Nonperturbatively Determined Relativistic Heavy Quark Action

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Outline

- Motivation
- Relativistic Heavy Quark Action (RHQ):
  - Review: What has been done in the past?
  - New Counting: Three-parameter action
  - Demostration
- Simulation Results:
  - Nonpertubative proposal
  - Numerical results
  - Comparisons with PT results
- Conclusion and Outlook

Motivations

- Flavor physics and CP violation play an important role in particle physics.
- Problem:
  - Too expensive to directly simulate heavy quarks by brute force
  - $(ma) \ll 1$ no longer true; $(ma)^n$ terms are significant
- Multiple fermion actions in lattice calculations:
  - NRQCD has no continuum limit
  - HQET cannot be applied to quarkonia

- Relativistic heavy quark (RHQ) action
  - Goes to SW action when $ma \ll 1$ and NRQCD in $ma \gg 1$ limit
  - Systematically absorb mass factor into the coefficients
  - Small cutoff effects: $\alpha_s \Lambda_{\mathrm{QCD}} a$ and $(\Lambda_{\mathrm{QCD}} a)^2$

Relativistic Heavy Quark Action (RHQ) - Review

- Operators up to dimension-5 constrained by parity and charge conjugation

Dimensions	Operators
3	$m\overline{\psi}\psi$
4	$\overline{\psi}\gamma_0 D_0\psi$, $\overline{\psi}\vec{\gamma}\cdot\vec{D}\psi$
5	$a\overline{\psi}D_0^2\psi$, $a\overline{\psi}(\vec{D})^2\psi$,
	$a\overline{\psi}\left[\gamma_0,\gamma_i\right]\left[D_0,D_i\right]\psi$, $a\overline{\psi}\left[\gamma^i,\gamma^j\right]\left[D^i,D^j\right]\psi$,
	$a\overline{\psi}\left[\gamma^i,\gamma^0\right]\left[D^i,D^0\right]\psi$

- The general relativistic heavy quark action:

$$S = \sum_n \overline{\psi_n}\left(D_{\mathrm{RHQ}} + m_0\right)\psi_n \tag{1}$$

$$\begin{aligned} D_{\mathrm{RHQ}} + m_0 &= m_0 + b_0\gamma_0 D_0 + b_1\vec{\gamma}\cdot\vec{D} - b_2 a D_0^2 - b_3 a(\vec{D})^2 \\ &\quad - \sum_i \frac{i}{2} b_4 a\sigma_{0i}F_{0i} - \sum_{i,j}\frac{i}{2} b_5 a\sigma_{ij}F_{ij} + b_6[\gamma_0 D_0, \vec{\gamma}\cdot\vec{D}], \end{aligned} \tag{2}$$

- 8 parameters: m_0, b_{0-6}

Review (cont'd)

- Counting redundant operators:
  - Spectrum-Conserving Transformations
  Fermilab action *A. El-Khadra , A. S. Kronfeld and P. B. Mackenzie (hep-lat/9604004)*
  - Application of the Field Equations
  Tsukuba action *S. Aoki, Y. Kuramashi and S. Tominaga (hep-lat/0107009)*
- Comparison of parameters between the Fermilab and Tsukuba actions

Action	b_0	b_1	b_2	b_3	b_4	b_5
Fermilab	1	ζ	1	$r_s\zeta$	$c_E\zeta$	$c_B\zeta$
Tsukuba	1	ν	1	r_s	c_E	c_B

- Difference:
  - $b_3 = b_1$ (Fermilab action)
  - b_3 is an independent coefficient (Tsukuba action)
  to restore the correct γ structure in continuum quark propagator
- One-loop coefficients available
- Four-parameter lattice action:

$$
\begin{aligned}
S &= a^4 \sum_n \overline{\psi_n} \Big(m_0 + b_0\gamma_0 D_0 + b_1 \vec{\gamma}\cdot\vec{D} - b_2 a D_0^2 - b_3 a (\vec{D})^2 \\
&\quad - \sum_i \frac{i}{2} b_4 a \sigma_{0i} F_{0i} - \sum_{i,j} \frac{i}{2} b_5 a \sigma_{ij} F_{ij} + b_6 [\gamma_0 D_0, \vec{\gamma}\cdot\vec{D}] \Big) \psi_n
\end{aligned}
\tag{3}
$$

Field Transformation Revisited

Considering $D_0 \approx m$

- Field transformation of

$$\begin{aligned} \psi' &= (1 - a^2[\gamma^i, \gamma^0][D^i, D^0]\xi_E)\psi \\ \overline{\psi}' &= \overline{\psi}(1 - a^2[\gamma^i, \gamma^0][\overleftarrow{D}^i, \overleftarrow{D}^0]\xi_E), \end{aligned} \tag{4}$$

$c_E \to c_E + 2ma\xi_E$ plus $O(a^2)$ term of $\{[\gamma^i, \gamma^0][D^i, D^0], \gamma^0 D^0\} = [\gamma^i, \gamma^0]\gamma^0[[D^i, D^0], D^0]$

- Field transformation

$$\begin{aligned} \psi' &= (1 - a^2[\gamma^i, \gamma^j][D^i, D^j]\xi_B)\psi \\ \overline{\psi}' &= \overline{\psi}(1 - a^2[\gamma^i, \gamma^j][\overleftarrow{D}^i, \overleftarrow{D}^j]\xi_B). \end{aligned} \tag{5}$$

$c_B \to c_B + 2ma\xi_B$ plus $O(a^1)$ term of $\{[\gamma^i, \gamma^j][D^i, D^j], \gamma^0 D^0\} = [\gamma^i, \gamma^j]\gamma^0\{[D^i, D^j], D^0\}$.

Simplified Heavy Quark Action (RHQ)

- "Naive" Power Counting: in terms of a
  $\Rightarrow D_0$ is treated the same as m_0, $O(1/a)$.
- In general, all the coefficients are functions of ma, $(aD_0)^2$, and g^2;
  that is, $C = \sum_{i,j,k} c_{ijk}\,(ma)^i(aD_0)^{2j}g^{2k}$

"Naive" Power Counting

- Operators listed by their order

n in a^n		Symbol	Operators
-1	$\mathcal{O}_{\text{Sym},-1}$	O^1	1
		O^2	$\gamma_0 D_0$
0	$\mathcal{O}_{\text{Sym},0}$	O^3	$\vec{\gamma}\cdot\vec{D}$
		O^4	$a[\vec{\gamma}\cdot\vec{D}, \gamma^0 D^0]$
1	$\mathcal{O}_{\text{Sym},1}$	O^5	$a(\vec{D})^2$
		O^6	$a^2\left\{(\vec{D})^2, \gamma^0 D^0\right\}$
		O^7	$a\left[\gamma^i, \gamma^j\right]\left[D^i, D^j\right]$
		O^8	$a^2\left\{\left[\gamma^i, \gamma^j\right]\left[D^i, D^j\right], \gamma^0 D^0\right\}$
		O^9	$a\left[\gamma^i, \gamma^0\right]\left[D^i, D^0\right]$
		O^{10}	$a^2\left[\left[\gamma^i, \gamma^0\right]\left[D^i, D^0\right], \gamma^0 D^0\right]$

- Effective action is $\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff},-1} + \mathcal{L}_{\text{eff},0} + \mathcal{L}_{\text{eff},1} + \ldots$, where

$$\mathcal{L}_{\text{eff},-1} = \overline{\psi}\Big(\frac{1}{a}C_L^1 + \gamma^0 D^0 C_L^2\Big)\psi \tag{6}$$

$$\mathcal{L}_{\text{eff},0} = \overline{\psi}\Big(\{\vec{\gamma}\vec{D}, C_L^3\} + a\{[\vec{\gamma}\vec{D}, \gamma^0 D^0], C_L^4\}\Big)\psi \tag{7}$$

$$\begin{aligned}\mathcal{L}_{\text{eff},1} = {} & a\overline{\psi}\Big(\vec{D}^2 C_L^5 + a\{\vec{D}^2, \gamma^0 D^0\}C_L^6 \\ & +[\gamma^i, \gamma^j][D^i, D^j]C_L^7 + a\{[\gamma^i, \gamma^j][D^i, D^j], \gamma^0 D^0\}C_L^8 \\ & +[\gamma^i, \gamma^0][D^i, D^0]C_L^9 + a[[\gamma^i, \gamma^0][D^i, D^0], \gamma^0 D^0]C_L^{10}\Big)\psi.\end{aligned} \tag{8}$$

Field Transformations

- The lagrangian can be written as the sum over all orders of these operators:

$$\mathcal{L}_{\mathsf{Sym}} = a^4 \overline{\psi} \left(\sum_w \mathcal{O}_{\mathsf{Sym},w} \cdot C_{L,w} \right) \psi, \tag{9}$$

where $w \in \{-1, 0, 1\}$ denotes the counting in a and $C_{L,w}$ is a vector of coefficients.

- Field transformations at all orders in a

$$\begin{aligned} \overline{\psi} &\to \overline{\psi} \left(1 + a \sum_w \mathcal{O}_{\mathsf{Sym},w} \cdot \overline{C}_{\mathsf{T},w} \right) \\ \psi &\to \left(1 + a \sum_w \mathcal{O}_{\mathsf{Sym},w} \cdot C_{\mathsf{T},w} \right) \psi \end{aligned} \tag{10}$$

- Use field transformations at each order of a and N to
  - Remove all contamination of aD_0 in the coefficients
  - Reduce the action to the form:

$$\mathcal{S}_{\mathsf{lat}} = a^4 \overline{\psi} \left[\frac{m_0}{a} + \gamma_0 D_0 + \zeta \vec{\gamma} \cdot \vec{D} + ia \frac{c_P}{2} \sigma_{\mu\nu} F^{\mu\nu} \right] \psi + O(a^2). \tag{11}$$

N. Christ, M. Li and H. Lin (hep-lat/??????)

Too much freedom introduced by transformation?

- Only certain of combinations of coefficients appear ($\overline{C}_{\mathrm{T},\,w} + C_{\mathrm{T},\,w}$, labeled as "s", or $\overline{C}_{\mathrm{T},\,w} - C_{\mathrm{T},\,w}$, labeled as "a", or both)
- Coefficients multiplied by D_0^2 are used to eliminate terms with $(D_0)^{2l}$
- Those multiplied by ma are used to remove ma dependence
- Lower-order transformation might mix with higher-order terms. For example,

$$\begin{aligned} D_0^2\vec{\gamma}\cdot\vec{D}\psi &= \left(\gamma^0 D^0\right)^2 \vec{\gamma}\cdot\vec{D}\psi \\ &= \gamma^0 D^0 \left(-\vec{\gamma}\cdot\vec{D}\gamma^0 D^0 + \left\{\gamma^0 D^0, \vec{\gamma}\cdot\vec{D}\right\}\right)\psi \\ &= \vec{\gamma}\cdot\vec{D}D_0^2\psi - \left[\left\{\gamma^0 D^0, \vec{\gamma}\cdot\vec{D}\right\}, \gamma^0 D^0\right]\psi. \end{aligned} \tag{12}$$

- Not trivial. Need to work out order by order.

$O(a^{-1})$ Example

- $O(a^{-1})$ Symanzik effective action

$$\mathcal{L}_{\mathrm{eff},-1} = \overline{\psi}\left(\frac{1}{a}C_{\mathrm{L}}^1 + \gamma^0 D^0 C_{\mathrm{L}}^2\right)\psi \tag{13}$$

- A $O(a^0)$ field transformation

$$\psi = \left(1 + C_{\mathrm{T}}^1 + a\gamma^0 D^0 C_{\mathrm{T}}^2\right)\psi' \tag{14}$$

$$\overline{\psi} = \overline{\psi}'\left(1 + \overline{C}_{\mathrm{T}}^1 - a\gamma^0 \overleftarrow{D}^0 \overline{C}_{\mathrm{T}}^2\right). \tag{15}$$

  on that action is needed to analyze $O(1/a)$ terms in Symanzik effective action.

- C_{L}^1, C_{L}^2, C_{T}^1, C_{T}^2, $\overline{C}_{\mathrm{T}}^1$ and $\overline{C}_{\mathrm{T}}^2$ are polynomial combinations of $(ma)^k$, $(aD^0)^{2l}$, g^{2n}.
- Proceed in a systematic fashion by identifying terms of the form $C = \sum_N c_N$ as being of order $N = k + 2l + 2n$.
- The Eqs. 14 become a sequence of simpler transformations where each involves a homogeneous polynomial of order N:

$$\psi = \prod_{N=0}\left(1 + (c_{\mathrm{T}}^1)_N + a\gamma^0 D^0 (c_{\mathrm{T}}^2)_N\right)\psi' \tag{16}$$

$$\overline{\psi} = \overline{\psi}'\prod_{N=0}\left(1 + (\overline{c}_{\mathrm{T}}^1)_N - a\gamma^0 \overleftarrow{D}^0 (\overline{c}_{\mathrm{T}}^2)_N\right). \tag{17}$$

$O(a^{-1})$ Example (cont'd)

- Claim: By proper choice of the transformation coefficients, it is possible to transform $\mathcal{L}_{\text{eff},-1}$ into the form:

$$\mathcal{L}_{\text{eff},-1} = \overline{\psi}\{m_r + \gamma^0 D^0\}\psi \tag{18}$$

- Proof:
  - At order $N = 0$, $(c_{\text{L}}^1)_0 = 0$ and $(c_{\text{L}}^2)_0 = 1$
  - At order $N = 1$, $(c_{\text{L}}^1)_1 = ma$ and $(c_{\text{L}}^2)_1 = 0$
  - Assuming the statement is true for $N = N_0$, i.e.

$$\mathcal{L}_{\text{eff},-1} = \overline{\psi}\left\{\frac{1}{a}(m_r a)_{N_0} + \gamma^0 D^0\right\}\psi. \tag{19}$$

The proof has similar form for $N = N_0 + 1$ after field transformation.
- Starting with

$$\mathcal{L}_{\text{eff},-1} = \overline{\psi}\left\{\frac{1}{a}\left((m_r a)_{N_0+1} + (c_{\text{L}}^2)_{N_0+2}\right) + \gamma^0 D^0(1 + (c_{\text{L}}^2)_{N_0+1})\right\}\psi. \tag{20}$$

$O(a^{-1})$ Example (cont'd)

- After next transformation of order $N_0 + 1$

$$\mathcal{L}_{\mathrm{eff},-1} = \overline{\psi'}\left\{\frac{1}{a}\left[(m_r a)_{N_0+1} + (c_{\mathrm{L}}^1)_{N_0+2} + (aD^0)^2\left((c_{\mathrm{T}}^2)_{N_0+1} + (\bar{c}_{\mathrm{T}}^2)_{N_0+1}\right)\right]\right.$$
$$\left. + \gamma^0 D^0\left[1 + (c_{\mathrm{L}}^2)_{N_0+1} + (c_{\mathrm{T}}^1)_{N_0+1} + (\bar{c}_{\mathrm{T}}^1)_{N_0+1}\right]\right\}\psi' \qquad (21)$$

end of proof

- Following this inductive procedure, we are thus able to express the $O(1/a)$ Symanzik lagrangian in the standard continuum form.

$O(a^0)$ and $O(a)$ Cases

- In $O(a^0)$ action, both $O(a^0)$ and new $O(a)$ transformations will be involved.
- In $O(a^1)$ action, up-to-$O(a^2)$ transformations will be involved, plus the extra coefficients from swapping the coefficient with non-$\gamma_0 D_0$ operators.
- No trivial pattern but to work out the product at each order in effective action

Products after Field Transformation

- Coefficients added to the unimproved lagrangian by field transformations:

Operators	Terms contributed	Coefficients
O^1	$O^1_L O^1_T$, $O^2_L O^2_T$ and $(\leftrightarrow)$	$mac^1_{T,s}$, $c^2_{T,s} D^2_0$
O^2	$O^1_L O^2_T$, $O^2_L O^1_T$ and $(\leftrightarrow)$	$mac^2_{T,s}$, $c^1_{T,s}$
O^3	$O^1_L O^3_T$, $O^2_L O^4_T$, $O^3_L O^1_T$, and $(\leftrightarrow)$	$mac^3_{T,s}$, $c^4_{T,a} D^2_0$, $c^1_{T,s}$
O^4	$O^1_L O^4_T$, $O^2_L O^3_T$, $O^3_L O^2_T$ and $(\leftrightarrow)$	$mac^4_{T,s}$, $c^3_{T,a}$, $c^2_{T,a}$
O^5	$O^1_L O^5_T$, $O^2_L O^6_T$, $O^3_L O^3_T$ and $(\leftrightarrow)$	$mac^5_{T,s}$, $c^6_{T,s} D^2_0$, $c^3_{T,s}$
O^6	$O^1_L O^6_T$, $O^2_L O^5_T$, $O^3_L O^4_T$ and $(\leftrightarrow)$	$mac^6_{T,s}$, $c^5_{T,s}$, $c^4_{T,a}$
O^7	$O^1_L O^7_T$, $O^2_L O^8_T$, $O^3_L O^3_T$ and $(\leftrightarrow)$	$mac^7_{T,s}$, $c^8_{T,s} D^2_0$, $c^3_{T,s}$
O^8	$O^1_L O^8_T$, $O^2_L O^7_T$, $O^3_L O^4_T$ and $(\leftrightarrow)$	$mac^8_{T,s}$, $c^7_{T,s}$, $c^4_{T,a}$
O^9	$O^1_L O^9_T$, $O^2_L O^3_T$, $O^2_L O^{10}_T$, $O^3_L O^2_T$ and $(\leftrightarrow)$	$mac^9_{T,s}$, $c^{10}_{T,s} D^2_0$, $c^3_{T,s}$, $c^2_{T,s}$
O^{10}	$O^1_L O^{10}_T$, $O^2_L O^9_T$, $O^2_L O^4_T$ and $(\leftrightarrow)$	$mac^{10}_{T,s}$, $c^9_{T,a}$, $c^4_{T,s}$

Counting D.o.F.

- The usage of the freedom introduced by field transformation

Coefficients	Choice of usage
$c^2_{T,s}$	eliminates $(aD_0)^{2n}$ in c^1_A; ma dependence remains
$c^1_{T,s}$	fixes c^2_A
$c^4_{T,a}$	eliminates $(aD_0)^{2n}$ in c^3_A
$c^3_{T,s}$	not used at $O(1)$; c^3_A ma dependence remains
$c^2_{T,a}$	zeros c^4_A
$c^4_{T,s}$ and $c^3_{T,a}$	free
$c^6_{T,s}$	eliminates $(aD_0)^{2n}$ in c^5_A
$c^3_{T,s}$	zeros remaining c^5_A
$c^5_{T,s}$	zeros c^6_A
$c^8_{T,s}$	eliminates $(aD_0)^{2n}$ in c^7_A with remaining ma dependence
$c^7_{T,s}$	zeros c^8_A
$c^{10}_{T,s}$	eliminates $(aD_0)^{2n}$ in c^9_A
$c^9_{T,s}$	fixes remaining c^9_A to be the same as c^7_A
$c^9_{T,a}$	zeros c^{10}_A

Summary of RHQ

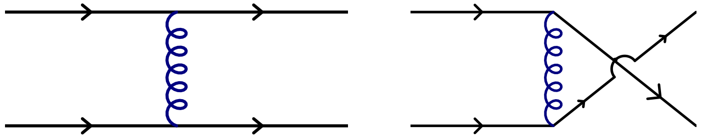
- Wilson terms are redundant operators; needed for solving the doubler problem
- Adjust the temporal and spatial coefficients differently for efficiency
- In summary,

$$\begin{aligned} \mathcal{S}_{\mathrm{lat}} &= a^4\overline{\psi}\left[\frac{m_r}{a} + \gamma_0 D_0 + \zeta(ma, g^2)\vec{\gamma}\cdot\vec{D}\right. \\ &\left. - aD_0^2 - a\zeta(ma,g^2)(\vec{D})^2 - ia\frac{c_{\mathrm{P}}(ma,g^2)}{2}\sigma_{\mu\nu}F^{\mu\nu}\right]\psi + O(a^2). \quad (22) \end{aligned}$$

- $c_E = c_B = c_{\mathrm{P}}$ in Fermilab action
- Three parameters : m_0, ζ, c_{P} to be tuned.

Tree-Level Coefficients

- Quark-quark scattering amplitude (proposed in *hep-lat/0107009*)



- The spinor is

$$u(p) = \begin{pmatrix} \phi \\ \frac{\zeta \vec{p}\ \vec{\sigma}}{E+m_0}\phi, \end{pmatrix} \tag{23}$$

  and ϕ denotes the 2-component spinor with normalization $\phi^\dagger\phi = 1$.

- - Start from four-parameter (Fermilab) action.
  - Correct the quark propagator and quark-gluon vertex with a single on-shell spinor redefintion $\psi \to (1+\delta\vec{\gamma}\cdot\vec{p})\psi$; $\bar{\psi} \to \bar{\psi}(1+\delta\vec{\gamma}\cdot\vec{p})$.
  - Reproduce Tsukuba tree-level calculation. *Done by Min Li.*

- - The vertex generated by the chromo-electric term is proportional to $\bar{u}(p')\sigma_{0i}u(p) \approx ap_i^2/m$;
  - set $c_E = c_B$ leaving the amplitude correct up to cutoff of $(ap_i)^2$.

Simulation Results

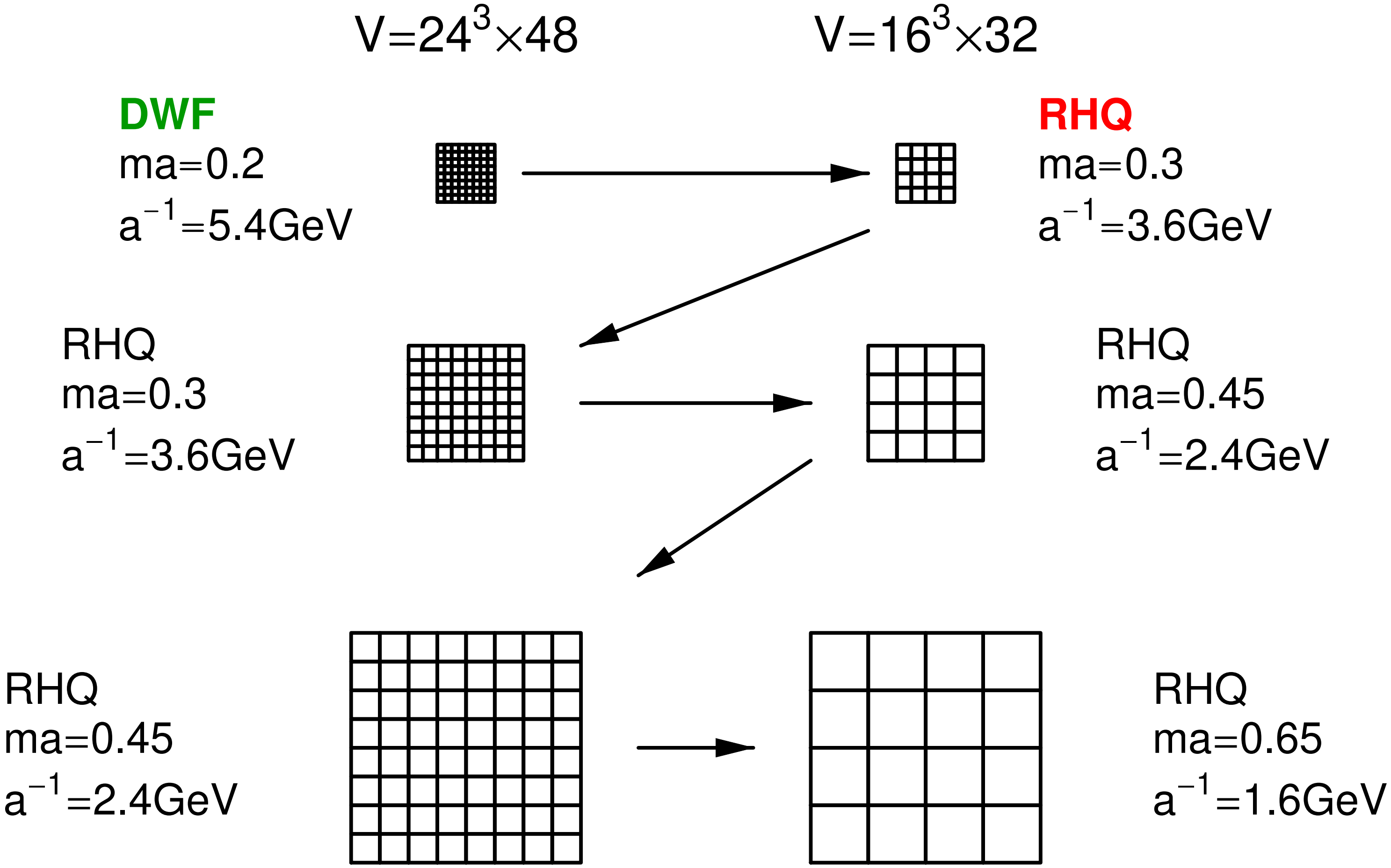
- Nonpertubative coefficient determination proposal
- Simulation details
- Numerical results: Four-parameter study to three-parameter ones
- Comparisons with perturbative results

NP Determined RHQ Coefficients

We propose an idea of matching on-shell quantities:

1. Spin-averaged: $m_{\text{sa}}^{hh} = \frac{1}{4}\left(m_{\text{PS}}^{hh} + 3m_{\text{V}}^{hh}\right)$, $m_{\text{sa}}^{hl} = \frac{1}{4}\left(m_{\text{PS}}^{hl} + 3m_{\text{V}}^{hl}\right)$
2. Hyperfine splitting: $m_{\text{hs}}^{hh} = m_{\text{V}}^{hh} - m_{\text{PS}}^{hh}$, $m_{\text{hs}}^{hl} = m_{\text{V}}^{hl} - m_{\text{PS}}^{hl}$
3. Spin-orbit averaged and splitting: $m_{\text{soa}}^{hh} = \frac{1}{4}\left(m_{\text{S}}^{hh} + 3m_{\text{AV}}^{hh}\right)$, $m_{\text{sos}}^{hh} = m_{\text{AV}}^{hh} - m_{\text{S}}^{hh}$
4. Dispersion relation: $E^2 = m_1^2 + \frac{m_1}{m_2}p^2$, where m_1 is the rest mass and m_2 is kinetic mass.

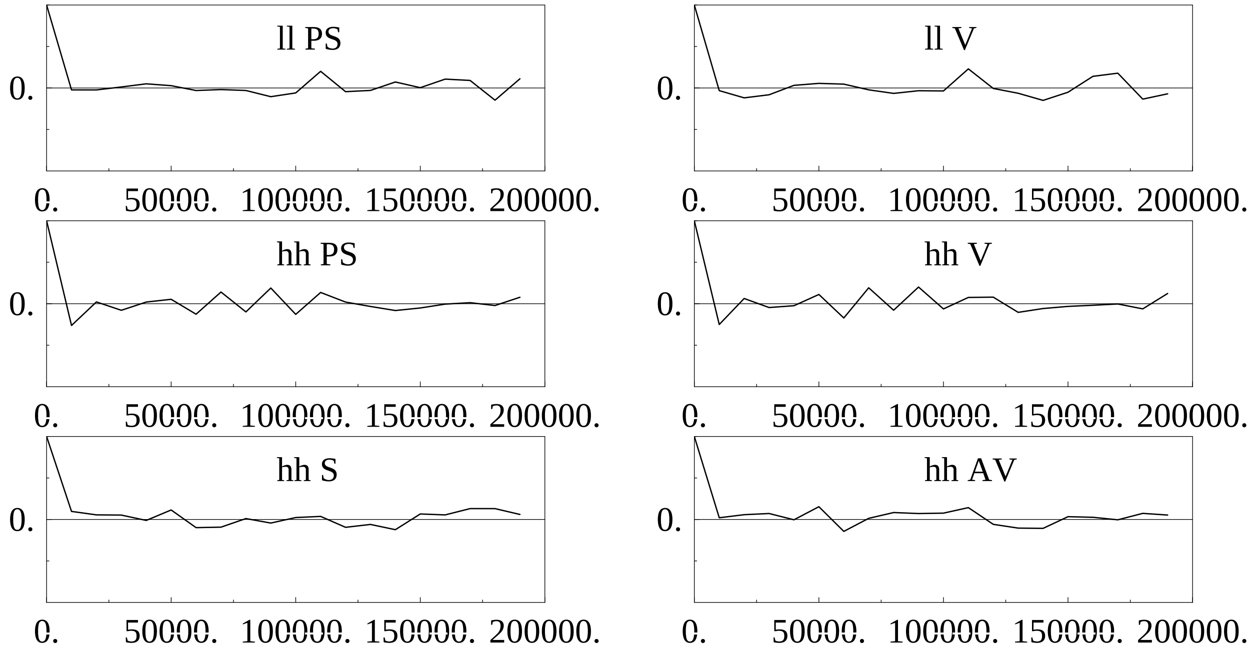
through an approach similar to the step-scaling technique by *Sommer et. al. hep-lat/0310035*.



V=24^3×48
V=16^3×32
DWF
ma=0.2
a^−1=5.4GeV
RHQ
ma=0.3
a^−1=3.6GeV
RHQ
ma=0.3
a^−1=3.6GeV
RHQ
ma=0.45
a^−1=2.4GeV
RHQ
ma=0.45
a^−1=2.4GeV
RHQ
ma=0.65
a^−1=1.6GeV

Matched Quenched Configurations

- Hardware: QCDOC 512-node machine at frequency 420 Hz, reproducibility checked.
- Wilson gauge action with heatbath algorithm.
- 20,000 sweeps for thermalization and 10,000 sweeps separation.
- Meson autocorrelation time:



Static Quark Potential

- APE smearing

$$U_k(n) \to P_{SU(3)}[U_k(n) + c_{smear} \sum_{\mathsf{l} \neq \mathsf{k}} U_{\mathsf{l}}(n) U_{\mathsf{k}}(n + \hat{\mathsf{l}}) U_{-\mathsf{l}}(n + \hat{\mathsf{l}} + \hat{\mathsf{k}})] \quad (24)$$

  to calculate Wilson loop

- The static quark potential

$$V(\vec{r}, t) = \log \left[\frac{\langle W(\vec{r}, t) \rangle}{\langle W(\vec{r}, t+1) \rangle} \right] = C - \frac{\alpha}{R} + \sigma R$$

  where R is the size of the Wilson loop.

- The lattice spacing is determined from

$$R^2 F(R)|_{R=r_0} = 1.65$$

  for $\beta < 6.57$ and

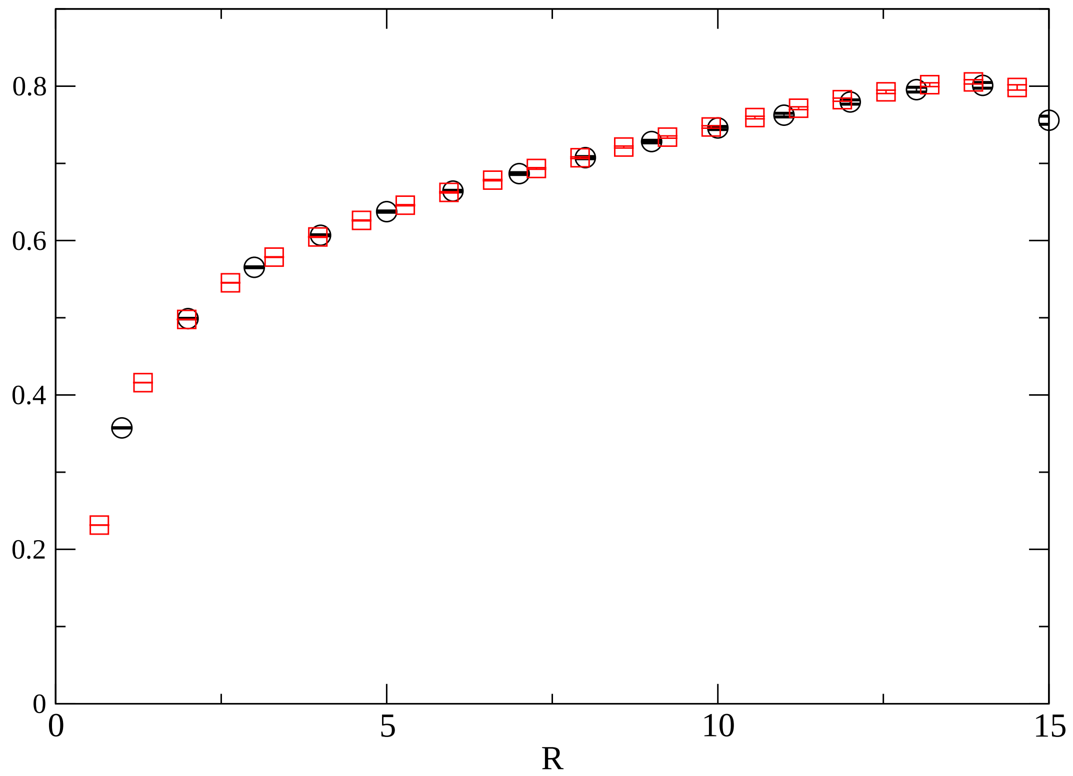
$$R^2 F(R)|_{R=r_c} = 0.65$$

  for $\beta > 6.57$ where $\frac{r_c}{r_0} = 0.5133(24)$.

- $a^{-1} = 4.8$ GeV at $\beta = 6.57$. Two reference scales are needed.

Lattice Spacing Ratio: λ

- Alternative check without fixing different scales of r_0 and r_c.
- Fit $\lambda(= a_2/a_1)$ to match potentials: $V_1(n_1) = V_2(n_2/\lambda)/\lambda + C$



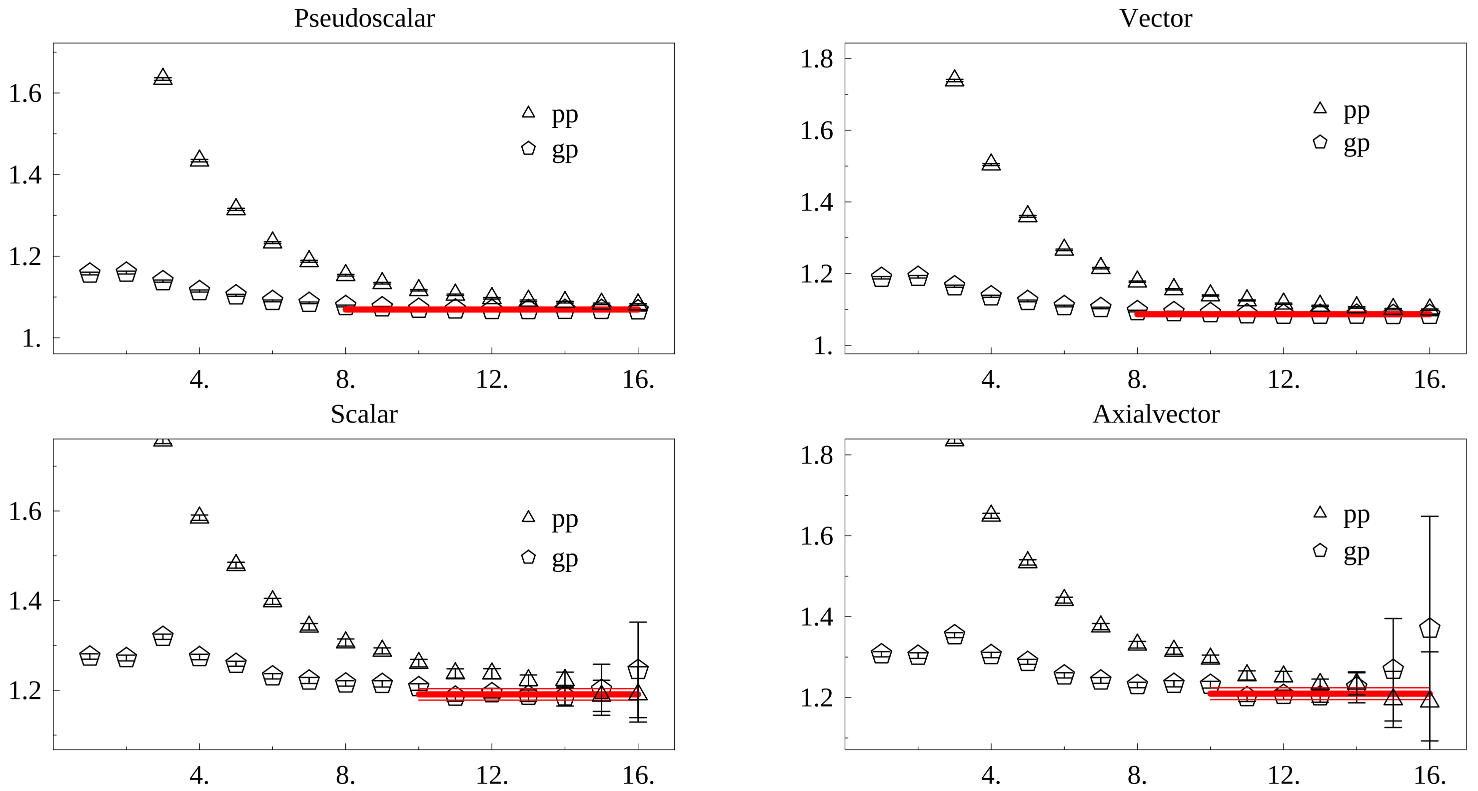
- $\lambda = 1.51(2)$.

Heavy Quark Source Smearing

- Coulomb gauge-fixed hydrogenic source smearing

$$\begin{aligned} \Psi_{gnd}(r) &= e^{-r/r_0} \\ \Psi_{ex}(r) &= (1 - r/2r_0)e^{-r/2r_0} \end{aligned} \tag{25}$$

- Effective Mass Plot (x-axis: T)



Fixed Parameters/Quantities

- Parameters:

Label	β	V	S_L	am_L	S_H	am_H	a^{-1}(SQ pot.)
Fine	6.638	$24^3 \times 48$	DWF	0.02	DWF	0.2	5.4 (GeV)
Coarse	6.351	$16^3 \times 32$	DWF	0.03	RHQ	Unknowns	3.6 (GeV)

with fixed L=0.9 fm, and $L_s = 12$, $M_5 = 1.5$ for DWF actions.

- Results:
  - Fine

	am_{PS}	am_V	m_1/m_2	am_{AV}	am_S
light-light	0.175(3)	0.233(5)	–	–	–
heavy-light	0.467(2)	0.485(3)	–	–	–
heavy-heavy	0.716(1)	0.728(1)	1.02(2)	0.810(5)	0.799(4)

  - Coarse

	am_{PS}	am_V	c^2
light-light	0.259(6)	0.328(10)	–

- 100 configurations.

More Parameters on the Coarse Lattices

Set	m_0	c_B	c_E	ζ
1	0.	1.552	1.458	1.013
2	0.07	1.547	1.424	1.001
3	0.0426	1.550	1.438	1.007
4	0.0426	1.550	1.438	1.1
5	0.0426	1.550	1.438	0.9
6	0.0330	1.609	1.538	1.044
7	0.0230	1.609	1.438	1.045
8	0.0230	1.609	1.538	1.044
9	0.0426	1.609	1.438	1.044
10	0.0426	1.550	1.438	1.007
11	0.	1.552	1.438	1.013
12	0.0328	1.511	1.438	1.036
13	0.0230	1.511	1.438	1.036
14	0.0328	1.511	1.538	1.036
15	0.01	1.700	1.574	1.022
16	0.00370	1.709	1.577	1.023
17	0.0138	1.715	1.579	1.025
18	0.02	1.719	1.580	1.0259
19	0.03	1.725	1.582	1.027
20	0.08	1.707	1.576	1.023
21	0.09	1.713	1.578	1.025

Set	m_0	c_B	c_E	ζ
22	0.1	1.719	1.580	1.0261
23	0.101	1.503	1.932	1.003
24	0.159	1.456	2.427	0.999
25	0.217	1.409	2.921	0.995
26	0.276	1.362	3.415	0.992
27	0.334	1.315	3.910	0.988
28	0.392	1.268	4.404	0.984
29	-0.132	1.964	0.620	1.047
30	-0.064	1.836	1.098	1.035
31	-0.0302	1.772	1.337	1.029
32	-0.00988	1.734	1.481	1.026
33	0.0173	1.683	1.673	1.021
34	0.0377	1.645	1.816	1.017
35	0.0432	1.584	0.635	1.044
36	0.0279	1.622	1.490	1.044
37	0.0381	1.597	1.587	1.044
38	0.0228	1.634	1.442	1.044
39	0.0124	1.729	1.235	1.043
40	0.0124	1.729	1.615	1.043
41	0.0124	1.729	0.856	1.043
42	0.027386	1.70923	1.307	1.039

Analysis

- Linear assumption

$$Y^{i,d}_{\text{coarse}} = A^d + J^d \cdot X^i_{\text{RHQ}}, \tag{26}$$

A: d-dimensional vector
J: $d \times 4$ matrix
$\{(X_{RHQ})^i_a\}_{1\le a\le 4} = \{m^i_0, c^i_B, c^i_E, \zeta^i\}$

- Steps:
  - Define correlated matrix:

$$(W_{\text{C}}^{-1})_{ij} = \sigma_{\text{coarse},\, i}\, \sigma_{\text{coarse},\, j}. \tag{27}$$

σ_{coarse} is the error estimate for the coarse lattice measurements.
  - Minimizing the

$$\chi^2_{\text{coarse}} = (A + J\cdot X_{\text{RHQ}} - Y_{\text{coarse}})^T \cdot W_{\text{C}} \cdot (A + J\cdot X_{\text{RHQ}} - Y_{\text{coarse}}) \tag{28}$$

to get a $d \times 4$ jacobian matrix J and a d-length vector of constant terms A.

Analysis (cont'd)

- Output coefficient obtained from

$$X_{\text{out}} = \left(J^T \cdot W_{\text{f}} \cdot J\right)^{-1} \cdot J^T \cdot W_{\text{f}} \cdot (Y_{\text{fine}} - A) \tag{29}$$

  where W_{f} is the full $d \times d$ correlation matrix as Eq. 27

- - Averaged Y_{fine} + jackknifed J and $A \Rightarrow \Delta X^{\text{c}}_{\text{out}}$;
  - Averaged J and A + jackknifed $Y_{\text{fine}} \Rightarrow \Delta X^{\text{f}}_{\text{out}}$;
  - $\Delta X^{\text{out}} = \sqrt{(\Delta X^{\text{c}}_{\text{out}})^2 + (\Delta X^{\text{f}}_{\text{out}})^2}$

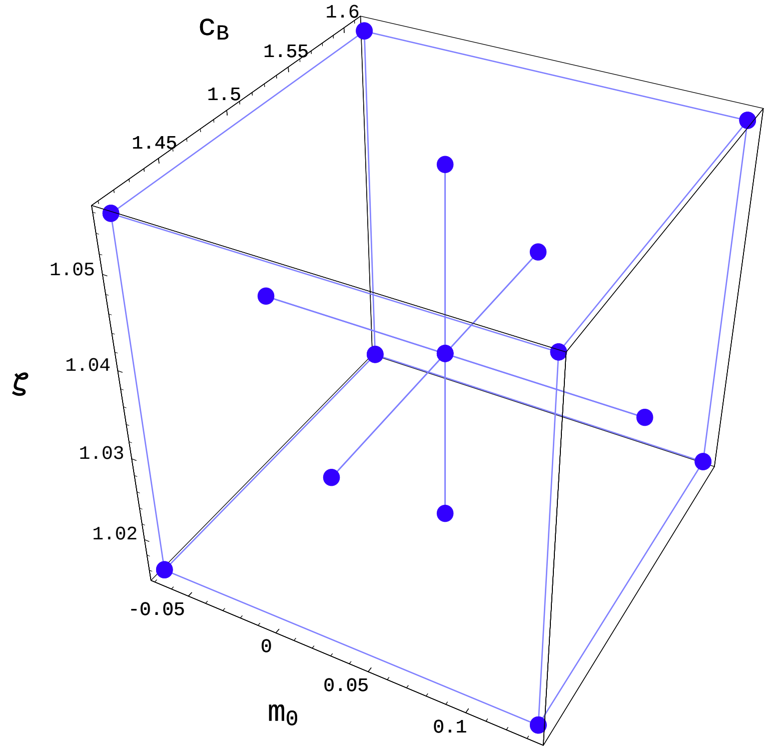
Four-Parameter Analysis

Various selections of data sets and corresponding output sorted by relative distance in data space; i.e. $\left(\frac{Y_{\text{fine}} - Y^i_{\text{coarse}}}{Y_{\text{fine}}}\right)^2$

Set	Set selection	$X_{\text{out}} = \{m_0, c_B, c_E, \zeta\}$	χ^2_{coarse}
I	{II, 23, 41, 63}	{-0.02(10), 1.68(19), 0.9(10), 1.041(23)}	316/24
II	{III, 7, 9, 10}	{ -0.03(11), 1.70(18), 0.9(9), 1.047(26)}}	102/21
III	{IV, 12, 62, 65}	{-0.04(12), 1.72(22), 0.8(10), 1.045(27)}	62/18
IV	{V, 3, 51, 61}	{ -0.03(12), 1.71(24), 0.9(11), 1.042(26)}	26/15
V	{VI, 6, 36, 37}	{0.00(9), 1.65(19), 1.1(8), 1.037(21)}	0.44/12
VI	{13, 14, 19, 20, 21 22, 38, 39, 42}	{0.03(7), 1.57(21), 1.5(7), 1.034(18)}	0.18/9

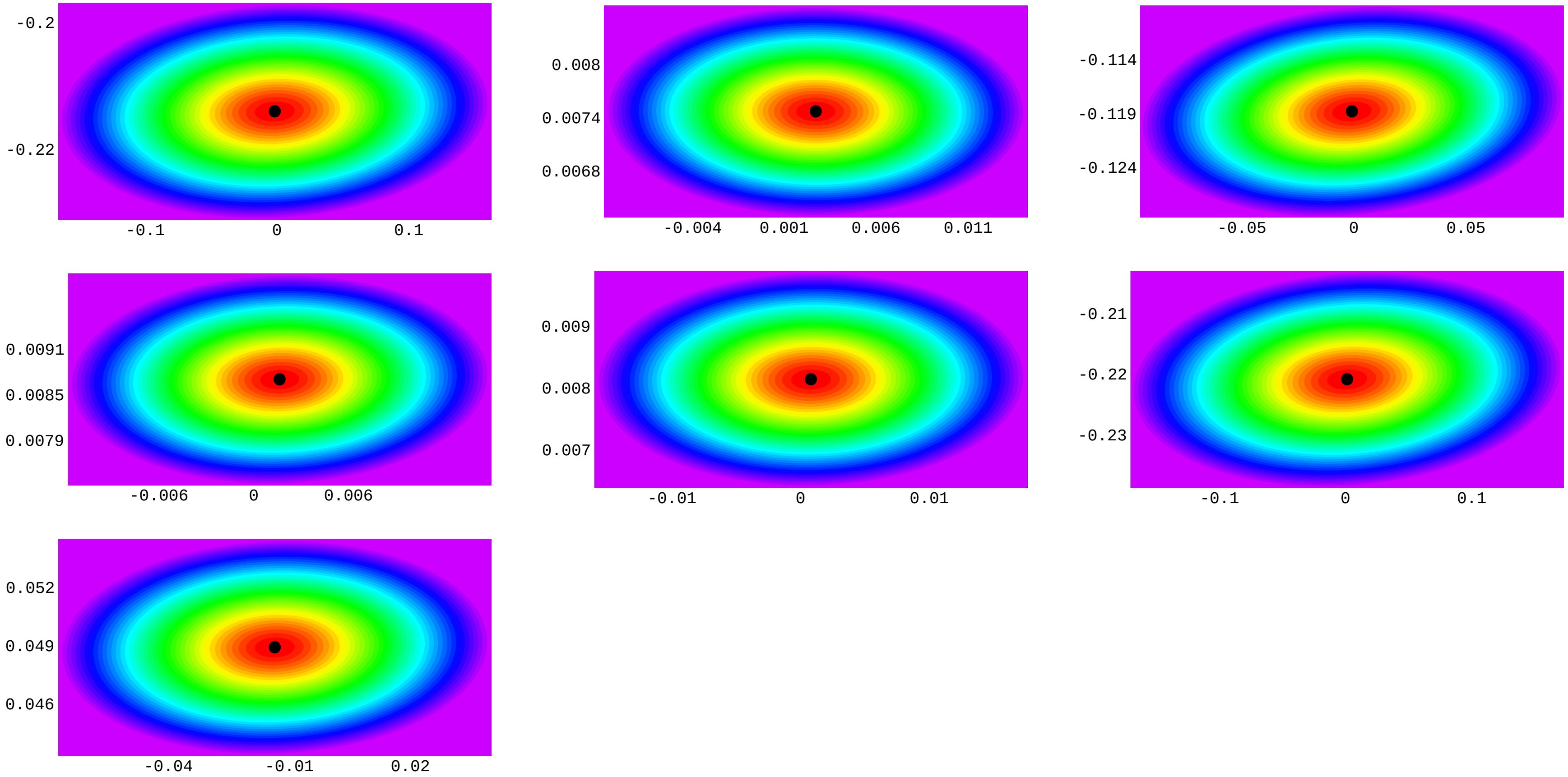
"Cartesian" Set

- Center: $X_{RHQ} = \{0.0328, 1.511, 1.538, 1.036\}$
- "Linear" region for requiring the linear fit can only differ up to a few percent: m_0: 0.1, c_B: 0.1, c_E: 0.3, ζ: 0.02.



χ^2_{coarse} Parameter Plot

- $J_{c_B - c_E} = \frac{\partial M}{\partial(c_B - c_E)}$ for the x-axis and $J_{c_B + c_E} = \frac{\partial M}{\partial(c_B + c_E)}$ for the y-axis. (from left to right: m^{hh}_{sa}, m^{hh}_{hs}, m^{hl}_{sa}, m^{hl}_{hs}, m^{hh}_{soa}, m^{hh}_{sos}, c^2)



- The minimized χ^2_{coarse} point have x-coordinates very close to zero $\Rightarrow c_B \approx c_E$.

Eigenspace

- The eigenvalues of the $J^T \cdot J$ matrix are:

$$\{9.58(15), 1.39(10), 0.000144(22), 0.000037(12)\}, \tag{30}$$

with corresponding eigenvectors

$$\begin{array}{llllll} & m_0. & c_B. & c_E. & \zeta. & \\ \{ & 0.832(4), & -0.1101(6), & -0.1081(7), & 0.533(6) & \}, \\ \{ & -0.523(7), & 0.062(6), & 0.086(3), & 0.846(4) & \}, \\ \{ & 0.182(7), & 0.81(7), & 0.56(10), & -0.004(6) & \}, \\ \{ & 0.040(23), & -0.57(10), & 0.82(7), & -0.0157(29) & \}. \end{array} \tag{31}$$

- The smallest eigenvalue is dominated by the eigenvector that has large component in c_E and that leads to large errorbar in the c_E coefficient.

Three-Parameter Action Analysis

- Take advantage of the original data

$$M^{n,d}_{\text{new}} = M^{n,d}_{\text{old}} + J^{n,d}_{c_E}(c_B - c_E), \tag{32}$$

- Repeat the analysis described in Eqs. 28 and 29:

$$X^{(3)}_{\text{out}} = \{m_0, c_{\text{P}}, \zeta\} = \{0.036(34), 1.53(10), 1.035(12)\}.$$

- Note: The overall errors on m_0 are only of the few percent level, can be translated into a 4% error from the heavy-light pseudoscalar mass.

Quadratic Effects

- The simple Taylor expansions up to second-derivative terms are

$$Y_{\text{coarse},qua} = Y_{\text{coarse},ave} + J^{(3)} \cdot (X_i - X_{14}) + \frac{1}{2}(X_i - X_{14}) \cdot Q \cdot (X_i - X_{14}) \quad (33)$$

- Consider quadratic terms

$$\begin{aligned} \chi^2_{\text{fine,qua}} &= \left(Y_{\text{fine}} - A^{(3)} - J^{(3)} \cdot X^{(3)}_{\text{out}} - 1/2(X^{(3)}_{\text{out}})^T \cdot Q^{(3)} \cdot X^{(3)}_{\text{out}}\right)^T \\ &\quad \cdot W_{\text{f}} \cdot \left(Y_{\text{fine}} - A^{(3)} - J^{(3)} \cdot X^{(3)}_{\text{out}} - 1/2(X^{(3)}_{\text{out}})^T \cdot Q^{(3)} \cdot X^{(3)}_{\text{out}}\right). \quad (34) \end{aligned}$$

The result is $\{m_0, c_P, \zeta\} = \{0.034(8), 1.50(3), 1.035(5)\}$ when considering quadratic effects.

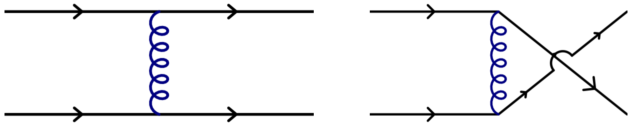
Comparison with Perturbative Results

- Straightforward comparison:
  - Fermilab action *M. Nobes and H. Trottier (hep-lat/0509128)*
  - Idea: quark scatters off background field
- Indirect comparison:
  - Tsukuba action *S. Aoki, Y. Kayaba and Y. Kuramashi (hep-lat/0309161)*
  - Idea: on-shell quark-quark scattering amplitude
- $c_x = c_x^{(0)} + g^2 c_x^{(1)}$

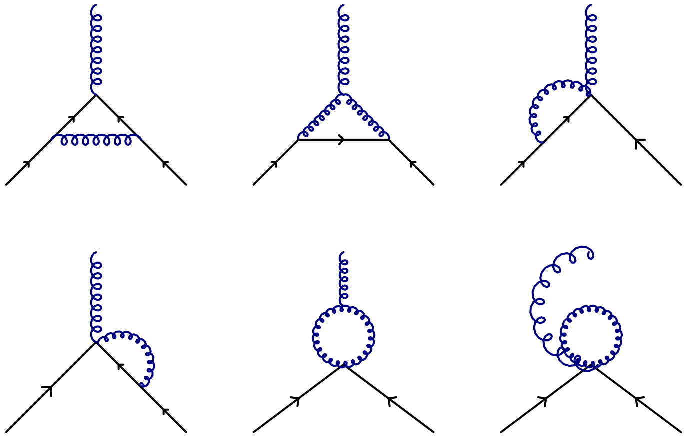
Tsukuba One-Loop Coefficients

- Quark-quark scattering amplitude

Tree-level



One-loop vertex



- Explicit coefficients with details listed in *hep-lat/0309161*
- Field transformation to connect it to the four-parameter RHQ:

$$\psi \to [1 + \delta \vec{\gamma} \cdot \vec{D}]\psi$$
$$\bar{\psi} \to \bar{\psi}[1 - \delta \vec{\gamma} \cdot \overleftarrow{D}]. \qquad (35)$$

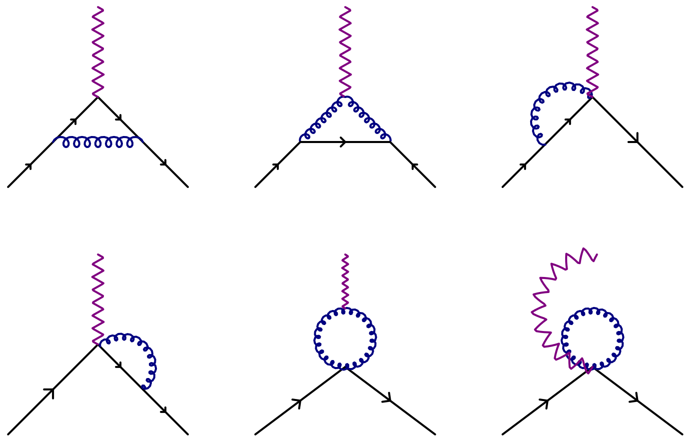
- In the continuum form, $\delta = \frac{r_s - \zeta}{2(ma - \zeta)}$ seems do the trick.
- On the lattice, they are slightly different because of the discretization for D_μ^2 and $D_\mu D_\mu$.

Fermilab One-Loop Coefficients

- Tree-level coefficients:

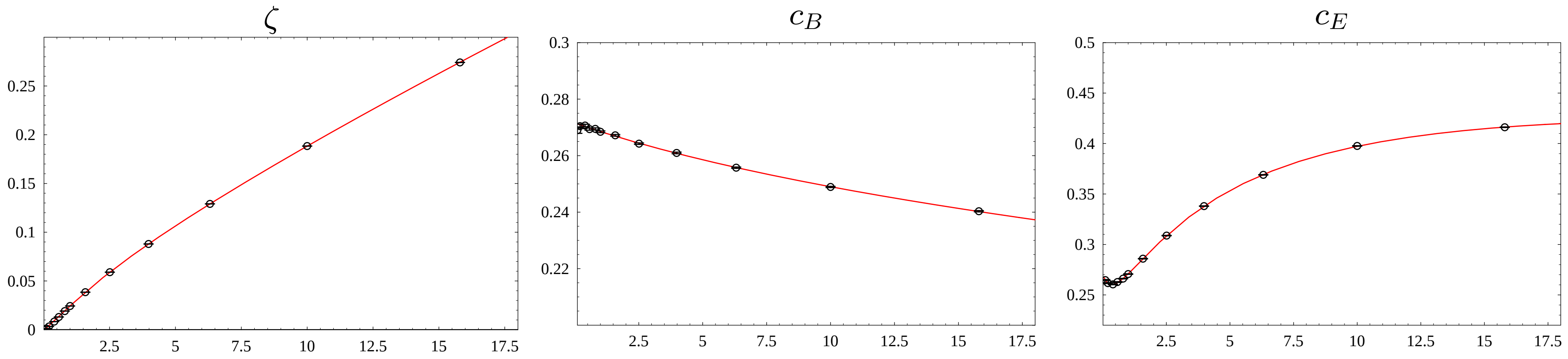
$$
\begin{aligned}
b_1 &= \zeta = \sqrt{\left(\frac{r_s m_0(2+m_0)}{4(1+m_0)}\right)^2 + \frac{m_0(2+m_0)}{2\ln(1+m_0)}} - \frac{r_s m_0(2+m_0)}{4(1+m_0)} \\
b_3 &= b5 = r_s\zeta \\
b_4 &= \left(\frac{\zeta^2-1}{m_0(2+m_0)} + \frac{r_s\zeta}{(1+m_0)} + \frac{r_s^2 m_0(2+m_0)}{4(1+m_0)^2}\right)\zeta
\end{aligned}
\quad (36)
$$

- One-loop level:



- Parameterize the one-loop Fermilab results from Table 6.1-2 in Nobes's thesis:

$$c_x^{[1]} = \frac{\sum_{i=0}^{3} a_i m_0^i}{1 + \sum_{i=1}^{3} d_i m_0^i} \tag{37}$$



	a_0	a_1	a_2	a_3	d_1	d_2	d_3
$\zeta^{[1]}$	0.00029923	0.00124977	0.163759	0.0258287	5.10243	1.65713	0.00633212
$c_E^{[1]}$	0.270419	0.431474	0.162718	0.00212438	1.87436	0.319194	0.00619183
$c_B^{[1]}$	0.271519	0.0122322	−0.000039117	0	0.0565955	0	0

- Massless limit of c_{SW} consistent with Tsukuba's result (0.26858825).

Summary Comparison

- Using J and A from data set to predict the corresponding data

Parameter	m^{hh}_{sa}	m^{hh}_{hs}	m^{hl}_{sa}	m^{hl}_{hs}	m^{hh}_{sos}	m^{hh}_{soa}	c^2
scaled Y_{fine}	1.0881(16)	0.0175(6)	0.725(3)	0.0267(13)	0.017(3)	1.211(8)	1.002(23)
$X^{(3)}_{\text{out}}$	1.0854(16)	0.0165(5)	0.716(4)	0.0239(16)	0.019(4)	1.206(13)	1.000(20)
$X^{Fermilab,\ tad}_{\text{out}}$	1.1700(15)	0.0124(4)	0.763(4)	0.0194(13)	0.014(4)	1.292(13)	0.943(16)
$X^{Tsukuba}_{\text{out}}$	1.1530(15)	0.0131(4)	0.754(4)	0.0201(13)	0.015(4)	1.274(13)	0.945(17)
$X^{Tsukuba,\ mod}_{\text{out}}$	1.0223(17)	0.0183(6)	0.684(4)	0.0258(17)	0.021(5)	1.142(13)	0.976(20)

- Example of SW action: $c^{\text{NP}}_{\text{SW}} = 1.544$ (by ALPHA) vs. $c^{\text{PT, one-loop}}_{\text{SW}} = 1.254$
- No surprise on the discrepancy between the perturbation coefficients and nonperturbative ones
- Modified coefficients suggested in

$$c^{\text{PT}}_{\{B,E\}}(m) \to c_{\{B,E\}}(m)^{\text{PT}} + \left(c^{\text{NP}}_{\text{SW}} - c^{\text{PT}}_{\{B,E\}}(m=0)\right) \tag{38}$$

- $X^{Tsukuba,\ mod}_{\text{out}}$ works better by shifting 0.3 clover term.

Systematic Uncertainties

- 4% from the heavy quark discretization effect from DWF on the fine lattice
- 1% from the remaining RHQ heavy quark discretization effect on the coarse lattice
- 1.3% from the lattice matching between fine and coarse lattice
- 0% from quadratic effects
- Added to statistical errorbars, the final coefficients are

$$X^{(3)}_{\text{out}} = \{m_0, c_{\text{P}}, \zeta\} = \{0.036(4), 1.54(20), 1.03(8)\}$$

Conclusion and Outlook

- It appears practical to determine m_0, c_P and ζ from finite-volume, nonperturbative matching.
- Loads of improvements can be done:
  - Increasing the statistics (faster if fewer constraints used)
  - Starting from finer lattice; thus smaller DWF discretization effect
- Immediately to be done:
  - Next-stage matching
  - More masses in the run
  - Using the coefficients for charm physics on the dynamical configurations
- Searching for $O(a)$-improved coefficients in the currents
- Application to full QCD:
  - Matching requires N_f=3 but not a physical value of m_{sea}.
  - Can make m_{sea} large and the calculation easy provided $m_{sea}a \ll 1$.
  - Must have m_{sea}/Λ_{QCD} equal for each pair of systems being matched.