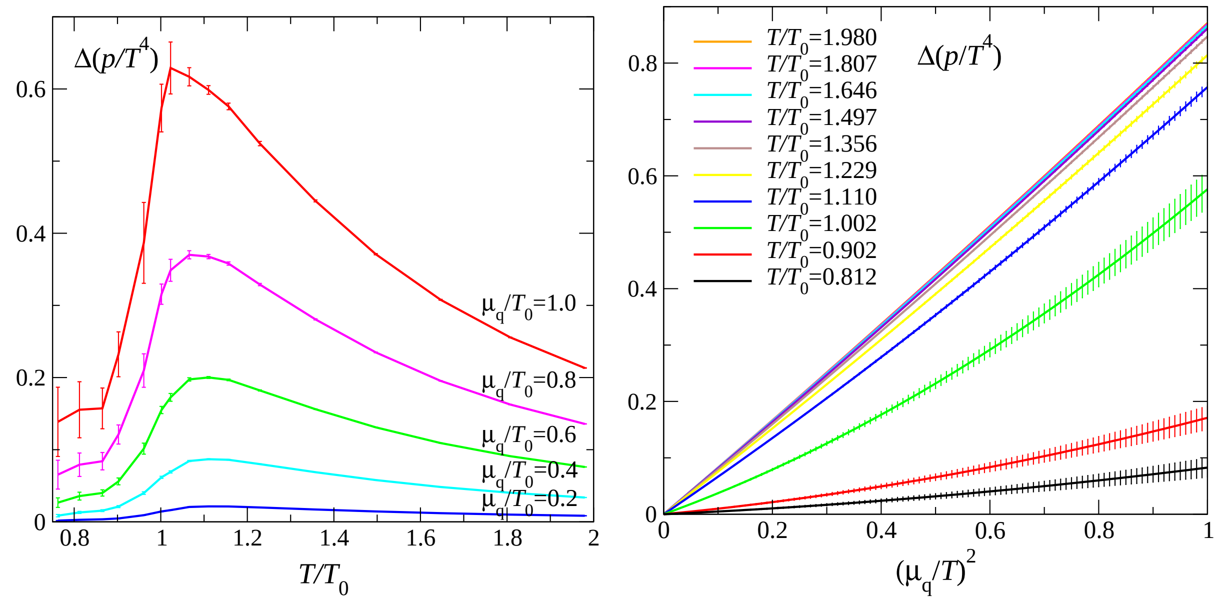
normalize with the leading order μ dependence: Δp^{SB}

almost universal temperature dependence for $\Delta p / \Delta p^{SB}$

Equation of state using Taylor expansion

$$\frac{p}{T^4}(\mu, T) = p_0(T) + c_2(T)\left(\frac{\mu}{T}\right)^2 + c_4(T)\left(\frac{\mu}{T}\right)^4 + \dots$$

the coefficients can be determined with $\mu = 0$ simulations



[Allton et al, '03]

Results are similar to the ones using the full technique
The overall normalization difference is also present
between the SB limits of the used actions
The convergence radius reaches the critical endpoint

The factorization method for finite μ

[Ambjorn et al., '02]

Tested for random matrix theory

$$Z = \int dW e^{-N\mathrm{Tr}(W^\dagger W)} \det D \text{ with } D = \begin{pmatrix} m & iW + \mu \\ iW^\dagger + \mu & m \end{pmatrix}$$

The key idea: Constrained partition functions

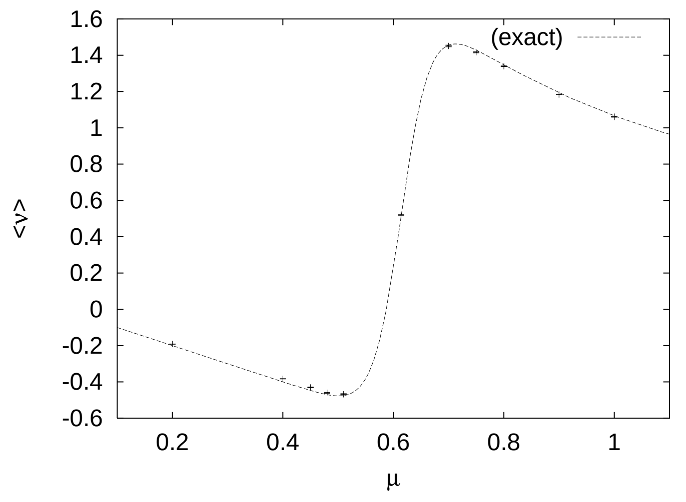
$$Z_i(x) = \int dW e^{-S_0} \delta(x - \nu_i)$$

with some observable ν_i

sample the configurations with given values of the observable

Simulations at many x values are required

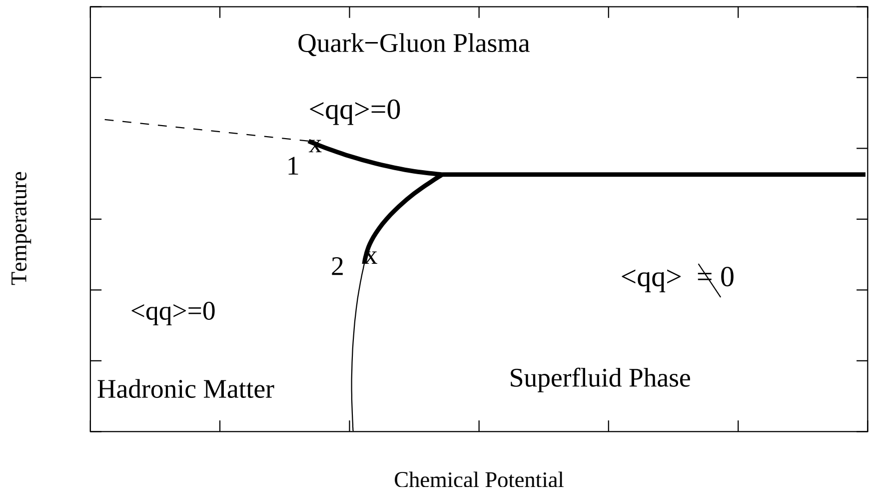
Good agreement with exact results $\rightarrow$ promising for QCD



QCD-like models at finite μ

SU(2): determinant is positive definite

$$S_f = \sum_{sites} \left\{ \bar{\chi}[\not{D}(\mu) + m]\chi + \frac{1}{2}\lambda[\chi^T \tau_2 \chi + \bar{\chi}\tau_2 \bar{\chi}^T] \right\}$$



[Kogut, Toublan, Sinclair , '02]

phase transitions observed both at finite T and $T = 0$ at finite μ

[Hands at al., '01; Kogut, Toublan, Sinclair, '03]

critical point $\rightarrow$ test possibility for all finite μ methods

hadron properties have been examined

using Wilson fermions [Muroya et al., '02]

sudden decrease of vector meson mass with increasing μ [Kogut, Toublan, Sinclair , '03]

QCD with isospin μ: $\mu_u = -\mu_d = 1/2\mu_I$

the phases of the u and d determinants cancel $\to$

full action is positive definite

Similar results as in the SU(2) case

phase transition observed at $T = 0$ at finite μ [Kogut, Sinclair, '02]

Four-fermion models:

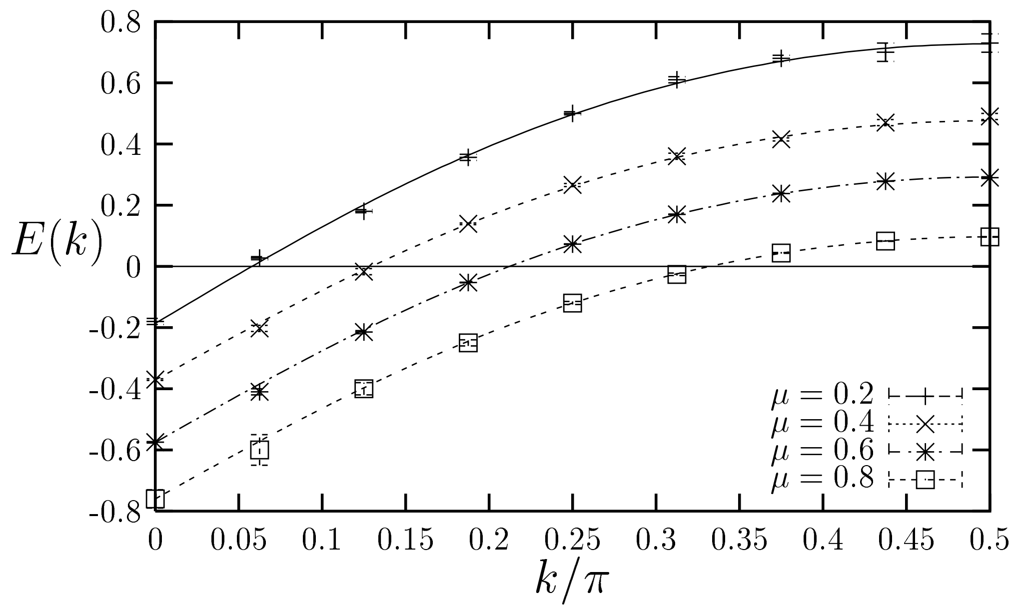
Gross-Neveu model in (2+1) dimensions [Hands et al., '03]

Fermi phenomena observed

meson correlation functions do not decay exponentially $\to$

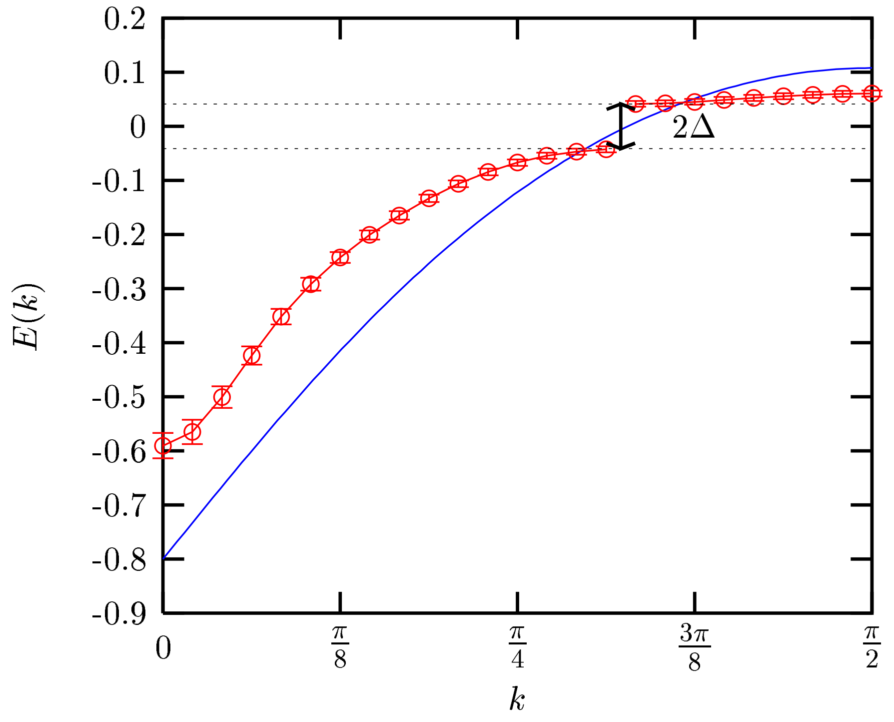
existence of massless particle-hole excitations

no mass gap in the dispersion relation



NJL model:
(2+1) dimensions: evidence for qq pairing
but no BCS condensation [Hands, Lucini, Morrison, 2000]

(3+1) dimensions [Hands, Walters, '02]



[Walters, FDQCD '03]

energy gap observed $\rightarrow$ evidence for BCS condensation
Strong coupling expansion: approximate Hamiltonian
four fermion interaction, analytically studied 1st PT [Fang, Luo, '03]

Summary, conclusions

1. Lattice QCD at finite T and $\mu = 0$

- in pure SU(3) theory both T_c and the EoS given in the continuum limit
- in full QCD continuum results for T_c and EoS not yet in the chiral and continuum limits, but can be expected in the near future
- Heavy quark potential shows the expected string breaking features
- Several results for correlation and spectral functions in particular with the help of MEM

2. Lattice QCD at finite μ

- Significant recent progress for $\mu \lesssim T$ using
  - exact reweighting
  - reweighting with Taylor expansion or
  - analytic continuation from imaginary μ
  - factorization method (hard to implement for real QCD)
- Phase transition observed at $T = 0$, finite μ using QCD-like models with positive definite action
  BUT not yet in real QCD