Some Recent Developments *in* Lattice Supersymmetry

There has been much interesting recent work on lattice supersymmetry, including

- The 2D Wess-Zumino model (K. Fujikawa; S. Catterall and S. Karamov)
- $N = 1$ Super Yang Mills theory in 4D (DESY-Münster-Roma collaboration)

This talk will focus on lattice Super Yang Mills theories:

- D. B. Kaplan, E. Katz and M. Unsal, "Supersymmetry on a spatial lattice," JHEP **0305**, 037 (2003) [arXiv:hep-lat/0206019];
- A. G. Cohen, D. B. Kaplan, E. Katz and M. Unsal, "Supersymmetry on a Euclidean spacetime lattice. I: A target theory with four supercharges," arXiv:hep-lat/0302017;
- A. G. Cohen, D. B. Kaplan, E. Katz and M. Unsal, "Supersymmetry on a Euclidean spacetime lattice. II: Target theories with eight supercharges," arXiv:hep-lat/0307012;
- Work in progress.

Why should one be interested in lattice supersymmetry?

Supersymmetric theories — especially nonabelian Yang-Mills theories — possess many fascinating nonperturbative phenomena:

chiral symmetry breaking	**confinement**
magnetic monopoles	**important instantons**
weak-strong duality	**conformal fixed points**
...	...

These theories also tell us about **string theory** and **quantum gravity** in the large N_c limit.

Until recently, there have not been any nonperturbative regulators for these theories.

What is the obstacle?

At a basic level, the problem is the SUSY algebra:

$$\{\overline{Q}, Q\} = 2P \longleftarrow$$ **generator of infinitesimal translations**

...and there are no infinitesimal translations on the lattice.

Oversimplistic argument.

Lattice QCD possesses only lattice symmetries, yet Poincaré symmetry emerges in the continuum limit without fine tuning.

Hypercubic lattice symmetry
$+$
Gauge symmetry
$\Longrightarrow$
NO relevant Poincaré violating operators

Poincaré symmetry arises as an "accidental" symmetry in the infrared limit of lattice QCD.

Perhaps supersymmetry can arise as an "accidental symmetry"?

A simple example of "accidental supersymmetry":

N=1 SYM in 3+1 dimensions

Continuum theory: gauge field plus a Weyl gluino in the adjoint representation. The theory possesses an exact discrete Z_{2N_c} chiral symmetry.

There are NO relevant SUSY violating operators consistent with gauge symmetry + Z_{2N_c} chiral symmetry

Therefore a lattice theory with the correct particle content and chiral symmetry (eg, overlap or domain wall fermions) will have a supersymmetric limit without fine tuning...

Simulation: **G. Fleming, J. Kogut, P. Vranas, Phys. Rev. D 64, 034510 (2001) [arXiv:hep-lat/0008009]**

...or one can forgo chiral symmetry and fine tune one operator (the gluino mass)

See: **I. Montvay, Int. J. Mod. Phys. A 17, 2377 (2002) [arXiv:hep-lat/0112007]**

SUSY theories with scalar fields are problematic. Scalar mass terms can violate SUSY...

...and the only symmetries known that forbid radiative generation of scalar mass terms are:

- **Shift symmetry;** $\phi \to \phi + const.$
  No good in general: ϕ would be a Goldstone boson with only derivative interactions.

- **Supersymmetry**

So there must be *<u>exact</u>* *supersymmetry on the lattice in order to obtain* *<u>accidental</u>* *supersymmetry in the IR ??!*

Perhaps one can find some partial supersymmetry that can be realized exactly on the lattice, which will forbid the relevant operators which spoil the full supersymmetry of the target theory?

Lots of reasons to be pessimistic:

- Difficult to recover Lorentz invariance;
- The lattice treats scalars, gauge bosons, and fermions so differently...how can one have an exact supersymmetry?
- How can one obtain the large chiral symmetries ("R-symmetries") of the continuum theory without domain wall/overlap fermions, or else fine tuning?

Nevertheless: A technique called "orbifolding" from string theory / deconstruction allows one to construct lattices which evade these problems.

First: an example — a lattice for SYM with 4 real supercharges in 2 Euclidean dimensions

Super Yang Mills theories in 2 Euclidean dimensions with 4 supercharges

The **continuum** theory is easily obtained by reducing $N = 1$ SYM in 4 to 2 Euclidean dimensions. The particle content is:

- a 2-component gauge field $\boldsymbol{v_m}$;
- a 2-component Dirac spinor $\boldsymbol{\psi}$;
- a complex scalar $\boldsymbol{s}$.

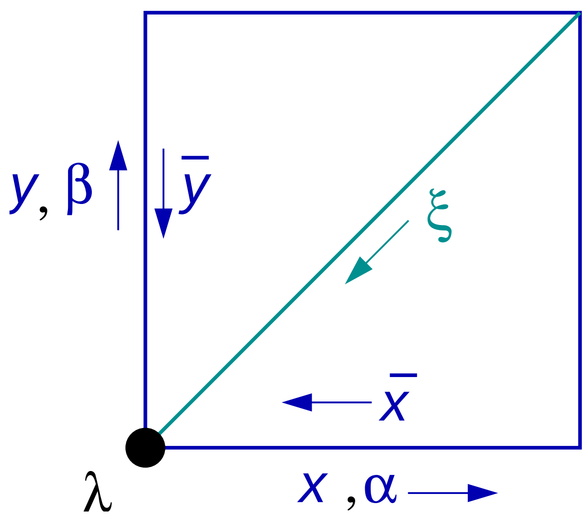
All fields transform as adjoints under the gauge symmetry. The Lagrangean is:

$$\mathcal{L} = \frac{1}{g_2^2}\,\mathrm{Tr}\left(\left|D_m s\right|^2 + \overline{\psi}\, D_m \gamma_m \psi + \tfrac{1}{4} v_{mn} v_{mn} \right.$$
$$\left. + \sqrt{2}(\overline{\psi}_L [s, \psi_R] + \overline{\psi}_R [s^\dagger, \psi_L]) - \tfrac{1}{2}[s^\dagger, s\,]^2 \right)$$

Note that the theory respects a chiral $U(1)$ symmetry, broken only by the anomaly.

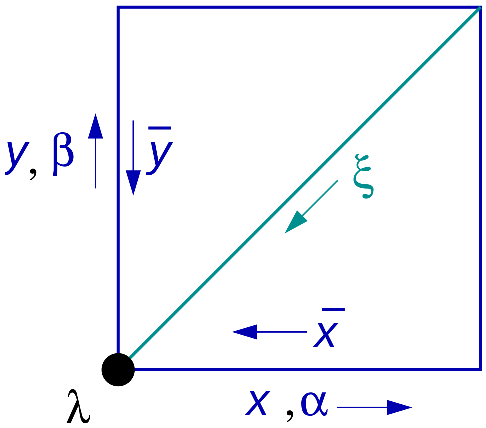
Super Yang Mills theories in 2 spacetime dimensions with **4** supercharges — *the lattice theory:*

The fundamental cell has the structure



- x and y are complex bosons
- α, β, λ, ξ are complex 1-component Grassmann variables
- Each field is a $k \times k$ matrix, and there is a $U(k)$ symmetry associated with each vertex (for a $U(k)$ gauge theory in the continuum);

The lattice action is...



The lattice action:

$$\frac{1}{g^2}\sum_{\mathbf{n}} \mathrm{Tr}\left[\tfrac{1}{2}\left(\overline{x}_{\mathbf{n}-\hat{\mathbf{e}}_1}x_{\mathbf{n}-\hat{\mathbf{e}}_1} - x_{\mathbf{n}}\overline{x}_{\mathbf{n}} + \overline{y}_{\mathbf{n}-\hat{\mathbf{e}}_2}y_{\mathbf{n}-\hat{\mathbf{e}}_2} - y_{\mathbf{n}}\overline{y}_{\mathbf{n}}\right)^2 \right.$$
$$+2\left|x_{\mathbf{n}}y_{\mathbf{n}+\hat{\mathbf{e}}_1} - y_{\mathbf{n}}x_{\mathbf{n}+\hat{\mathbf{e}}_2}\right|^2$$
$$\left. +\sqrt{2}\left(\Delta_{\mathbf{n}}(\lambda,\overline{x},\alpha) + \Delta_{\mathbf{n}}(\lambda,\overline{y},\beta) - \Delta_{\mathbf{n}}(\xi,y,\alpha) + \Delta_{\mathbf{n}}(\xi,x,\beta)\right)\right]$$

where:

$$\begin{aligned}
\Delta_{\mathbf{n}}(\lambda,\overline{x},\alpha) &= \lambda_{\mathbf{n}}\left(\overline{x}_{\mathbf{n}-\hat{\mathbf{e}}_1}\alpha_{\mathbf{n}-\hat{\mathbf{e}}_1} - \alpha_{\mathbf{n}}\overline{x}_{\mathbf{n}}\right)\ ,\\
\Delta_{\mathbf{n}}(\lambda,\overline{y},\beta) &= \lambda_{\mathbf{n}}\left(\overline{y}_{\mathbf{n}-\hat{\mathbf{e}}_2}\beta_{\mathbf{n}-\hat{\mathbf{e}}_2} - \beta_{\mathbf{n}}\overline{y}_{\mathbf{n}}\right)\ ,\\
\Delta_{\mathbf{n}}(\xi,y,\alpha) &= \xi_{\mathbf{n}}\left(y_{\mathbf{n}}\alpha_{\mathbf{n}+\hat{\mathbf{e}}_2} - \alpha_{\mathbf{n}}y_{\mathbf{n}+\hat{\mathbf{e}}_1}\right),\\
\Delta_{\mathbf{n}}(\xi,x,\beta) &= \xi_{\mathbf{n}}\left(x_{\mathbf{n}}\beta_{\mathbf{n}+\hat{\mathbf{e}}_2} - \beta_{\mathbf{n}}x_{\mathbf{n}+\hat{\mathbf{e}}_2}\right)
\end{aligned}$$

Action has "classical flat directions". Expand about

$$\langle x\rangle = \langle\overline{x}\rangle = \langle y\rangle = \langle\overline{y}\rangle = \frac{1}{\sqrt{2}\,a}\times \mathbf{1}_k \quad\longleftarrow\quad (k\times k \text{ unit matrix})$$

a will be the lattice spacing...

The bosonic action:

$$x \equiv \frac{1}{\sqrt{2}\,a} + \frac{s_1 + iv_1}{\sqrt{2}}\,, \qquad y \equiv \frac{1}{\sqrt{2}\,a} + \frac{s_2 + iv_2}{\sqrt{2}}\,.$$

★ First term in the lattice action:

$$\frac{1}{g^2}\sum_{\mathrm{n}} \mathrm{Tr}\left[\tfrac{1}{2}\left(\overline{x}_{\mathrm{n}-\hat{e}_1} x_{\mathrm{n}-\hat{e}_1} - x_{\mathrm{n}}\overline{x}_{\mathrm{n}} + \overline{y}_{\mathrm{n}-\hat{e}_2} y_{\mathrm{n}-\hat{e}_2} - y_{\mathrm{n}}\overline{y}_{\mathrm{n}}\right)^2\right]$$

$$\to \frac{1}{g^2a^2}\int d^2z \frac{1}{2}\mathrm{Tr}\,\left(D_1 s_1 + D_2 s_2\right)^2 + O(a)\,.$$

where $D_m = \partial_m + i[v_m, \cdot]$

★ Second term in the lattice action:

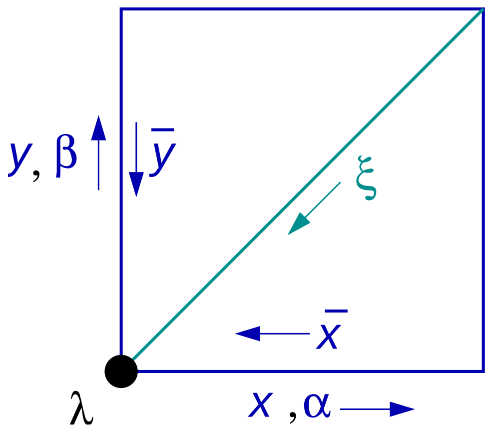
$$\frac{1}{g^2}\sum_{\mathrm{n}} 2\,\mathrm{Tr}\left|x_{\mathrm{n}} y_{\mathrm{n}+\hat{e}_1} - y_{\mathrm{n}} x_{\mathrm{n}+\hat{e}_2}\right|^2$$

$$\to \frac{1}{g^2a^2}\int d^2z\, \frac{1}{2}\left(D_1 s_2 - D_2 s_1\right)^2 + \frac{1}{4}[s_1, s_2]^2 + \frac{1}{2}\left(v_{12}\right)^2$$

Each term individually violates Euclidean rotation symmetry (as expected for link variables!) but when added together they yield:

$$\frac{1}{g^2a^2}\int d^2z\, \frac{1}{2}|D_m s|^2 - \frac{1}{2}[s^\dagger, s]^2 + \frac{1}{4}v_{mn}v_{mn} + O(a)$$

$s = \frac{s_1 + is_2}{\sqrt{2}}$ becomes the complex scalar with an internal $U(1)$ symmetry; $v_{1,2}$ become the gauge bosons.



The fermions:

With a similar analysis of the fermionic part of the action, one finds that the four 1-component fermions arrange themselves into a single Dirac fermion:

$$\begin{pmatrix} \lambda \\ \xi \end{pmatrix} \to \psi, \qquad i\,(\alpha \;\; \beta) \to \overline{\psi}$$

in the gamma matrix basis

$$\gamma_1 = \sigma_3\ , \quad \gamma_2 = \sigma_1\ , \quad \gamma_3 = \sigma_2\ .$$

One finds:

- No fermion doublers
- Full target theory recovered as $a \to 0$ provided that

$$g^2 a^2 \to (g_2)^2\ , \quad fixed$$

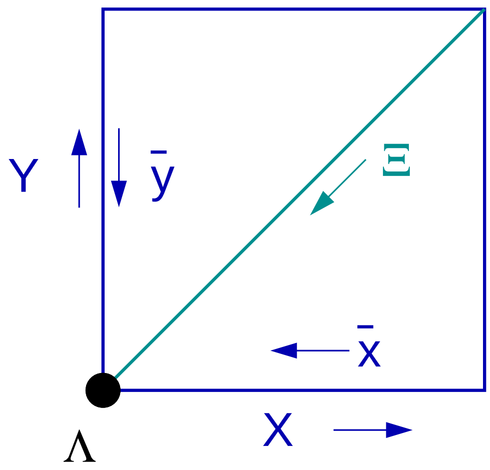
Is the theory stable under radiative corrections?

Yes. Lattice possesses one exact supersymmetry which protects it. In a superfield notation (θ is a Grassmann parameter):

Supercharge: $\boldsymbol{Q = \frac{\partial}{\partial\theta}}$.

Superfields:

$$\mathbf{X_n} = \boldsymbol{x}_{\mathbf{n}} + \sqrt{2}\,\boldsymbol{\theta}\,\boldsymbol{\alpha}_{\mathbf{n}}$$
$$\mathbf{Y_n} = \boldsymbol{y}_{\mathbf{n}} + \sqrt{2}\,\boldsymbol{\theta}\,\boldsymbol{\beta}_{\mathbf{n}}$$
$$\boldsymbol{\Lambda}_{\mathbf{n}} = \boldsymbol{\lambda}_{\mathbf{n}} - \left((\overline{\boldsymbol{x}}_{\mathbf{n}-\hat{\mathbf{e}}_1}\boldsymbol{x}_{\mathbf{n}-\hat{\mathbf{e}}_1} - \boldsymbol{x}_{\mathbf{n}}\overline{\boldsymbol{x}}_{\mathbf{n}}) + (\overline{\boldsymbol{y}}_{\mathbf{n}-\hat{\mathbf{e}}_2}\boldsymbol{y}_{\mathbf{n}-\hat{\mathbf{e}}_2} - \boldsymbol{y}_{\mathbf{n}}\overline{\boldsymbol{y}}_{\mathbf{n}}) + i\boldsymbol{d}_{\mathbf{n}}\right)\boldsymbol{\theta}$$
$$\boldsymbol{\Xi}_{\mathbf{n}} = \boldsymbol{\xi}_{\mathbf{n}} + 2\left(\overline{\boldsymbol{x}}_{\mathbf{n}+\hat{\mathbf{e}}_1}\overline{\boldsymbol{y}}_{\mathbf{n}} - \overline{\boldsymbol{y}}_{\mathbf{n}+\hat{\mathbf{e}}_2}\overline{\boldsymbol{x}}_{\mathbf{n}}\right)\boldsymbol{\theta}$$



- The lattice action may be rewritten in a manifestly supersymmetric fashion;
- the exact supersymmetry plus global symmetries forbid relevant operators that could spoil the continuum limit.

What fixes $\langle x\rangle = \langle y\rangle = \frac{1}{\sqrt{2}\,a}$*?*

One may add to the lattice action small soft SUSY violating operators

$$\frac{1}{g^2}\sum_{\mathbf{n}} \mu^2 a^2\,\mathrm{Tr}\left[\left(x_{\mathbf{n}}^2 - \frac{1}{2a^2}\right)^2 + \left(y_{\mathbf{n}}^2 - \frac{1}{2a^2}\right)^2\right]$$

with $\mu \to 0$ **as the volume of the lattice gets large.**

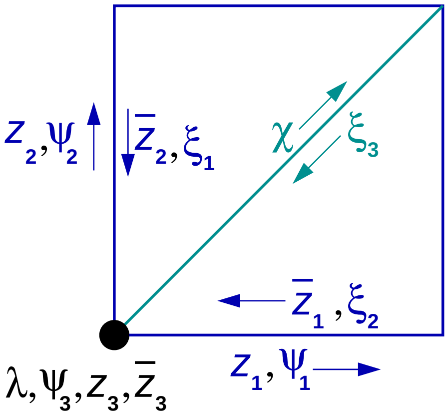
Another example...

Lattice for SYM with **8** supercharges in 2 dimensions

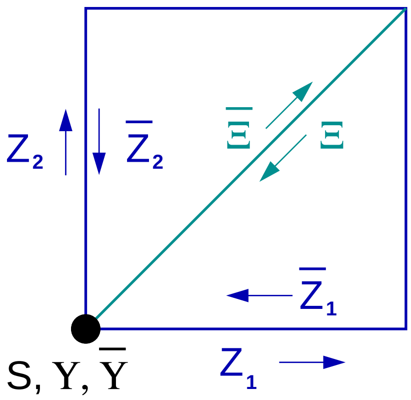
$$\mathcal{L}_{cont.} = \frac{1}{g_2^2}\mathrm{Tr}\left[\frac{1}{4}v_{mn}v_{mn} + \frac{1}{2}(D_m s_\mu)^2 + \overline{\Psi}_i\gamma_m D_m\Psi_i \right.$$
$$\left. + \overline{\Psi}_i[s_0,\ \Psi_i] + i\overline{\Psi}_i\gamma_3\tau^a_{ij}[s_a,\ \Psi_j] - \tfrac{1}{4}[s_\mu,\ s_\nu]^2\right].$$

$$m,n = 1,2; \quad i,j = 1,2; \quad a,b = 1,2,3; \quad \mu,\nu = 0,\ldots,3.$$

Components



Superfields



Lattice possesses 2 exact supercharges. Superfields are functions of Grassmann variables θ, $\overline{\theta}$. *For example:*

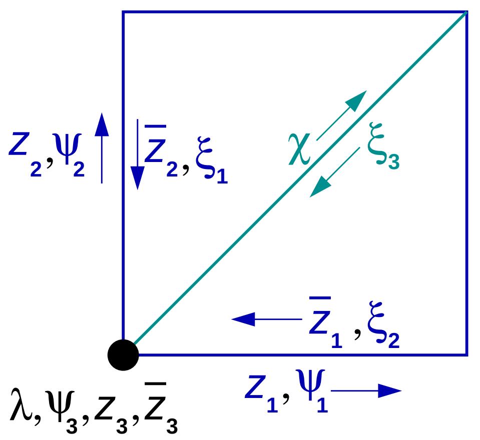
$$\mathbf{Z}_{i,\mathbf{n}} = z_{i,\mathbf{n}} + \sqrt{2}\,\theta\psi_{i,\mathbf{n}} - \sqrt{2}\,\theta\overline{\theta}(\overline{z}_{3,\mathbf{n}}z_{i,\mathbf{n}} - z_{i,\mathbf{n}}\overline{z}_{3,\mathbf{n}+\hat{\mathbf{e}}_i})$$

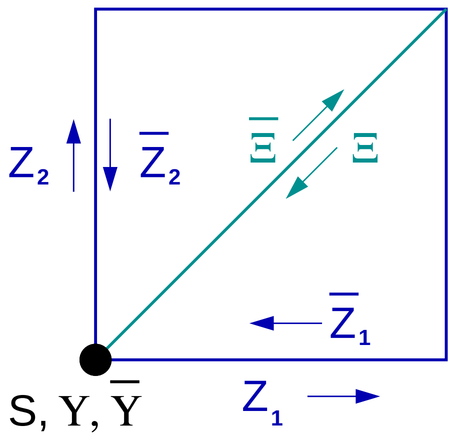
$$\Xi_{\mathbf{n}} = \xi_{3,\mathbf{n}} + \sqrt{2}\,\theta\Big(G_{\mathbf{n}} - \sqrt{2}\,\epsilon_{ij}\,\overline{z}_{i,\mathbf{n}+\hat{\mathbf{e}}_j}\overline{z}_{j,\mathbf{n}}\Big) - \sqrt{2}\,\theta\overline{\theta}(\overline{z}_{3,\mathbf{n}+\hat{\mathbf{e}}_1+\hat{\mathbf{e}}_2}\xi_{3,\mathbf{n}} - \overline{z}_{3,\mathbf{n}}\xi_{3,\mathbf{n}})\ ,$$

$$\Upsilon_{\mathbf{n}} = \lambda_{\mathbf{n}} - \theta\Big(\overline{z}_{i,\mathbf{n}-\hat{\mathbf{e}}_i}z_{i,\mathbf{n}-\hat{\mathbf{e}}_i} - z_{i,\mathbf{n}}\overline{z}_{i,\mathbf{n}} + [\overline{z}_{3,\mathbf{n}}, z_{3,\mathbf{n}}] + id_{\mathbf{n}}\Big) - \sqrt{2}\theta\overline{\theta}[\overline{z}_{3,\mathbf{n}}, \lambda_{\mathbf{n}}]\ .$$

*Lattice for SYM with **8** supercharges in 2 spacetime dimensions continued*

Components *Superfields*





Lattice action:

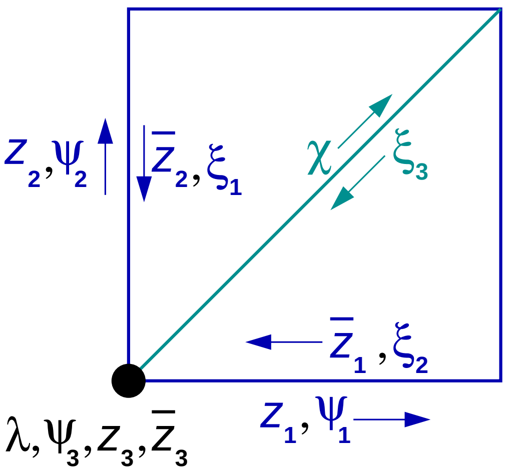
$$S = \frac{1}{g^2}\sum_{\mathbf{n}} \mathrm{Tr}\left[\int d\theta d\overline{\theta}\,\left(\frac{1}{2}\overline{\mathbf{\Upsilon}}_{\mathbf{n}}\mathbf{\Upsilon}_{\mathbf{n}} + \frac{1}{\sqrt{2}}\mathbf{S}_{\mathbf{n}}(\mathbf{Z}_{i,\mathbf{n}}\overline{\mathbf{Z}}_{i,\mathbf{n}} - \overline{\mathbf{Z}}_{i,\mathbf{n}-\hat{\mathbf{e}}_i}\mathbf{Z}_{i,\mathbf{n}-\hat{\mathbf{e}}_i}) - \frac{1}{2}\overline{\mathbf{\Xi}}_{\mathbf{n}}\mathbf{\Xi}_{\mathbf{n}}\right)\right.$$
$$\left. + \int d\theta\,\left(\epsilon_{ij}\,\mathbf{\Xi}_{\mathbf{n}}\,\mathbf{Z}_{i,\mathbf{n}}\mathbf{Z}_{j,\mathbf{n}+\hat{\mathbf{e}}_i}\right) - \int d\overline{\theta}\,\left(\epsilon_{ij}\,\overline{\mathbf{\Xi}}_{\mathbf{n}}\,\overline{\mathbf{Z}}_{i,\mathbf{n}+\hat{\mathbf{e}}_j}\overline{\mathbf{Z}}_{j,\mathbf{n}}\right)\right]$$

$$= \frac{1}{g^2}\sum_{\mathbf{n}} \mathrm{Tr}\left[\tfrac{1}{2}\left(\overline{z}_{i,\mathbf{n}-\hat{\mathbf{e}}_i}z_{i,\mathbf{n}-\hat{\mathbf{e}}_i} - z_{i,\mathbf{n}}\overline{z}_{i,\mathbf{n}} + [\overline{z}_{3,\mathbf{n}},\ z_{3,\mathbf{n}}]\right)^2\right.$$
$$+ 2\left(\left|\epsilon_{ij}z_{i,\mathbf{n}}z_{j,\mathbf{n}+\hat{\mathbf{e}}_i}\right|^2 + \left|z_{i,\mathbf{n}}z_{3,\mathbf{n}+\hat{\mathbf{e}}_i} - z_{3,\mathbf{n}}z_{i,\mathbf{n}}\right|^2\right)$$
$$+ \sqrt{2}\Big(\Delta_{\mathbf{n}}(\lambda, \overline{z}_a, \psi_a) - \Delta_{\mathbf{n}}(\chi, \overline{z}_a, \xi_a) + \epsilon_{abc}\Delta_{\mathbf{n}}(\psi_a, z_b, \xi_c)\Big)$$

Supercharges:

$$Q = \frac{\partial}{\partial\theta} + \sqrt{2}\,\overline{\theta}[\overline{z}_3,\ \cdot\ ]\ ,\quad \overline{Q} = \frac{\partial}{\partial\overline{\theta}} + \sqrt{2}\,\theta[\overline{z}_3,\ \cdot\ ]\ ,\quad \{Q, \overline{Q}\} = 2\sqrt{2}[\overline{z}_3,\ \cdot\ ]$$

*Lattice for SYM with **8** supercharges in 2 spacetime dimensions continued:*



In the continuum limit:

$$\boldsymbol{\langle z_i \rangle = \frac{1}{\sqrt{2}\,a},\ a \to 0,\ g^2 a^2 \to g_2^2\ \textbf{fixed}}$$

- The imaginary parts of the complex link bosons $z_{1,2}$ become the two gauge field components v_m;
- The real parts of the link bosons $z_{1,2}$ and the complex site boson z_3 become the four real scalars s_μ;
- The two Dirac fermions have components:

$$\Psi_1 = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}\ ,\quad \Psi_2 = \begin{pmatrix} \lambda \\ -\xi_3 \end{pmatrix}\ ,\quad \overline{\Psi}_1 = \begin{pmatrix} -\chi & \psi_3 \end{pmatrix}\ ,\quad \overline{\Psi}_2 = \begin{pmatrix} \psi_1 & \psi_2 \end{pmatrix}\ .$$

- *No fine tuning needed*

How were these lattices constructed? In brief:

For a SYM **target theory** in $\boldsymbol{d}$ Euclidean dimensions with a $U(\boldsymbol{k})$ gauge symmetry and $\boldsymbol{Q}$ supercharges,

- Begin with the desired target theory in the continuum, but with the much bigger gauge group $U(\boldsymbol{k}\boldsymbol{N}^{\boldsymbol{d}})$;
- Reduce the theory to zero dimensions $\Longrightarrow$ a matrix model, with $\boldsymbol{Q}$ supersymmetries, and a $U(\boldsymbol{k}\boldsymbol{N}^{\boldsymbol{d}})\times G_R$ global symmetry
  - *For* $\boldsymbol{Q}=4$, $G_R=SO(4)\times U(1)$;
  - *For* $\boldsymbol{Q}=8$, $G_R=SO(6)\times SU(2)$;
  - *For* $\boldsymbol{Q}=16$, $G_R=SO(10)$.

  G_R is an "R-symmetry" which does not commute with the supercharges.
- Identify a particular $(Z_{\boldsymbol{N}})^{\boldsymbol{d}}$ subgroup of the $U(\boldsymbol{k}\boldsymbol{N}^{\boldsymbol{d}})\times G_R$ symmetry. Remove from the matrix model any field which transforms nontrivially under this discrete symmetry ("orbifolding").

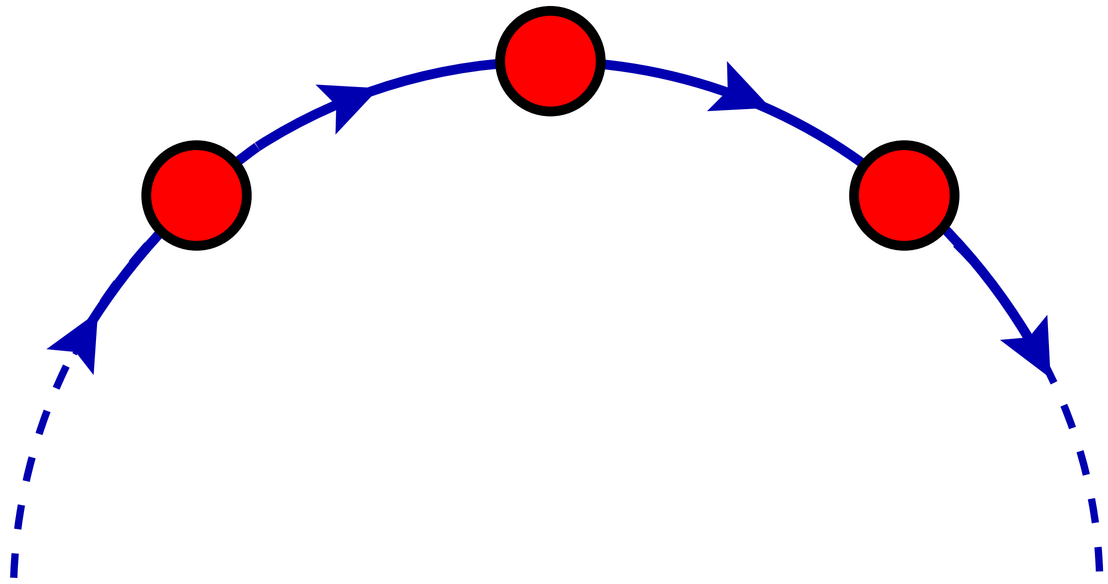
The resulting matrix model may be recognized as a $\boldsymbol{N}^{\boldsymbol{d}}$-site lattice possessing a $U(\boldsymbol{k})$ gauge symmetry, and $(\boldsymbol{Q} \times 2^{-\boldsymbol{d}})$ exact supercharges.

If one takes the continuum limit as discussed in the 2D example, the tree level theory will always yield the desired target theory in the $a \to 0$ limit. (No undesired relevant operators at tree level)

A quick tour of the remaining lattices which can be constructed in this manner...

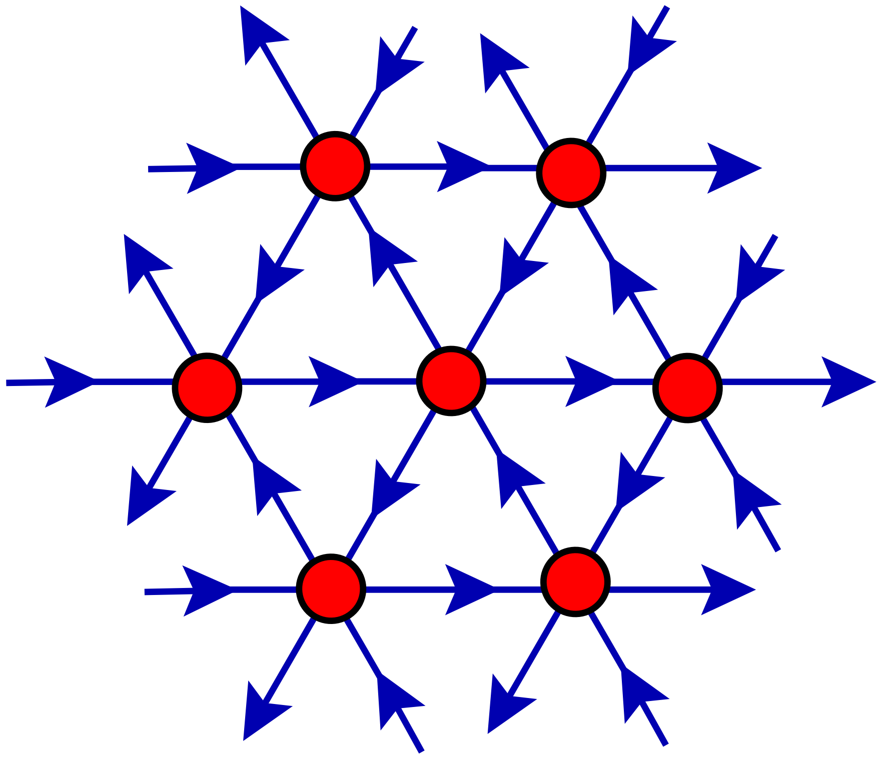
Super Yang Mills theories in $0 + 1$ dimensions

Quantum Mechanics with **2, 4, 8** *or* **16** *supercharges, in a discretized Euclidean Lagrangean formulation*



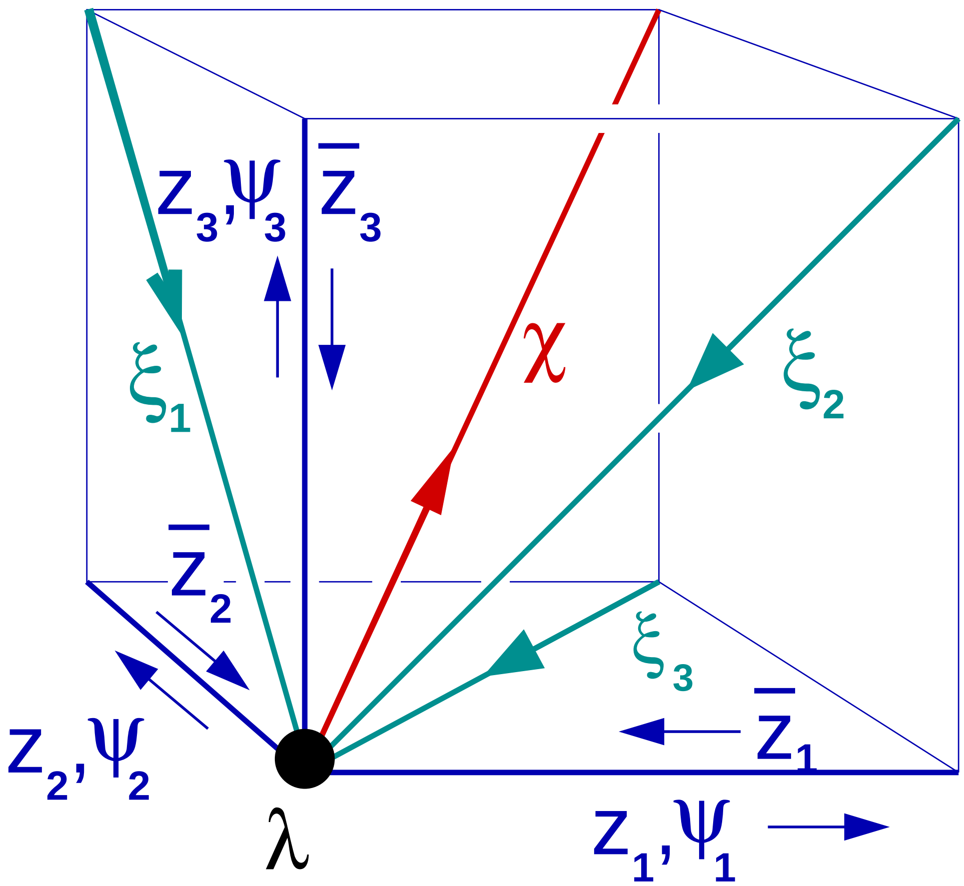
- Superfields on both sites and links
- Number of exact supercharges $= \frac{1}{2} \times$ number of target theory
- *The* 16 *supercharge case is "Matrix Theory": claimed to be equivalent to M-theory in the large* N_c *limit.*

Super Yang Mills theories in 2 dimensions with **16** supercharges



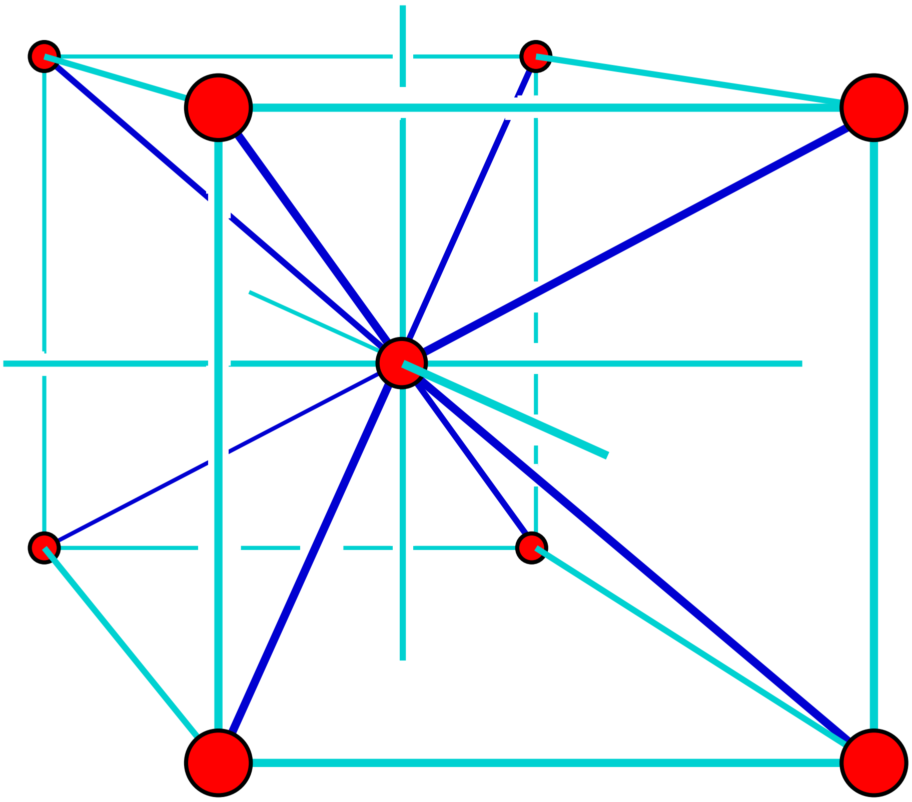
- Lattice possesses 4 exact supercharges
- Superfields on links and sites
- Site-link interactions and triangle plaquette interactions among links
- *No fine tuning required*

Super Yang Mills theories in 3 dimensions with **8** supercharges



- Lattice possesses 1 exact supercharge
- *Two operators may require logarithmic fine tuning; can be computed exactly by performing two 1-loop diagrams.*

Super Yang Mills theories in 3 dimensions with **16** supercharges



- Lattice possesses 2 exact supercharges
- Continuum theory is expected to have IR conformal fixed point, $SO(8)$ chiral symmetry (Seiberg)
- *No fine tuning expected.*

Super Yang Mills theories in 4 dimensions with **16** supercharges

- Lattice has A_4^* structure
  the lattice generated by the fundamental weight vectors of $SU(5)$

- Lattice possesses 1 exact supercharge

- Target theory is $N = 4$ SYM, a conformal field theory

- *Fine tuning issues still unresolved*

Closing Remarks:

The orbifold technique allows one to construct supersymmetric lattices with unusual and beautiful properties. It is hoped that they will be useful for both theoretical, and eventually, numerical investigations.

The treatment of dynamical massless (or light) fermions remains a hurdle for numerical studies.

Possible directions to pursue for now:

- Simulation of the 1D and 2D examples;
- Development of a Nicolai map for these lattices, as an alternative to computing the fermion determinant;
- Determining whether there are other supersymmetric lattices that could be constructed in similar manners;
- Studies of the lattice for $N = 4$ SYM in 4D for a better understanding of dualities...