A Practical Gauge Invariant Construction of Abelian Chiral Gauage Theories on the Lattice

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Introduction

Path-Integral measure for Ginsparg-Wilson fermions

- Dirac (QCD)

$$\prod_x d\psi(x) d\bar{\psi}(x)$$

- Weyl (EW, GUT, etc.)

$$\mathcal{D}[\psi_L]\mathcal{D}[\bar{\psi}_L] \quad \Longleftarrow$$ Gauge-field-dependent !

Local, Gauge-invariant, Smooth construction
Only for $U(1)$ theories (*Lüscher, NP B549 (1999) 295*)

Introduction (2)

Numerical evaluation of observables in the $U(1)$ theories

$$\langle \mathcal{O}[\psi_L, \bar{\psi}_L] \rangle_F [U(x,\mu)] = \int \mathcal{D}[\psi_L]\mathcal{D}[\bar{\psi}_L] \mathrm{e}^{-S_W} \mathcal{O}$$

not straightforward to follow the above construction
(cf. Complex phase problem in gauge-field path-integral)

- Our proposal
  - $U(x,\mu) \in Z_N$ (N, large enough)
  - Explicit formula for the path-integral measure
  - Numerical evaluations possible (in principle)
    Ready in 2 dim.

Weyl fermion on the lattice

- Chiral projection

$$\hat{\gamma}_5 = \gamma_5(1-2aD) \qquad (\hat{\gamma}_5)^2 = 1 \qquad \gamma_5 D + D\hat{\gamma}_5 = 0$$

$$\hat{\gamma}_5 \psi_L(x) = -\psi_L(x) \qquad \bar{\psi}_L(x)\gamma_5 = +\bar{\psi}_L(x)$$

- Path Integral Measure

$$\psi_L(x) = \sum_i v_i(x) c_i \qquad \bar{\psi}_L(x) = \sum_k \bar{c}_k \bar{v}_k(x)$$

$$\hat{\gamma}_5 v_i(x) = -v_i(x) \quad (i = 1, \cdots, N_-) \qquad (v_i, v_j) = \delta_{ij}$$

$$\bar{v}_k(x)\gamma_5 = +\bar{v}_k(x) \quad (k = 1, \cdots, N_0) \qquad \left(\bar{v}_k^\dagger, \bar{v}_l^\dagger\right) = \delta_{kl}$$

$$\mathcal{D}[\psi_L]\mathcal{D}[\bar{\psi}_L] \equiv \prod_i dc_i \prod_k d\bar{c}_k \qquad \text{cf. } \prod_x d\psi(x) d\bar{\psi}(x)$$

Weyl fermion on the lattice (2)

- Phase ambiguity ($U(1)$ bundle)

$$v_i(x) \longrightarrow \sum_j v_j(\mathcal{Q}^{-1})_{ji} \qquad (\mathcal{Q} : \text{Unitary})$$

$$\mathcal{D}[\psi_L] \longrightarrow \mathcal{D}[\psi_L] \det \mathcal{Q} \qquad (\text{pure phase factor})$$

Weyl fermion on the lattice (3)

- Partition function (or Effective action)

$$
\begin{aligned}
e^{\Gamma_{\text{eff}}[U]} &= \int \mathcal{D}[\psi_L]\mathcal{D}[\bar{\psi}_L]\; e^{-a^4 \sum_x \bar{\psi}_L(x) D \psi_L(x)} \\
&= \int \prod_i dc_i \prod_k d\bar{c}_k\; e^{-\bar{c}_k \left(a^4 \sum_x \bar{v}_k(x) D v_i(x)\right) c_i}
\end{aligned}
$$

- Variation of the Effective action : $\delta_\eta U_\mu(x) = \eta_\mu(x) U_\mu(x)$

$$
\begin{aligned}
\delta_\eta \Gamma_{\text{eff}} &= \mathrm{Tr}\left\{ (\delta_\eta D) \hat{P}_L D^{-1} P_R \right\} + \sum_i (v_i, \delta_\eta v_i) \\
&= -\left\langle a^4 \sum_x \bar{\psi}_L(x) \delta_\eta D \psi_L(x) \right\rangle_F + (\text{terms from measure})
\end{aligned}
$$

Integrability condition, Gauge anomaly cancellation

Gauge-field dependence of the measure

$$\mathfrak{L}_\eta = i \sum_i (v_i, \delta_\eta v_i)$$

$$\sum_i (v_i, \delta_\eta v_i) \longrightarrow \sum_i (v_i, \delta_\eta v_i) + \delta_\eta \ \ln\det \mathcal{Q}$$ “Connection of the $U(1)$ bundle”

- Integrability condition

$$\delta_\eta \mathfrak{L}_\zeta - \delta_\zeta \mathfrak{L}_\eta = i \mathrm{Tr} \left\{ \hat{P}_L [\delta_\eta \hat{P}_L, \delta_\zeta \hat{P}_L] \right\}$$ “Curvature”

- Gauge anomaly cancellation: $\eta_\mu(x) = -\nabla_\mu \omega(x)$

$$\delta_\omega \Gamma_{\mathrm{eff}} \ = \ i \sum_x \omega(x) \mathrm{tr} \left\{ P_R - \hat{P}_L \right\} (x,x) + \sum_i (v_i, \delta_\omega v_i)$$

Lüscher's approach

Construct the measure term

$$\mathfrak{L}_\eta = \sum_x \eta_\mu(x) j_\mu(x); \qquad j_\mu(x) = j_\mu(x)[U(x,\mu)]$$

s.t.

- local w.r.t. $U(x,\mu)$
- integrable
- canceling gauge anomaly

Lüscher's approach (2)

- Use of the infinite lattice

$$\mathfrak{L}_\eta = \mathfrak{L}_\eta^\star + \Delta\mathfrak{L}_\eta$$

$$D(x,y) = \sum_{n=-\infty}^{\infty} D_\infty(x, y+nL) = D_\infty(x,y) + \sum_{n\neq 0} D_\infty(x, y+nL)$$

- The choice of the measure term in the infinite lattice
  $U_t(x,\mu) = \mathrm{e}^{itA_\mu(x)}$ $t \in [0,1]$

$$\begin{aligned} \mathfrak{L}_\eta^\star &= i\int_0^1 dt \mathrm{Tr}\left\{\hat{P}_L[\partial_t \hat{P}_L, \delta_\eta \hat{P}_L]\right\} \\ &\quad + \delta_\eta \int_0^1 dt \sum_x A_\mu(x) k_\mu(x)|_{A\to tA} \end{aligned}$$

Topological structure of $U(1)$ gauge field space

$$\| 1 - P(x,\mu,\nu) \| < \epsilon \qquad \text{(Admissibility condition)}$$

- $U(x,\mu)$ on a finite L^D lattice
  Non-trivial, classified by the magnetic flux

$$m_{\mu\nu} = \frac{1}{2\pi} \sum_{s,t} F_{\mu\nu}(x + s\hat{\mu} + t\hat{\nu})$$

- $U(x,\mu)$ in the infinite lattice
  Trivial

$$U(x,\mu) = \mathrm{e}^{iA_\mu(x)}, \quad |A_\mu(x)| \leq \pi(1 + 8 \| x \|),$$
$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$$

Contractible

$$U_t(x,\mu) = \mathrm{e}^{itA_\mu(x)} \ (t \in [0,1]) \quad \longrightarrow \quad U_0(x,\mu) = 1$$

Lüscher's approach (3)

- Local cohomology problem

$$k_\mu(x) = k_\mu(x)[U(x,\mu)]; \quad (\text{local w.r.t. } U(x,\mu))$$

$$\partial^*_\mu k_\mu(x) = \mathrm{tr}\left\{ P_R - \hat{P}_L \right\}(x,x)$$

  - Poincaré lemma (in the infinite lattice)

$$d^* f(x) = 0 \quad \Longrightarrow \quad f(x) = d^* g(x)$$

  -

$$\begin{aligned} q(x) &= \mathrm{tr}\{\gamma_5(1-D)(x,x)\} \qquad (a=1) \\ &= \frac{1}{32\pi^2}\epsilon_{\mu\nu\lambda\rho}F_{\mu\nu}(x)F_{\lambda\rho}(x+\hat{\mu}+\hat{\nu}) + \partial^*_\mu k_\mu(x) \end{aligned}$$

A Practical Implementation of the Gauge Invariant Measure

- Use of the infinite lattice
  - Topological structure of gauge field space
  - Poincaré lemma
- Use of the continuus interpolation

$$U_t(x,\mu) = \mathrm{e}^{itA_\mu(x)} \qquad t \in [0,1]$$

  - Cohomological analysis of gauge anomaly
  - Construction of the measure term

Hard to implement numerically !?

- a finite L^D lattice
- discrete interpolation

A Practical Implementation of the Gauge Invariant Measure(2)

- $U(x,\mu)$ on a finite L^D lattice

$$F_{\mu\nu}(x) = \frac{2\pi m_{\mu\nu}}{L^2} + \widetilde{F}_{\mu\nu}(x)$$

$$U(x,\mu) = V_{[m]}(x,\mu) \times \tilde{U}(x,\mu)$$

$\tilde{U}(x,\mu)$: actual dynamical degrees of freedom

$$\tilde{U}(x,\mu) = \mathrm{e}^{i\tilde{A}_\mu(x)}, \quad \tilde{A}_\mu(x) \text{ periodic}$$

$$|\tilde{A}_\mu(x)| \leq \pi(1 + 8 \parallel x \parallel) \leq \pi(1 + 2L + 6L^2)$$

$$\tilde{F}_{\mu\nu}(x) = \partial_\mu \tilde{A}_\nu(x) - \partial_\nu \tilde{A}_\mu(x)$$

$$U_t(x,\mu) = \mathrm{e}^{it\tilde{A}_\mu(x)} \times V_{[m]}(x,\mu) \; (t \in [0,1]) \quad \longrightarrow \quad U_0(x,\mu) = V_{[m]}(x,\mu)$$

A Practical Implementation of the Gauge Invariant Measure(3)

- Discrete interpolation within Z_N subspace of $\{U(x,\mu)\}$:

$$U(x,\mu)^{(k)} = \mathrm{e}^{iA_\mu(x)^{(k)}} \cdot V_{\mu[m]}(x) \quad (k = 0, 1, 2, \cdots, M)$$

$$A_\mu(x)^{(k+1)} - A_\mu(x)^{(k)} = \pm \left(\frac{2\pi}{N}\right) \delta_{\mu,\mu_k}\delta_{x,x_k} = \eta_\mu(x)^{(k)}$$

(Minimal variation in Z_N)

A Practical Implementation of the Gauge Invariant Measure(4)

- For a finite difference: $U'_\mu(x) = \mathrm{e}^{i\eta_\mu(x)} U_\mu(x)$, $\mathrm{e}^{i\omega(x)} \in Z_N$

$$i \sum_i (v_i, \delta_\eta v_i) \quad \Longleftrightarrow \quad \mathrm{e}^{-i\mathfrak{L}_\eta} \equiv \frac{\det(v_i, v'_j)}{|\det(v_i, v'_j)|}$$

$$\mathrm{Tr}\left\{\hat{P}_L[\delta_\eta \hat{P}_L, \delta_\zeta \hat{P}_L]\right\} \quad \Longleftrightarrow \quad \mathrm{e}^{i\mathfrak{F}_{\eta\zeta}} = \det\left(1 - \hat{P}_0 + \hat{P}_0 \hat{P}_\eta \hat{P}_{\eta+\zeta} \hat{P}_\zeta P_0\right) /|..|$$

$$\mathrm{tr}\left\{P_R - \hat{P}_L\right\}(x,x) \quad \Longleftrightarrow \quad \det\left(1 - P_R + P_R \mathrm{e}^{i\omega(x)}\right) \times \det\left(1 - \hat{P}_0 + \hat{P}_0 \mathrm{e}^{-i\omega(x)} \prod_i \hat{P}_i \hat{P}_0\right) /|..|$$

  - $U(1)$ bundle $\Rightarrow U(1)$ lattice gauge field
  - Z_N subspace is a natural choice
    $\Longrightarrow Z_N$ gauge invariance

Local Cohomology Problem over Z_N subspace of gauge fields

- Gauge anomaly under Z_N gauge transformation

$$\mathrm{e}^{iq(x)} = \det\left(1 - P_R + P_R \mathrm{e}^{i\omega(x)}\right) \times \det\left(1 - \hat{P}_0 + \hat{P}_0 \mathrm{e}^{-i\omega(x)} \prod_i \hat{P}_i \hat{P}_0\right) / |..|$$

$$\prod_x \mathrm{e}^{iq(x)} = \mathrm{e}^{i\frac{2\pi}{N}Q} \qquad \text{(Topological)}$$

- Local Cohomology Problem over Z_N subspace

$$q(x) = \frac{1}{32\pi^2} \epsilon_{\mu\nu\lambda\rho} \tilde{F}_{\mu\nu}(x) \tilde{F}_{\lambda\rho}(x + \hat{\mu} + \hat{\nu}) + \partial^*_\mu k_\mu(x)$$

  - Can be solved within Z_N subspace
  - Explicit formula of $k_\mu(x)$, applicable to numerical computation

Z_N Chiral Gauge Theory with Exact Gauge Invariance

- The choice of the measure

$$v_j(x) = \begin{cases} w_j(x)\,\mathrm{e}^{i\phi} & (j=1) \\ w_j(x) & (j \neq 1) \end{cases} \qquad (w_j(x) : \text{arbitrarily chosen basis})$$

$$\begin{aligned} \mathrm{e}^{i\phi} &= \det\left(w_i, \hat{P} \prod_{k=0}^{M} \hat{P}^{(k)} \hat{P}^{(0)} w_j^0 \right) / |..| \times \\ & \prod_{k=0}^{M} \left\{ \prod_{l=0}^{k} \mathrm{e}^{-i\mathfrak{F}_{kl}} \right\} \times \mathrm{e}^{-i\sum_x \tilde{A}_\mu(x) k_\mu(x)} \end{aligned}$$

$$\mathrm{e}^{i\mathfrak{F}_{kl}} = \det\left(1 - \hat{P} + \hat{P}\hat{P}_{+\eta^{(k)}}\hat{P}_{+\eta^{(k)}+\zeta^{(l)}}\hat{P}_{+\zeta^{(l)}}\hat{P}\right) / |..|$$

Possible Applications

- Numerically demanding in four dim.
- Ready in two dim.
  - two-dim. chiral Schwinger models
    - Composite massless fermion in 11112 model (cf. WB phase choice *Narayanan-Neuberger-Y.K.*)
    - Check of the cluster property
- Analysis of non-abelian theories ...