Calculating q^* for matching factors of operators with anomalous dimensions

Weonjong Lee & Steve Sharpe (in progress)

$$O_{cont}(\mu) = Z(\mu, a)\ O_{latt}(a)$$

$$Z(\mu,a) = 1 + \alpha_V(q^*)\left[\frac{-\gamma^{(0)}\ln \mu a + C_1}{4\pi}\right] + O(\alpha^2)$$

anom. dim.

What scale q^* should one pick?
($\alpha_V \sim 0.2$)

- Application of scale setting methods of BLM (Brodsky, Lepage, Mackenzie) & LM (Lepage, Mackenzie) to matching factors

- We need for calculating weak matrix elements using improved staggered fermions (non-perturbative matching difficult)

Technical note:

- In general, operator matching involves mixing

$$O_{cont,i}(\mu) = Z(\mu, a)_{ij}\; O_{latt,j}(a)$$

- Focus today on examples without mixing

spin, taste

	C_1	$C_1/4\pi$
$O_{S\times S,\,naive} = \bar{Q}^{(1)}(\mathbb{1} \otimes \mathbb{1})\, Q^{(2)}$	-39	~ -3
$O_{S\times S,\,HYP} = \bar{Q}^{(1)}(\mathbb{1} \otimes \mathbb{1})\, Q^{(2)}$	-0.1	~ 0
$O_{S\times T,\,HYP} = \bar{Q}^{(1)}(\mathbb{1} \otimes \xi_\mu \xi_\nu)\, Q^{(2)}$	6.5	~ 0.5
$O_{S\times P,\,HYP} = \bar{Q}^{(1)}(\mathbb{1} \otimes \xi_5)\, Q^{(2)}$	12.9	~ 1

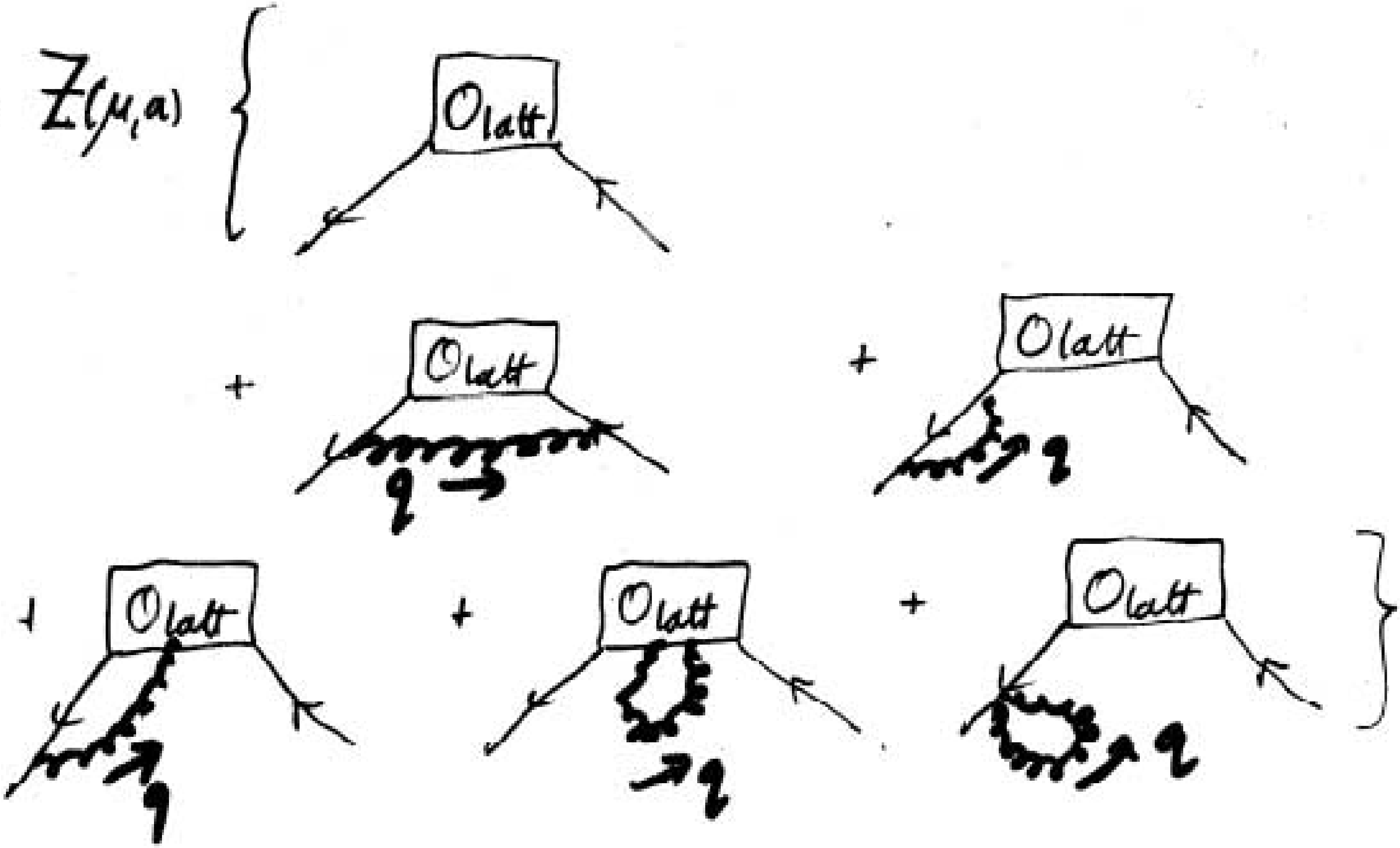
(all flavor non-singlets)

- all have $\gamma^{(0)}_a - 8$

- methods presumably generalize to cases involving mixing

Diagrams contributing to 1-loop matching:

$= Z(\mu, a)$



- only gluon loops – q is momentum on gluon line
- IR regulation by "gluon mass" λ : $\frac{1}{q^2} \to \frac{1}{q^2+\lambda^2}$

Importance of determining q^*

- Consider finite lattice quantity ($\gamma = 0$) e.g. Z_V, Z_A, W_{MCL}

$$Z_V = 1 + \underbrace{\alpha\, \tilde{c}_1}_{\text{known, of } O(1)} + \underbrace{\alpha^2 \tilde{c}_2}_{\text{not known,}} + \ldots$$

- LM: use physical coupling, e.g. α_V $\left[V(q) \equiv -C_F \frac{4\pi \alpha_V(q)}{q^2} \right]$,

  use scale $q^* =$ "typical" momentum flowing in gluon

  A-priori: $\frac{0.5}{a} \lesssim q^* \lesssim \frac{2\pi}{a}$

$$\Rightarrow \Delta Z_V \approx \tilde{c}_1 \Delta\alpha$$
$$\approx \tilde{c}_1 \alpha^2 \underbrace{\left[\frac{\beta_0}{4\pi}\, 2\, \Delta \ln q^* \right]}_{\approx\, 2 \;-\; 4.5}$$

$\beta_0 = 11 - 2N_f/3$

$\Delta \ln q^* = \ln \pi$ $\qquad$ $\Delta \ln q^* = \ln 4\pi$

$\Rightarrow$ larger uncertainty than naive

$$\alpha^2 \tilde{c}_2 \quad \text{with} \quad |\tilde{c}_2| \approx 1$$

- LM, Horada et al, ... works in many examples
  - correct q^* reduces errors to naive $O(\alpha^2)$ level

Why we need q^* !

- We use "horizontal matching" Gupta, Bhattacharya, Sharpe

$$Z^{\text{hor-1loop}}(\mu=2\,\text{GeV}, a) = Z^{\text{cont}}(2\,\text{GeV}, \text{"}q^*\text{"})\, Z^{\text{1-loop match}}(\text{"}q^*\text{"}, a)$$

continuum running using 2-loop anom. dims.

$$\left(1 + \frac{\alpha_V(q^*)}{4\pi}\left[-\gamma^{(0)} \ln q^* a + c_1\right]\right)$$

- Nominal dependence on "q^*" is at $O(\alpha_V(q^*)^2)$

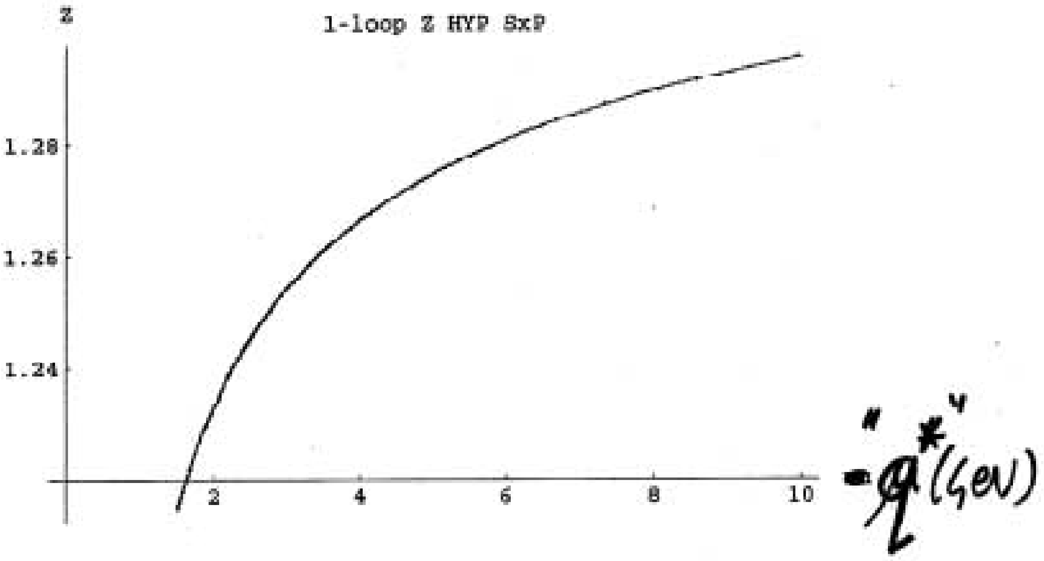
→ Figure

- For $\frac{1}{a} \le q^* \le \pi/a$, $\Delta Z \sim (3\text{-}4)\,\alpha_V^2$

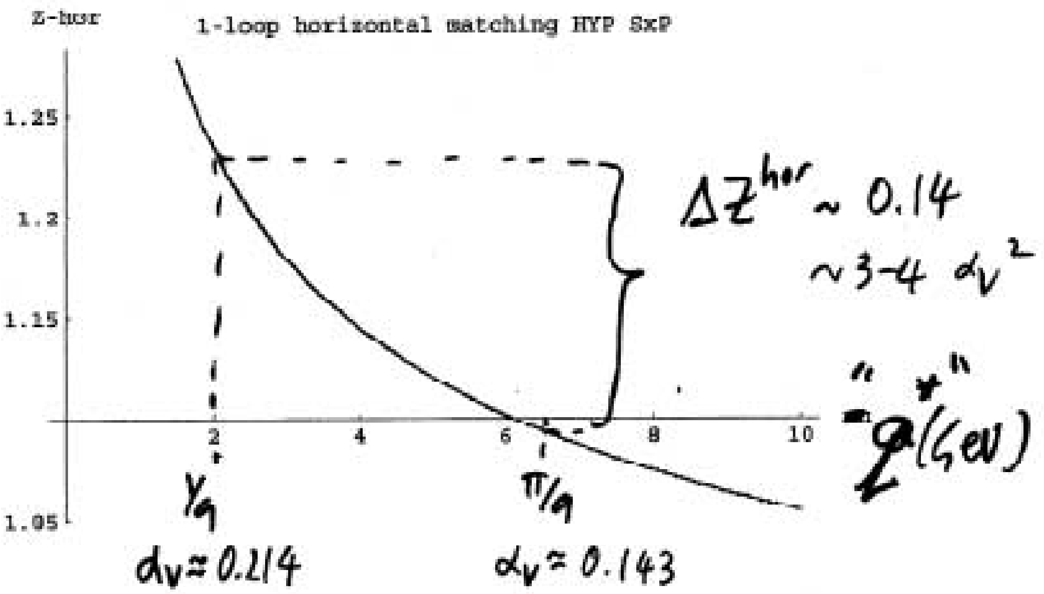
- Can we reduce to $\sim \pm 1\, \alpha_V^2$?

Old analysis method applied to HYP $S \times P$

1-loop match
$Z(``q^*", a)$



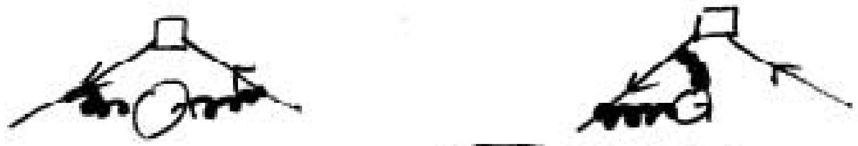
$Z^{\text{hor-1-loop}}(2\,\text{GeV}, a)$



- $a^{-1} = 2\,\text{GeV}$

BLM method of determining q^*

- requires that N_f dependent part of 2-loop coef. is <u>known</u>



N_f independent

$$\text{"}Z_V\text{"} = 1 + c_1 \frac{\alpha_V(\frac{1}{a})}{4\pi} + \left[c_2 + \overbrace{\left(11 - \frac{2N_f}{3}\right)}^{\beta_0} d_2\right]\left(\frac{\alpha_V}{4\pi}\right)^2 + \dots$$

known   unknown   known

finite qty ($\gamma = 0$)

important to use physical coupling (affects value of d_2)

- assume large 2-loop contribution comes from renormalization of coupling in 1-loop graphs & that fermion loops give size of this contribution — gluon & ghost contributions coming in ratio given by β_0

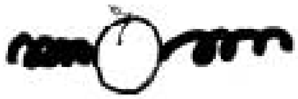
* since $\frac{\alpha_V}{4\pi}(q^*) = \frac{\alpha_V}{4\pi}\left(\frac{1}{a}\right) - \beta_0 \, 2\ln q^* a \left(\frac{\alpha}{4\pi}\right)^2$

find
$$\boxed{\text{"}Z_V\text{"} \simeq 1 + c_1 \frac{\alpha_V(q^*)}{4\pi} + c_2\left(\frac{\alpha}{4\pi}\right)^2} + \dots$$

with
$$\boxed{q^* = \frac{1}{a}\exp\left(\frac{-d_2}{2c_1}\right)}$$

What if don't know N_f-dep. part of 2-loop term?

- LM: calculate it approximately, assuming

 $\rightarrow \dfrac{1}{q_L^2}\underbrace{\left(\dfrac{2N_f}{3}\dfrac{\alpha_V}{4\pi}\, q_L^2 \ln q_L^2\right)}_{\text{approx form of loop integral}}\dfrac{1}{q_L^2}$

q_L^2 is a discretization of q^2a^2 (e.g. $q_L^2 = \sum_\mu 4\sin^2 q_\mu a/2$)

<u>and taking</u>, 1-loop α to be $\alpha_V(\frac{1}{a})$

- Explicitly:

gluon

<u>if</u> 1-loop term $= \dfrac{\alpha_V(\frac{1}{a})}{4\pi}\displaystyle\int_q^{\text{latt}} f_L(q)\,\frac{1}{q_L^2} = \frac{\alpha_V(1/a)}{4\pi}\,c_1$

then N_f dep. part of 2-loop term $\approx \dfrac{2N_f}{3}\left[\dfrac{\alpha_V(\frac{1}{a})}{4\pi}\right]^2 \displaystyle\int_q^{\text{latt}} f_L(q)\,\frac{\ln q_L^2}{q_L^2} = -\frac{2N_f}{3}\left[\frac{\alpha}{4\pi}\right]^2 d_2$

- Apply BLM with approximate d_2

$$2\ln q^* a = \frac{-d_2}{c_1} = \frac{\int_q^{\text{latt}} f_L(q)\,\ln q_L^2/q_L^2}{\int_q^{\text{latt}} f_L(q)\,1/q_L^2} = \langle \ln q_L^2\rangle$$

New issues when applying BLM/LM to matching factors $Z(\mu, a)$:

- 1-loop contribution to Z is the difference between log divergent matrix elements calculated with two different regulators
  (not a single finite lattice integral as for Z_V)
  - IR divergences must cancel when insert "$\ln q^2$"
  - what is the correct form of "$\ln q^2$" in continuum regulator?
  - how deal with double poles that "$\ln q^2$" introduces?
- Is there any relationship between q^* & μ?

Two results:

- applying BLM to anom. dim. $\gamma(g)$
  - conceptually nice, but does not solve α^2 problem
- apply LM to $Z(\mu, a)$ for $\ln(\mu a) << 1$
  - removes large $O(\alpha^2)$ terms from horizontal matching

BLM - like Method

Based on Ji's formula

$$Z(\mu,a) = \lim_{g'\to 0} \exp\left\{ - \int_{g'}^{g_c(\mu)} d\tilde{g}\, \frac{\gamma_c(\tilde{g})}{\beta_c(\tilde{g})} + \int_{g'}^{g_L(a)} d\tilde{\tilde{g}}\, \frac{\gamma_L(\hat{g})}{\beta_L(\hat{g})} \right\}$$

- Valid for any coupling constant schemes, as long as adjust γ (and β) appropriately
- Explicitly IR insensitive

In practice, truncate $\overset{\gamma\beta}{\underset{\wedge}{\gamma}}$, usually at 2 loops:

$$Z(\mu,a) \simeq \left(\frac{\alpha_C(\mu)}{\alpha_L(a)}\right)^{\frac{\gamma^{(0)}}{2\beta_0}} \frac{\left(1 + \frac{\alpha_L(a)}{4\pi} J_L\right)}{\left(1 + \frac{\alpha_C(\mu)}{4\pi} J_C\right)}$$

Buras

$$J_{L,C} = \frac{\gamma^{(0)}\beta_1}{2\beta_0^{\,2}} - \frac{\gamma_{L,C}^{(1)}}{2\beta_0}$$

accurate up to $(1 + O(\alpha^2))$

IDEA: apply BLM to $\gamma_{C,L}(g)$!

Step 1: use V-scheme in cont. & latt.

e.g. $$\gamma_c(g_V^{(\mu)}) = \frac{\alpha_V(\mu)}{4\pi}\gamma^{(0)} + \left(\frac{\alpha}{4\pi}\right)^2 \underbrace{\gamma^{(0)}(c_\gamma + \beta_0 d_\gamma)}_{\gamma^{(1)}} + \ldots$$

Step 2: reexpand using $\alpha_V(\mu^*)$ to remove d_γ

$$\gamma_c = \frac{\alpha_V(\mu^*)}{4\pi}\gamma^{(0)} + \left(\frac{\alpha}{4\pi}\right)^2 \gamma^{(0)} c_\gamma$$

$$\underbrace{\mu^* = \mu e^{-d_\gamma/2}}_{\text{"}q^*\text{ for anom dim" in continuum}} \qquad \left(d_\gamma = -\tfrac{5}{6} \text{ for scalar bilinear}\right)$$

Find: 2 loop term SAME in lattice & cont.

$$\gamma_L(g(a)) = \frac{\alpha_V(\frac{1}{a^*})\gamma^{(0)}}{4\pi} + \left(\frac{\alpha}{4\pi}\right)^2 \gamma^{(0)} c_\gamma$$

$$\frac{1}{a^*} = \frac{1}{a} e^{-d_\gamma/2}\, e^{\underbrace{\frac{J_L - J_C}{\gamma^{(0)}}}_{\frac{\gamma_c^{(1)} - \gamma_L^{(1)}}{2\gamma^{(0)}\beta_0} = \frac{c_1}{\beta_0}}}$$ ← depends on operator

↑ optimal scale for lattice anom. dim. Differs from that for continuum

Result: reorganization of horizontal matching, with scales chosen to align cont. & lattice ops:

$$Z_{BLM}(\mu,a) = \left(\frac{\alpha_V(\mu^*)}{\alpha_V(1/a^*)}\right)^{\frac{\gamma^{(0)}}{2\beta_0}} \frac{\left[1+\frac{\alpha_V(1/a^*)}{4\pi}J'\right]}{\left[1+\frac{\alpha_V(\mu^*)}{4\pi}J'\right]}\left(1+O(\alpha^2)\right)$$

$$J' = \frac{\gamma^{(0)}\beta_1}{2\beta_0^2} - \frac{\gamma^{(0)}c_\sigma}{2\beta_0}$$

Special μ?

If $\mu^* = 1/a^*$ then $Z_{BLM} = 1 + O(\alpha^2)$

$\Rightarrow \mu = \frac{1}{a}\exp\left(\frac{J_L - J_C}{\gamma^{(0)}}\right)$

Can also apply to continuum running:

$$Z^{cont}_{BLM}(\mu_1,\mu_2) = \left(\frac{\alpha_V(\mu_1^*)}{\alpha_V(\mu_2^*)}\right)^{\frac{\gamma^{(0)}}{2\beta_0}} \frac{\left[1+\frac{\alpha_V(\mu_2^*)}{4\pi}J'\right]}{\left[1+\frac{\alpha_V(\mu_1^*)}{4\pi}J'\right]}$$

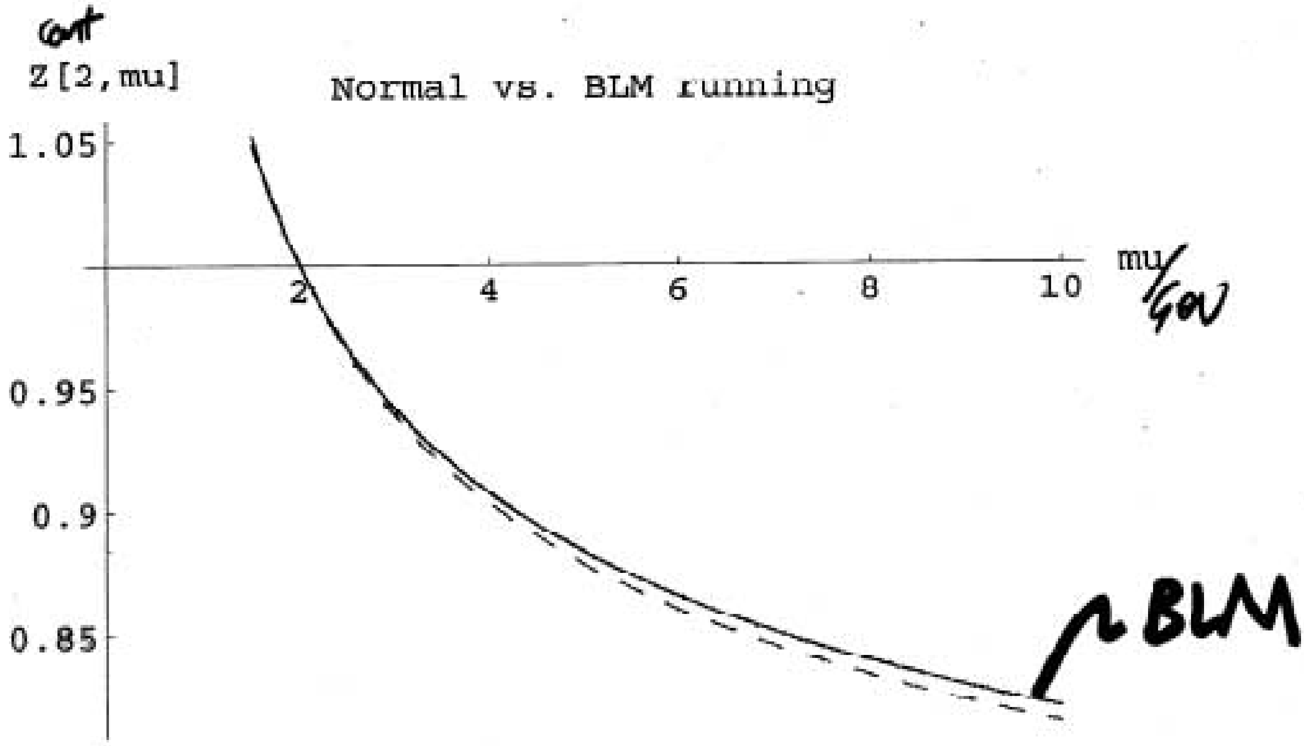
$$\mu_i^* = \mu_i\, e^{-d\gamma/2}$$

Numerical example:

Operator	μ^*/μ	$(\frac{1}{a^*})/(\frac{1}{a})$	μa for $\mu^* = 1/a^*$
S×S, naive	1.52	202 !	133 !
S×S, HYP	1.52	1.54	1.016 ← ok?
S×T, HYP	1.52	0.675	0.445 ⇒ μ too small if $\frac{1}{a} \approx 2$ GeV ?
S×P, HYP	1.52	0.303	0.200

too small ?

Normal running vs. "BLM" running



Continuum running for scalar bilinear

— WORKS FINE

- To go further with BLM requires N_f dep. terms of 3-loop anom. dims. — NOT AVAILABLE
- Can use LM-like method to approximately calculate N_f dependent $O(\alpha^2)$ term in Z
- Our method requires $\ln \mu a \ll 1$ so can perturbatively expand Z:

$$Z(\mu, a) = 1 + \frac{\alpha_V(\mu)}{4\pi}\left(-\gamma^{(0)} \ln \mu a + c_1\right)$$

Ratio of Shape

$$+\left(\frac{\alpha}{4\pi}\right)^2 \Big\{ \frac{\gamma^{(0)}}{2}\left(\gamma^{(0)} - 2\beta_0\right)\left[\ln \mu a\right]^2$$

$$-\left[\gamma^{(0)} c_1 + \gamma_L^{(1)}\right] \ln \mu a$$

$$+ C_2 + \beta_0 d_2 \Big\} + O(\alpha^3)$$

unknown: assume $\pm 8\pi^2$

evaluate approximately following LM; ABSORB into scale of leading order term

$$\Rightarrow q^* = q^*(\mu a)$$

- Need higher order LM (HLM - Hornbostel, Lepage, Morningstar) if leading $O(\alpha)$ term vanishes

On lattice side of matching, follow LM exactly – need to be careful about IR divergences

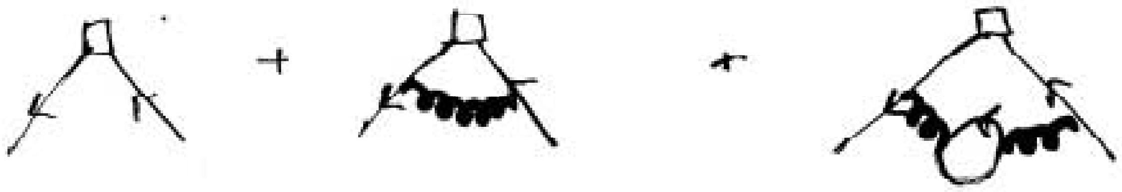
gluon

$$\mathcal{M}^{\text{latt}}_{1\text{-loop}} = \mathcal{M}_0\left[1+\frac{\alpha}{4\pi}\underbrace{\int_q^{\text{latt}}\left(\frac{-\gamma^{(0)}}{2q_L^2}+\tilde{f}_L(q)\right)\frac{1}{q_L^2+\lambda^2a^2}}_{\gamma^{(0)}\ln a\lambda \;+\; \text{const.} \;+\; O(\lambda)}\right]$$

IR divergent **IR finite**

approximate N_f-dependent 2-loop term by (plus take $\alpha \to \alpha_V(1/a)$ in 1-loop term)

$$\mathcal{M}_0\,\frac{2N_f}{3}\left(\frac{\alpha}{4\pi}\right)^2\underbrace{\int_q^{\text{latt}}\left(\frac{-\gamma^{(0)}}{2q_L^2}+\tilde{f}_L\right)\frac{\ln q_L^2}{q_L^2+\lambda^2a^2}}_{\gamma^{(0)}[\ln a\lambda]^2 \;+\; \text{const.}'}$$



$+ \ldots$ $\qquad$ $+ \ldots$

- Continuum side of matching: $(\varepsilon = (4-n)/2)$

$$\underset{1\text{-loop}}{M^C\left(\tfrac{1}{\mu}\right)} = M_0\left(1 + \frac{\alpha(\mu)}{4\pi} \lim_{n\to 4} \underbrace{\left[\int_q^{\text{cut}} \frac{-\gamma^{(0)}}{2q^2}(1 - v_1\varepsilon + \ldots)\frac{1}{q^2+\lambda^2} - \text{pole}\right]}_{-\gamma^{(0)}\ln(\mu/\lambda) \;+\; \text{const}_1'' \;+\; O(\lambda)}\right)$$

- To obtain approximate N_f-dep. 2-loop term replace

$$\ln q^2/\mu^2 \quad \text{with} \quad \frac{1}{\varepsilon}\left[1 - \left(\frac{q^2}{\mu^2}\right)^{-\varepsilon}(1 + l_1\varepsilon + l_2\varepsilon^2)\right] = \ln q^2/\mu^2 + l_1 + O(\varepsilon)$$

more accurate representation of 

- Gives double poles (OK - since 2-loop) but λ indep$^{\underline{t}}$ so can subtract using $\overline{MS}$

- Result for approx N_f-dep 2-loop term is:

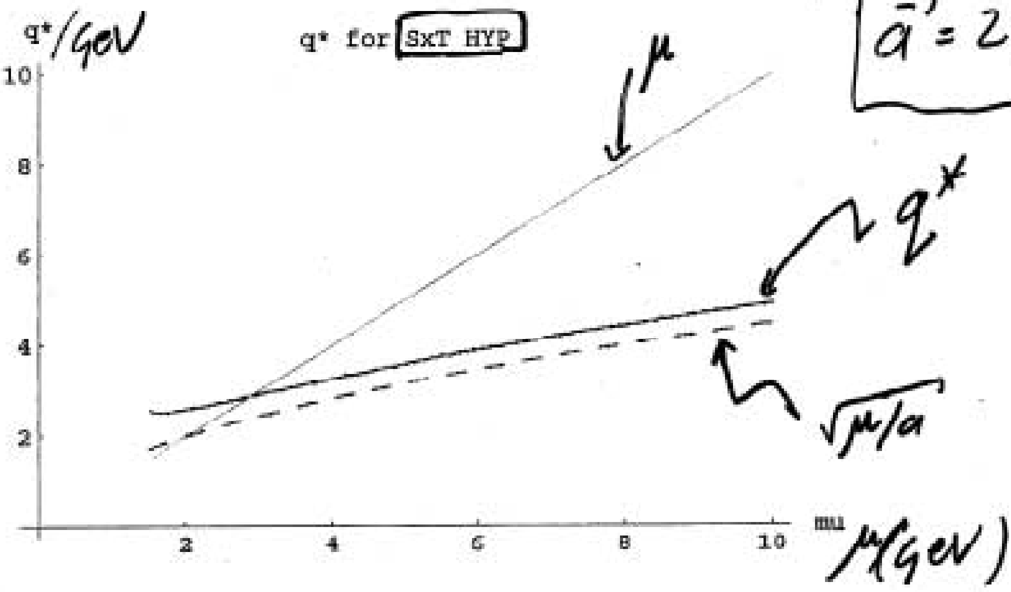
$$M_0\left(\frac{\alpha}{4\pi}\right)^2\left(\frac{2N_f}{3}\right)\left[\gamma^{(0)}\left[\ln \lambda/\mu\right]^2 + l_1 \ln \lambda/\mu + \text{const}_1'''' + O(\lambda)\right]$$

- Leading & subleading IR logs cancel if $l_1 = 0$
- Matching factor has expected μa dependence at 2-loop + obtain approx. value for d_2

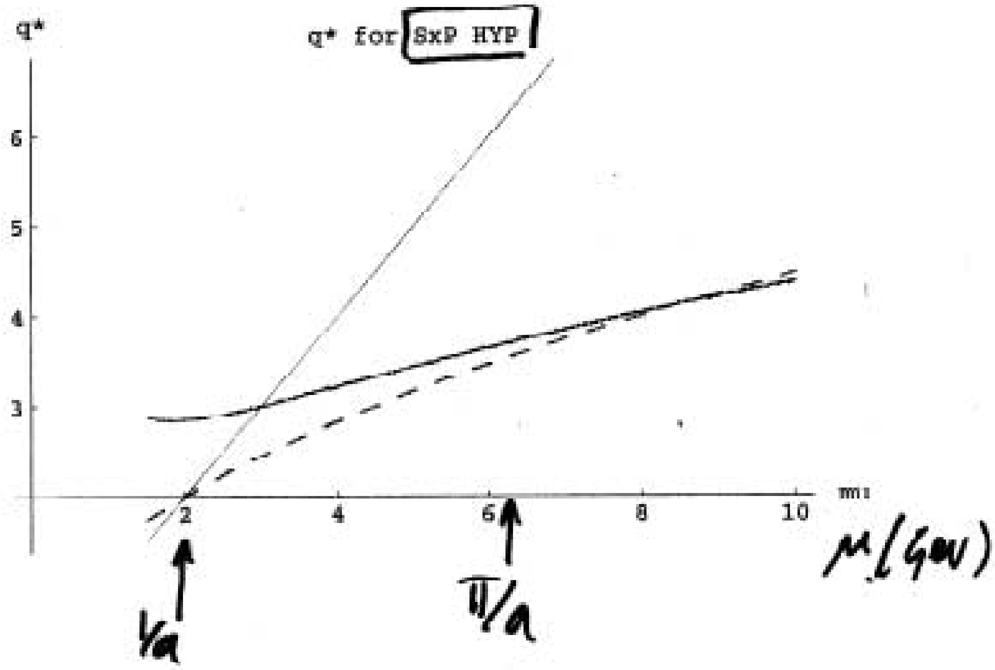
q^*, μ and $\sqrt{\mu/a}$ for HYP $S \times T$, $S \times P$

$\bar{a}^{-1} = 2\,\text{GeV}$

$\frac{c_1}{4\pi} \sim 0.4$



$\frac{c_1}{4\pi} \sim 1$



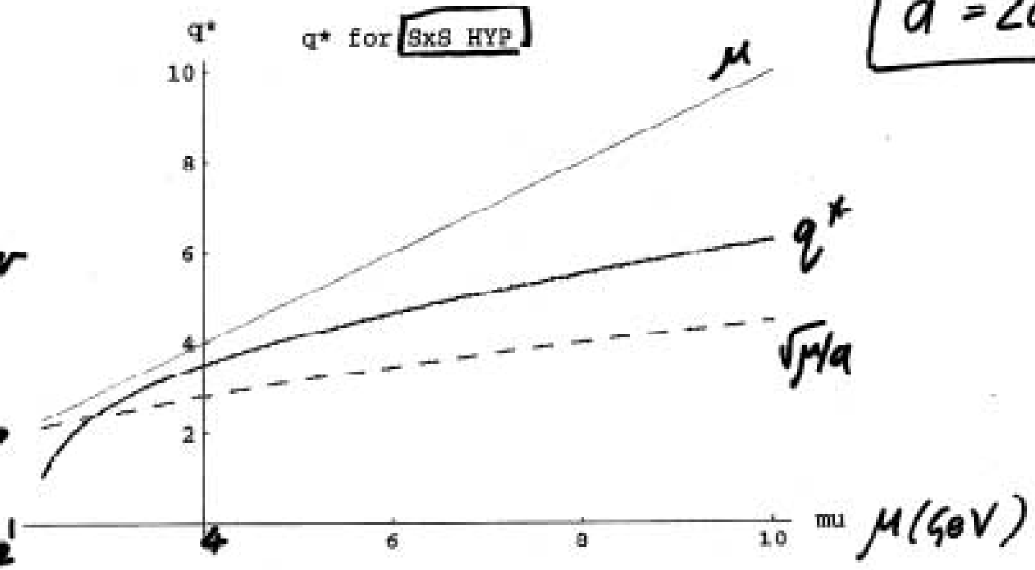
q^* is reasonable

q^{*}, μ and $\sqrt{\mu/a}$ for HYP and naive $S \times S$

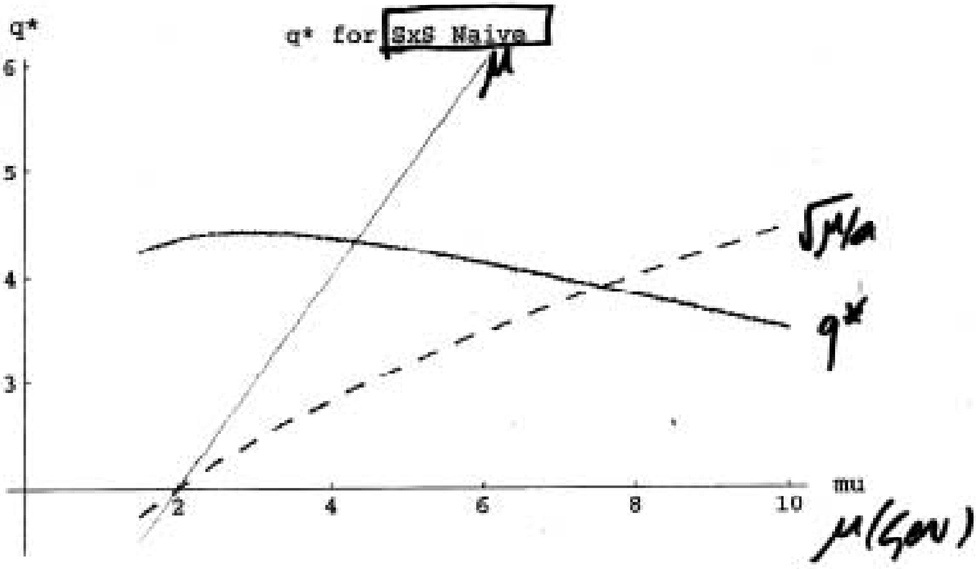
$\boxed{a^{-1} = 2\,\mathrm{GeV}}$

$\frac{C_1}{4\pi} \sim 0$

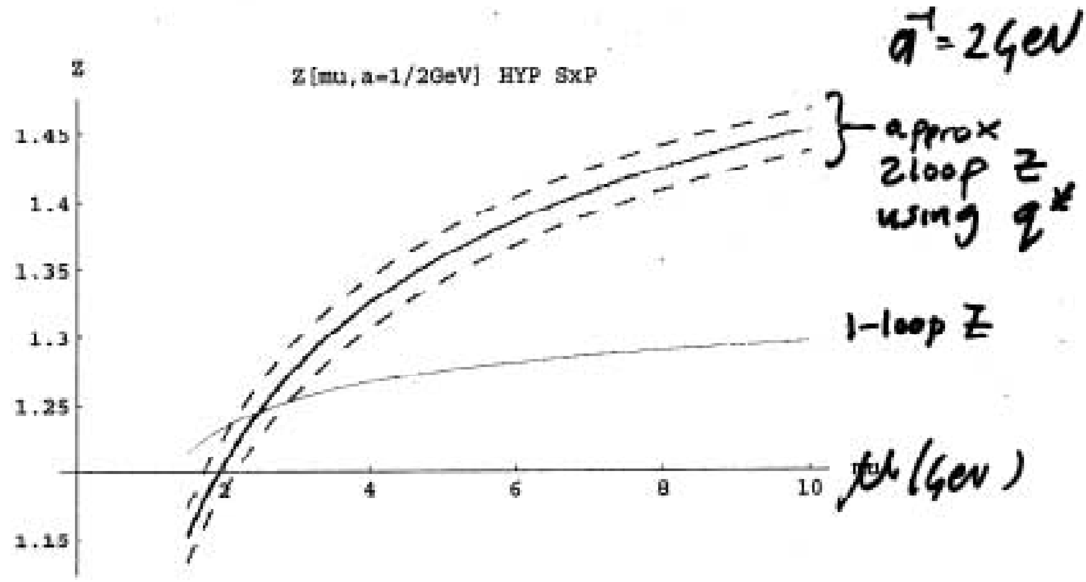
Need higher order LM

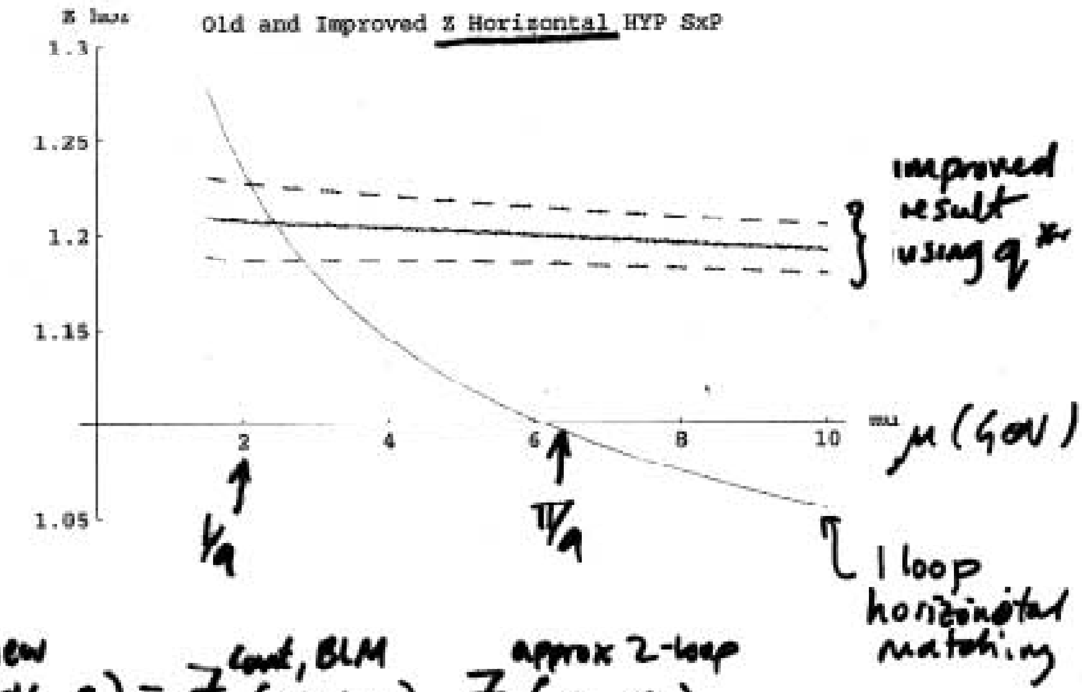


$\frac{C_1}{4\pi} \sim -3$



New analysis method applied to HYP $S \times P$





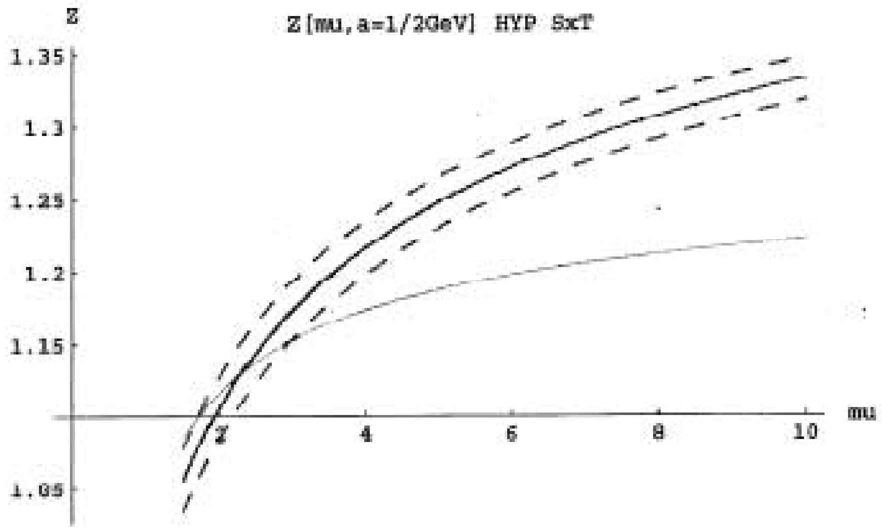
$$Z^{hor,new}(2GeV, a) = Z^{cont,BLM}(2GeV, \mu)\ Z^{approx\ 2\text{-}loop}(\mu, a)$$

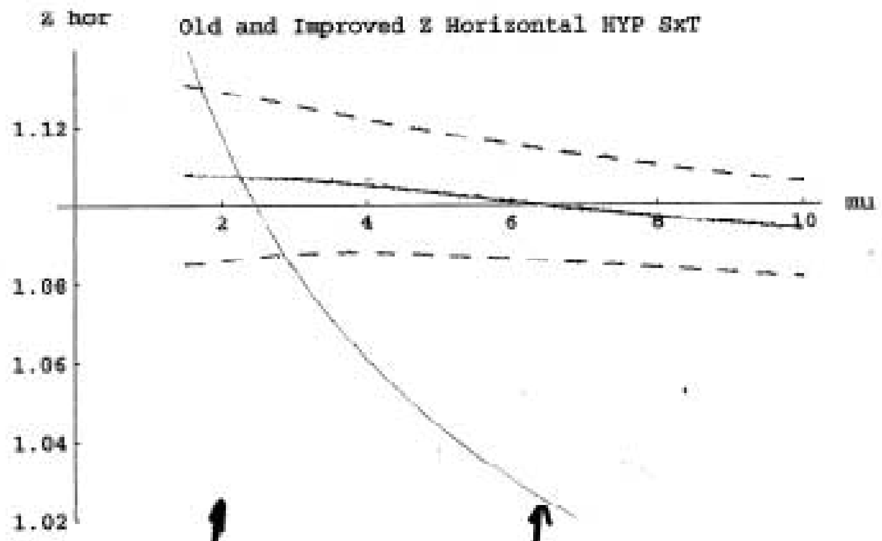
using q^*

- approx. indep. of horizontal matching scale !
- $O(\alpha^2)$ errors dominate

New analysis method applied to HYP $S \times T$

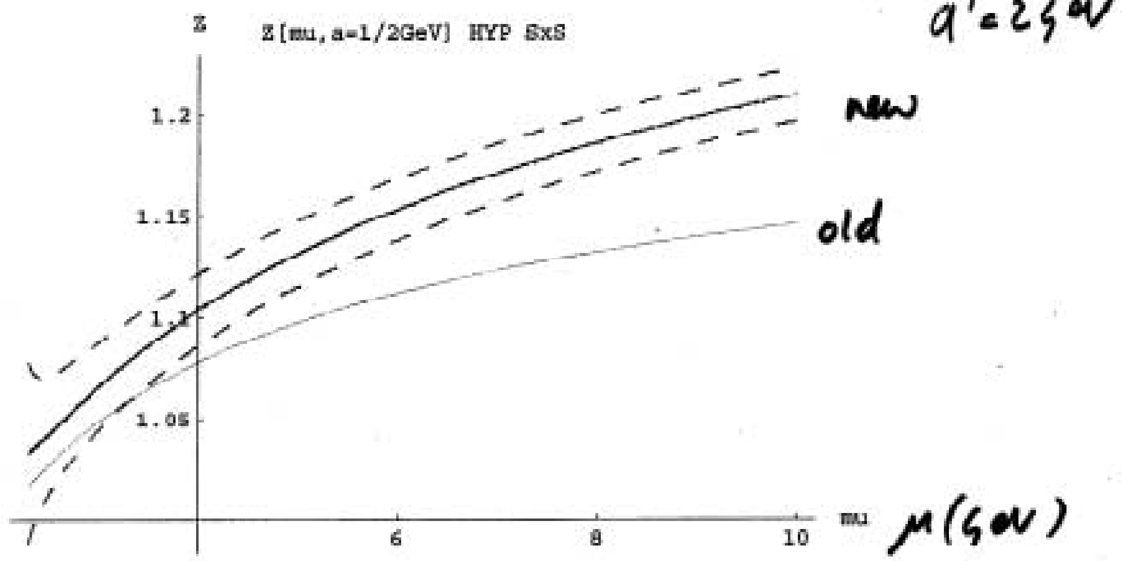
$a^{-1} = 2\,\mathrm{GeV}$

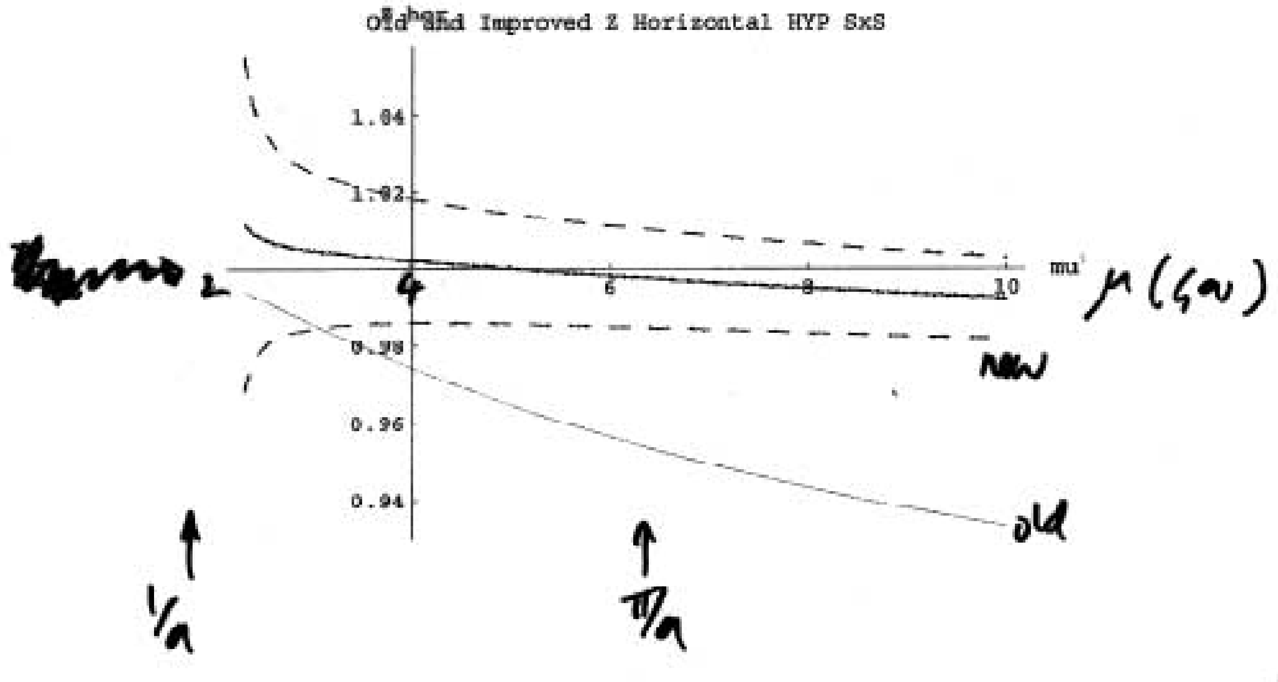




$\uparrow$ π/a

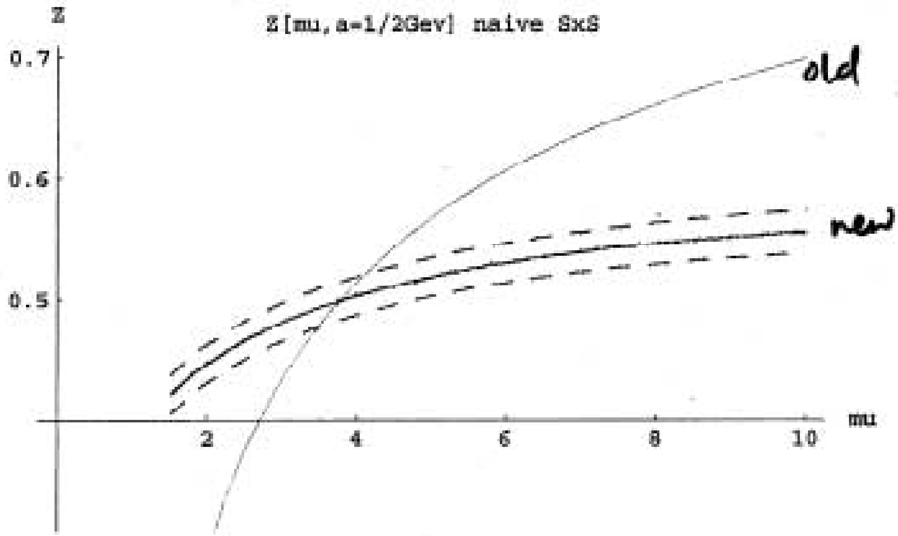
New analysis method applied to HYP $S \times S$

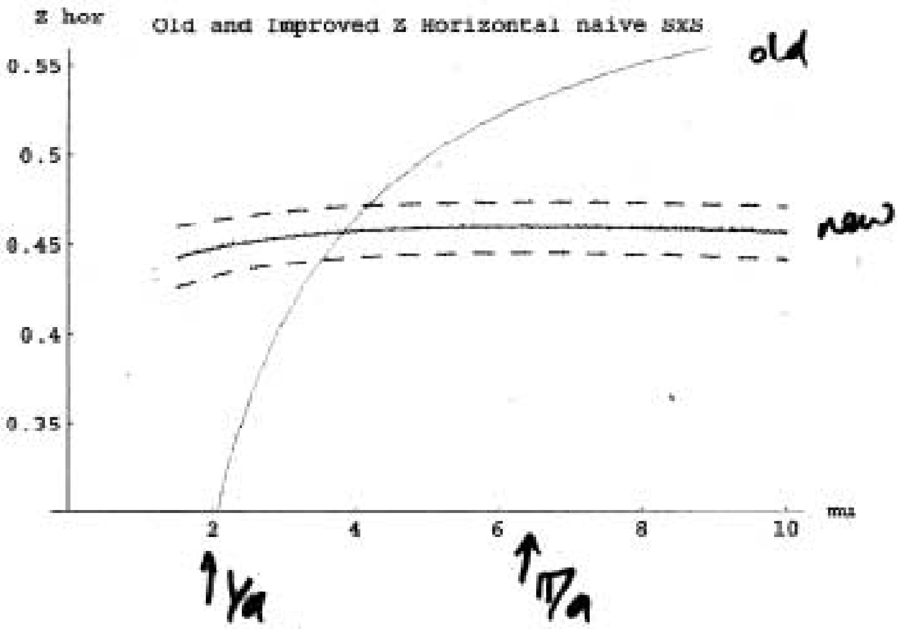




New analysis method applied to naive $S \times S$

$a^{-1} = 2\,GeV$





- too good to be true
  - large $O(\alpha)$ term $\Rightarrow$ large $O(\alpha^2)$?
- need to check against non-pert. result.