

Determination of running coupling α_s
for $N_f = 2 + 1$ QCD
with Schrödinger functional scheme

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Purpose of this project

- Determine the fundamental parameter of $N_f = 2 + 1$ QCD

$$\mathcal{L} = -\frac{1}{4g^2} F_{\mu\nu}^a F_{\mu\nu}^a + \bar{\psi}_i (\gamma_\mu D_\mu + \textcolor{red}{m_i}) \psi_i$$

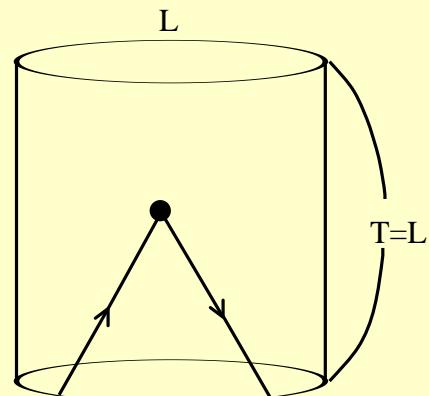
- Strong coupling $\textcolor{red}{g}$
 - Low energy input m_π , m_K , m_Ω are measured by PACS-CS
- Quark masses $\textcolor{red}{m_i}$
 - Bare quark masses are measured by PACS-CS
- We adopt low energy hadron masses as input.
 - Comparison with high energy input.
 - Need RG running from low to high energy region.

Goal of this project

- Evaluate $\alpha_S(M_Z)$ by input of hadron masses (m_π, m_K, m_Ω).
- NP renormalization factor of quark mass (Z_m).

Method

- Schrödinger functional scheme (Lüscher et al, Alpha)
for non-perturbative renormalization

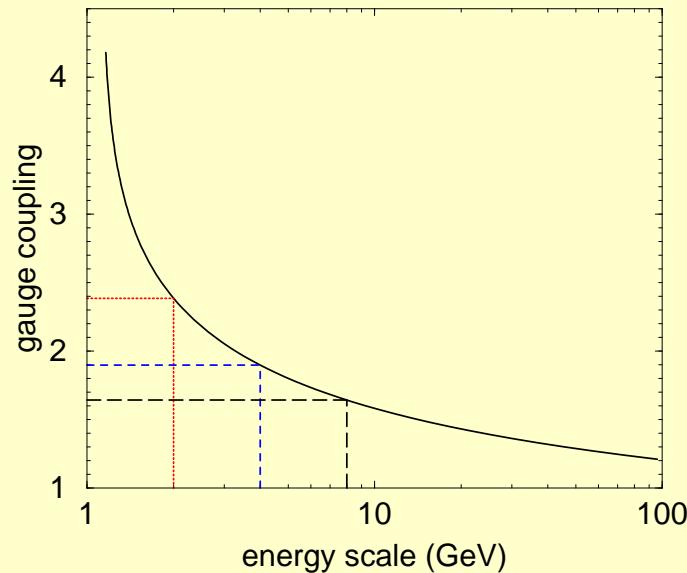


- Finite volume of L^4
- Appropriate boundary condition
- Renormalization scale $\sim 1/L$

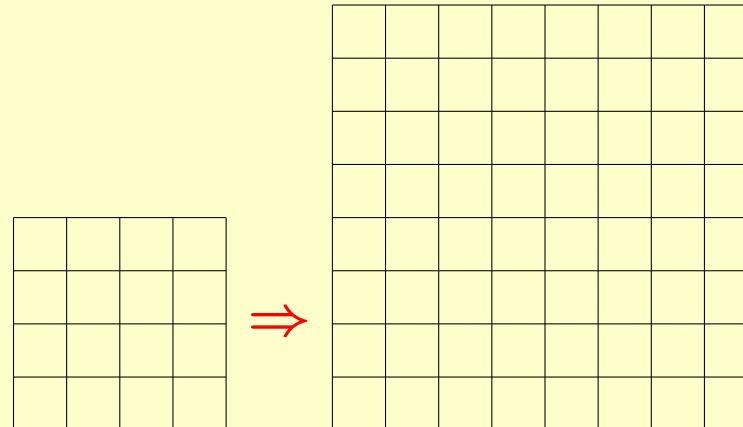
- Covers from low to high energy region.

Step Scaling Function

- Renormalization group flow $g(L) \rightarrow g(2L)$
when one changes the renormalization scale $L \rightarrow 2L$



implemented easily on lattice



- Follow the renormalization group flow in discretized way.

Numerical setup

- Iwasaki gauge action $\beta = 2.1 \sim 6.2$
 - tree level boundary improvement
- Wilson fermion with clover term.
 - non-perturbative c_{SW}
 - one loop boundary improvement
 - $\theta = \pi/5$
- RHMC/HMC algorithm for 3rd/two flavour(s)
- CPS++ code ([RBC Collaboration](#))
- Machines
 - PC cluster kaede at Tsukuba: (~ 180 PU)
 - SR11000 at Tokyo: (~ 64 PU)
 - PACS-CS: (256 PU) \times 1 month
 - RSICC at Riken (128 PU)
 - T2K at Tsukuba: (2560 cores)
 - T2K at Tokyo: (128 cores)

Configurations

- Take the continuum limit by three box sizes.

L/a	4	6	8
$2L/a$	8	12	16

$$\Sigma(u, a/L) = \bar{g}^2(2L) \Big|_{\bar{g}^2(L)=u, m=0}$$

- Tuning of β and κ for fixed physical box size.

\bar{g}^2	1.001	1.249	1.524	1.840	2.129	2.632	3.418
4^4	110K	170K	230K	170K	210K	320K	100K
8^4	40K	40K	86K	134K	50K	74K	380K
6^4	153K	150K	170K	170K	110K	144K	120K
12^4	42K	51K	48K	35K	40K	34K	119K
8^4	98K	86K	122K	98K	74K	122K	134K
16^4	31K	52K	42K	23K	40K	51K	54K

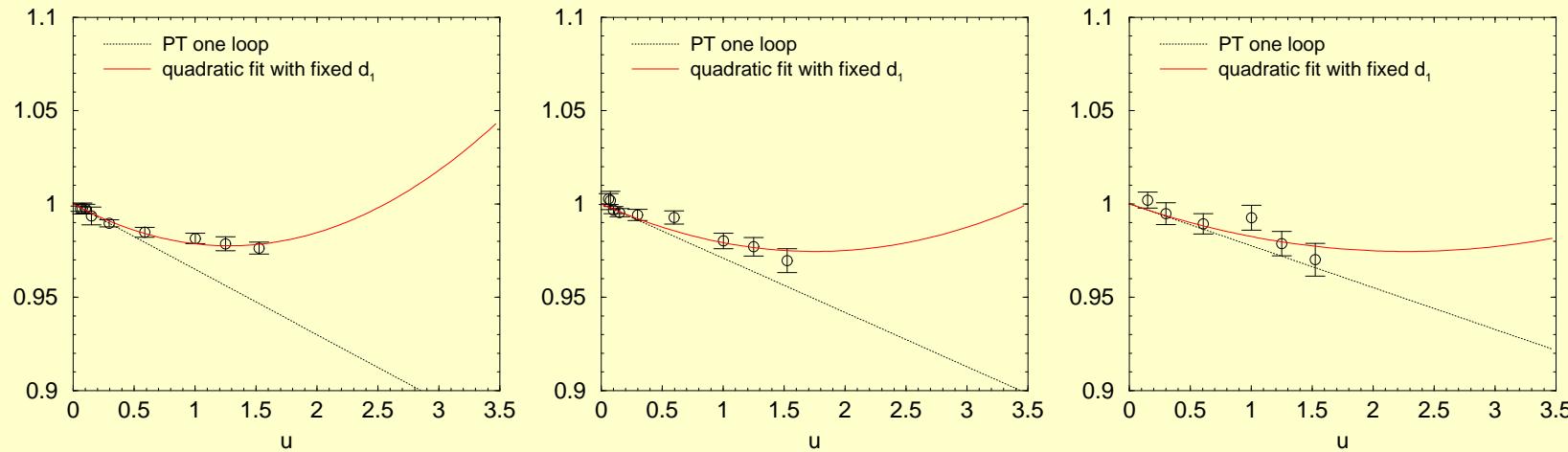
Perturbative improvement

- Deviation from the continuum perturbative SSF

$$\frac{\Sigma(u, a/L) - \sigma_{\text{PT}}(u)}{\sigma_{\text{PT}}(u)} = \delta_1(a/L)u + \delta_2(a/L)u^2 + \dots$$

L/a	4	6	8	10	12
δ_1	-0.03506	-0.02909	-0.02243	-0.01832	-0.01553

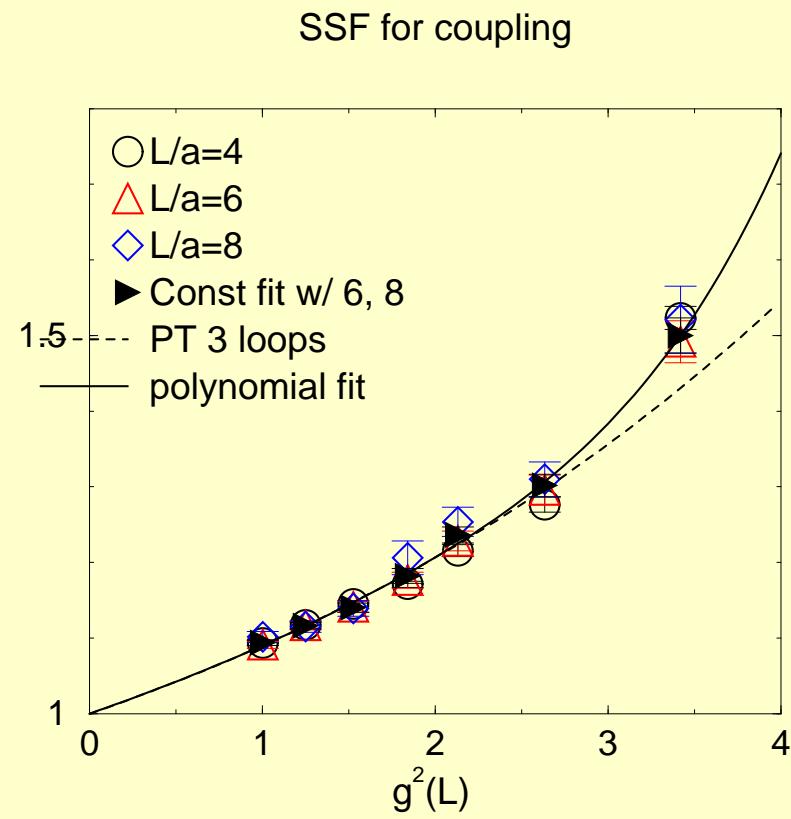
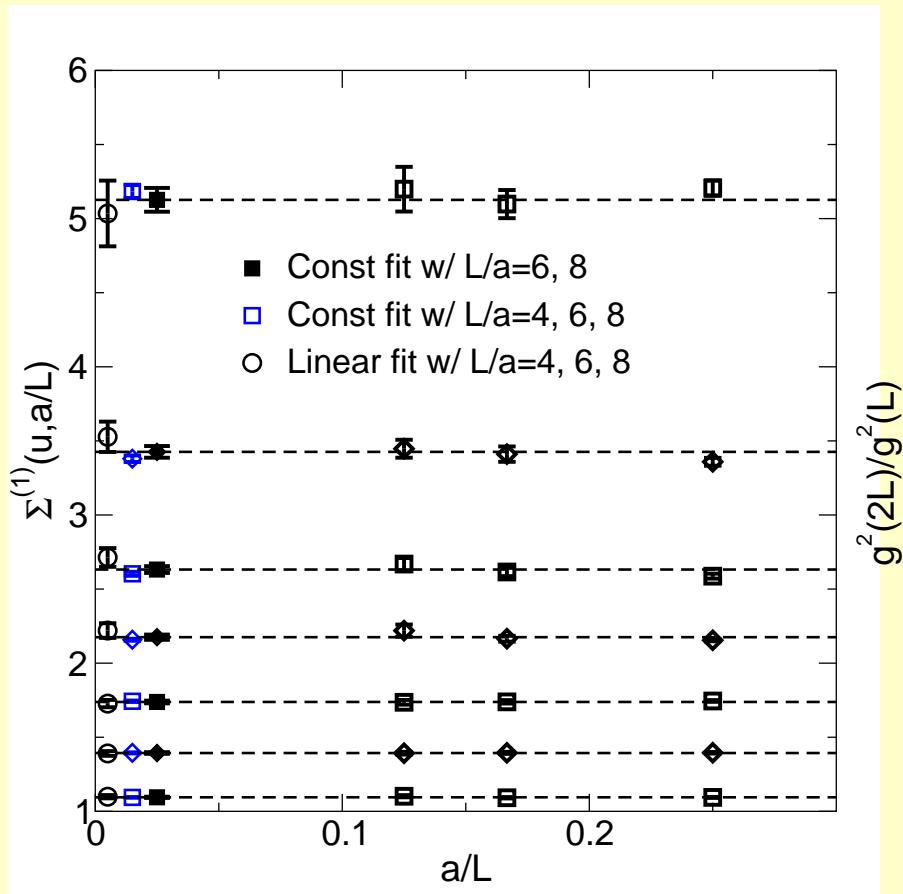
- “Perturbative” behavior of $1 + \delta(a/L)$: (high β simulation)



SSF

- “Two loops” improvement

$$\Sigma^{(2)} \left(u, \frac{a}{L} \right) = \frac{\Sigma(u, a/L)}{1 + \delta_1(a/L)u + d_2(a/L)u^2}$$



Polynomial fit of NP SSF

$$\sigma(u) = u + s_0 u^2 + s_1 u^3 + s_2 u^4 + s_3 u^5 + s_4 u^6,$$
$$s_2 = 0.002265, \quad s_3 = -0.00158, \quad s_4 = 0.000516.$$

Introduction of physical scale

- Lattice spacing as an intermediate scale:
 - CP-PACS Collaboration ($m_\pi \sim 500$ MeV)

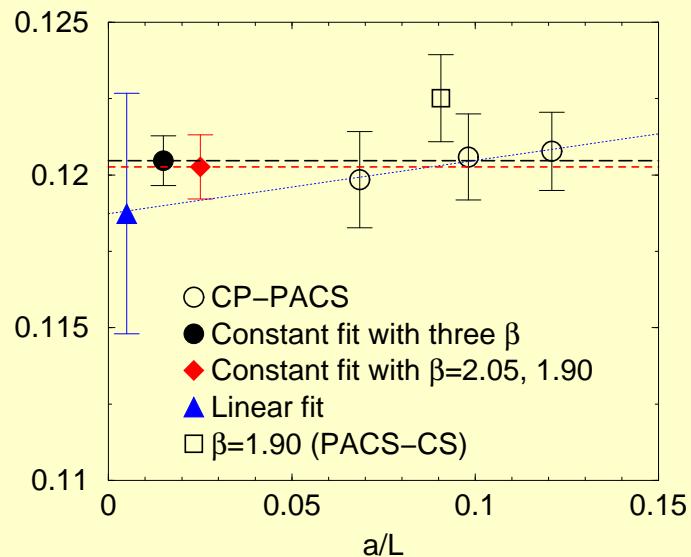
β	a (fm)	L_{\max}/a	$1/L_{\max}$ (MeV)	$\bar{g}^2(L_{\max})$
1.83	0.1209(16)	4	408.0(5.4)	5.565(54)
1.90	0.0982(19)	4	502.3(9.7)	4.695(23)
2.05	0.0685(26)	6	480(18)	4.740(79)

- PACS-CS Collaboration ($m_\pi \sim 155$ MeV)

β	a (fm)	L_{\max}/a	$1/L_{\max}$ (MeV)	$\bar{g}^2(L_{\max})$
1.90	0.0907(13)	4	544.0(7.8)	4.695(23)

$$\alpha_{\overline{\text{MS}}}(M_Z)$$

- NP running from L_{\max} to $1/\mu = 2^{-5}L_{\max}$ by SSF.
- 2-loops matching between SF and $\overline{\text{MS}}$ scheme at μ .
- 4-loops β -fn. in $\overline{\text{MS}}$ scheme.
- 3-loops matching between $\alpha_{\overline{\text{MS}}}^{(3)}(m_c)$ and $\alpha_{\overline{\text{MS}}}^{(4)}(m_c)$.
- 3-loops matching between $\alpha_{\overline{\text{MS}}}^{(4)}(m_b)$ and $\alpha_{\overline{\text{MS}}}^{(5)}(m_b)$.



CP-PACS:

$$\alpha_s(M_Z) = 0.12047(81)(-48)(-173)$$

3-loops vs. 2-loops matching

const. vs. linear extrapolation

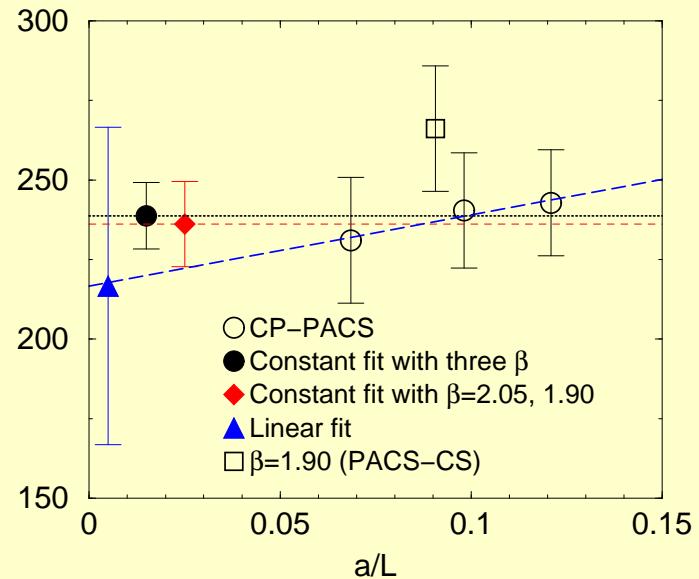
PACS-CS:

$$\alpha_s(M_Z) = 0.1225(14)(-5)$$

Λ parameter

$$\Lambda = \mu (b_0 \bar{g})^{-\frac{b_1}{2b_0^2}} \exp\left(-\frac{1}{2b_0 \bar{g}}\right) \exp\left(-\int_0^{\bar{g}} dg \left(\frac{1}{\beta} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g}\right)\right)$$

- NP running from L_{\max} to $1/\mu = 2^{-5}L_{\max}$ by SSF.
- 3-loops β -fn. above μ .
- 3-loops matching at threshold m_c and m_b to get $\Lambda_{\overline{\text{MS}}}^{(5)}$.



CP-PACS:

$$\Lambda_{\overline{\text{MS}}}^{(5)} = 239(10)(-6)(-22) \text{ MeV}$$

3-loops vs. 2-loops matching

const. vs. linear extrapolation

PACS-CS:

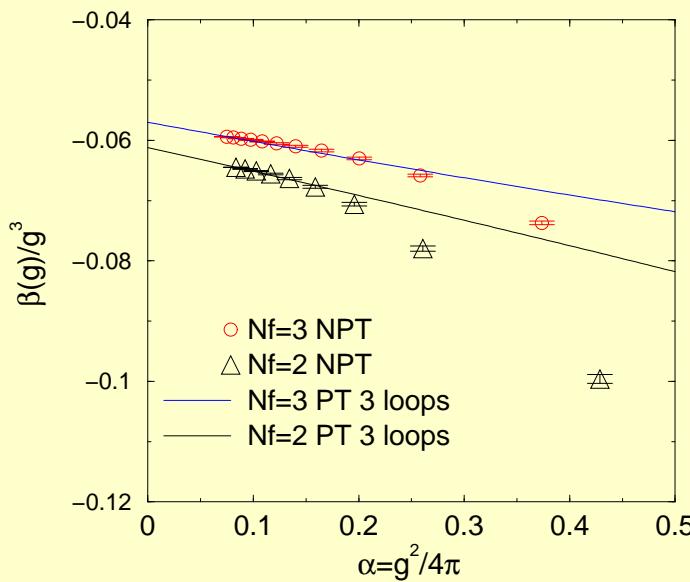
$$\Lambda_{\overline{\text{MS}}}^{(5)} = 266(20)(-7) \text{ MeV}$$

Non-perturbative β -function

$$\beta\left(\sqrt{\sigma(u)}\right) = \beta(\sqrt{u}) \sqrt{\frac{u}{\sigma(u)}} \frac{\partial \sigma(u)}{\partial u}.$$

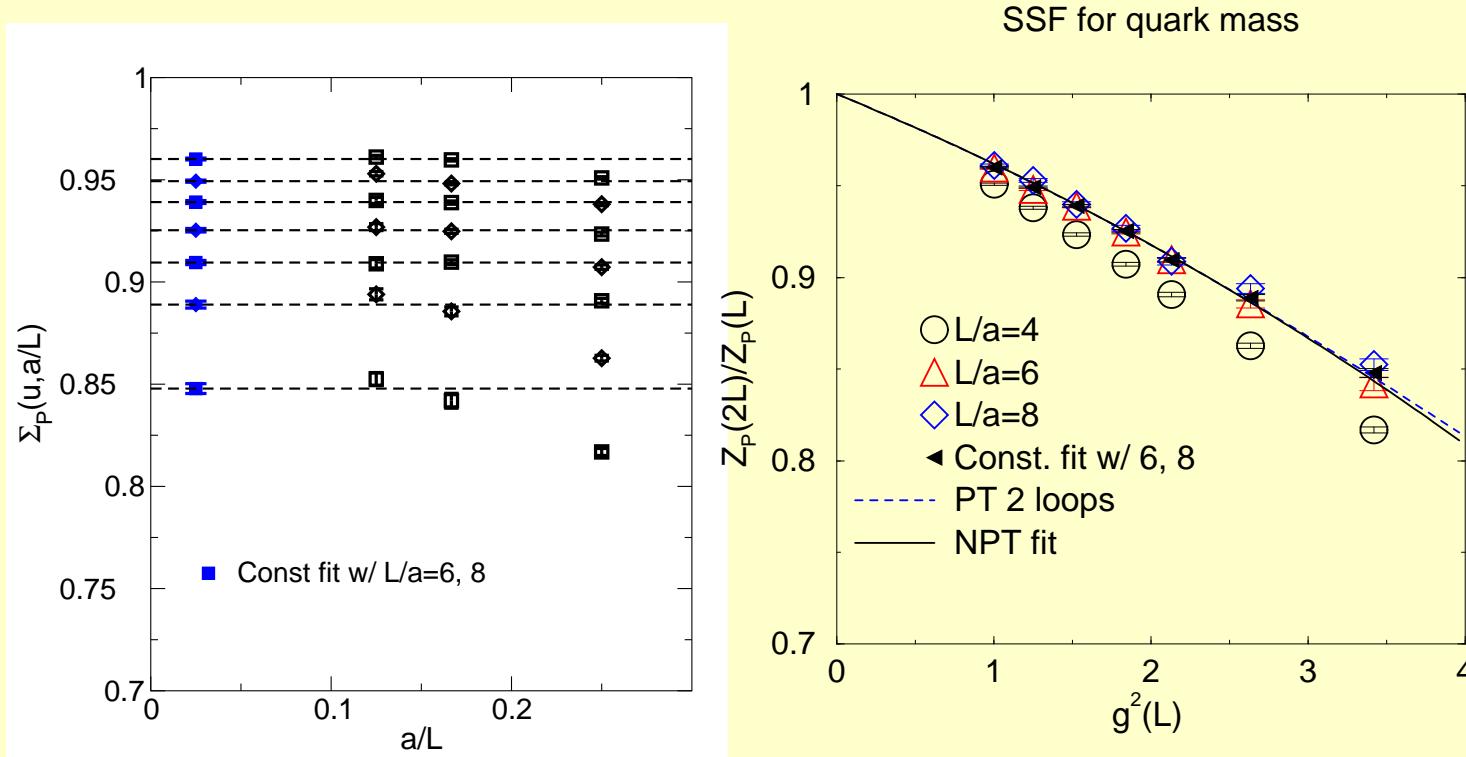
- Polynomial fit (fixed to 3-loop coeff.)

$$\sigma(u) = u + s_0 u^2 + s_1 u^3 + s_2 u^4 + s_3 u^5 + s_4 u^6,$$
$$s_3 = -0.000673, \quad s_4 = 0.0003434.$$



SSF of Z_P

$$\Sigma_P \left(u, \frac{a}{L} \right) = \frac{Z_P(g_0, 2L/a)}{Z_P(g_0, L/a)} \Big|_{\bar{g}^2(L)=u, m=0}$$



- Scaling behavior is not good for $L/a = 4$.
- Continuum limit with two data points 6 and 8.

Non-perturbative renormalization factor Z_m

- Renormalization factor for the RGI mass.

$$M = Z_M(g_0)m_0(g_0),$$

$$Z_M(g_0) = \frac{Z_A(g_0)}{\underbrace{Z_P(g_0, a/L_{\max})}_{Z_m^{(\text{SF})}(g_0, a/L_{\max})}} \frac{\overline{m}(2^n/L_{\max})}{\underbrace{\overline{m}(1/L_{\max})}_{\text{NP run in SF}}} \frac{M}{\underbrace{\overline{m}(2^n/L_{\max})}_{2 \text{ loops run}}}$$

$Z_m^{(\text{SF})}(g_0, a/L_{\max})$ NP run in SF 2 loops run

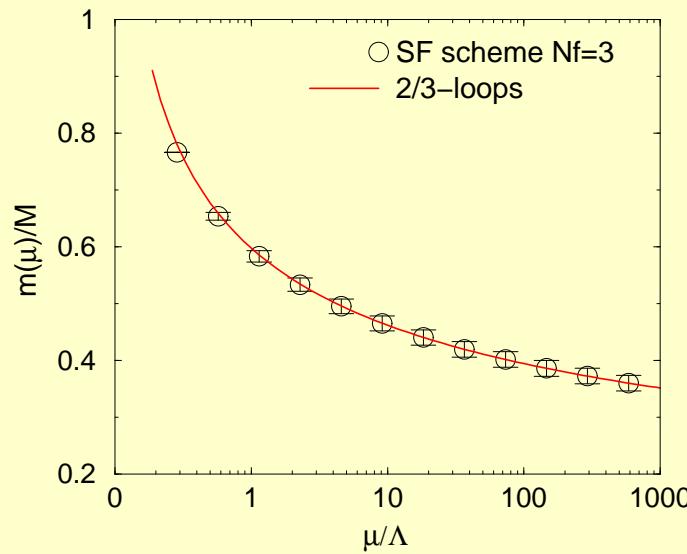
- For PACS-CS result at $\beta = 1.90$ (example)
 - $L_{\max} = 544.0(7.8)$ MeV
 - $M/\overline{m}(1/L_{\max}) = 1.305(33)$
 - $Z_P(g_0, a/L_{\max}) = 0.62987(28)$
 - $Z_A(m = 0.04) = 0.8227(60)$, $Z_A(m = 0.01) = 0.8437(60)$
 - $Z_M(g_0) = 1.705(45)$
- Running back from M with 4-loops RG fn. in $\overline{\text{MS}}$ scheme.
 - $Z_m^{\overline{\text{MS}}}(\mu = 2 \text{ GeV}, \beta = 1.90) = 1.353(39)$ (Preliminary!)

(cf. $Z_m^{\text{tad}} = 1.13813$) $\rightarrow 1.388(39)$

Non-perturbative running mass $\overline{m}(\mu)$

$$\frac{\overline{m}(1/L_n)}{M} = \underbrace{\prod_{i=1}^n \sigma_P(u_i)}_{\text{SSF of } Z_P} \underbrace{\frac{\overline{m}(1/L_{\max})}{M}}_{\text{Input}}$$

$$\frac{\overline{m}(1/L_n)}{M} = \left(2b_0\bar{g}^2(L_n)\right)^{\frac{d_0}{2b_0}} \exp\left(\int_0^{\bar{g}(L_n)} dg \left(\frac{\tau(g)}{\beta(g)} - \frac{d_0}{b_0 g}\right)\right)$$



α_s

- Running coupling $\alpha_s(M_Z)$ from hadron mass input.
 - CP-PACS: $\alpha_s(M_Z) = 0.12047(81)(-48)(-173)$
 - PACS-CS: $\alpha_s(M_Z) = 0.1225(14)(-5)$
- Scaling behavior seems to be good.
- PT behavior of the deviation $\Sigma(u, a/L) - \sigma_{\text{PT}}(u)$ is non-trivial
 - “Two loops” coeff. is comparable to one loop.
 - Iwasaki action may not be a good setup for SF scheme.

Z_m

- $Z_m^{\overline{\text{MS}}}(\mu = 2 \text{ GeV}, \beta = 1.90) = 1.353(39)$ (Preliminary!)
 - $Z_m^{\text{PT,tad}} = 1.13813$
- Scaling behavior of quark mass SSF seems not so good.
 - We may need perturbative improvement.

Future work

- May adopt more suitable gauge action
 - Better PT behavior (Lüscher-Weisz action)

Schrödinger functional scheme

(Lüscher et al, Alpha)

- Dirichlet boundary condition at $t = 0, T$.

$$U_k(x)|_{x_0=0} = \exp(aC_k), \quad C_k = \frac{i}{L} \begin{pmatrix} \phi_1 & & \\ & \phi_2 & \\ & & \phi_3 \end{pmatrix}$$

- Unique global minimum at background field B_μ .
- Mass gap in fermionic mode
(quark mass can be set to zero).
- Renormalized coupling

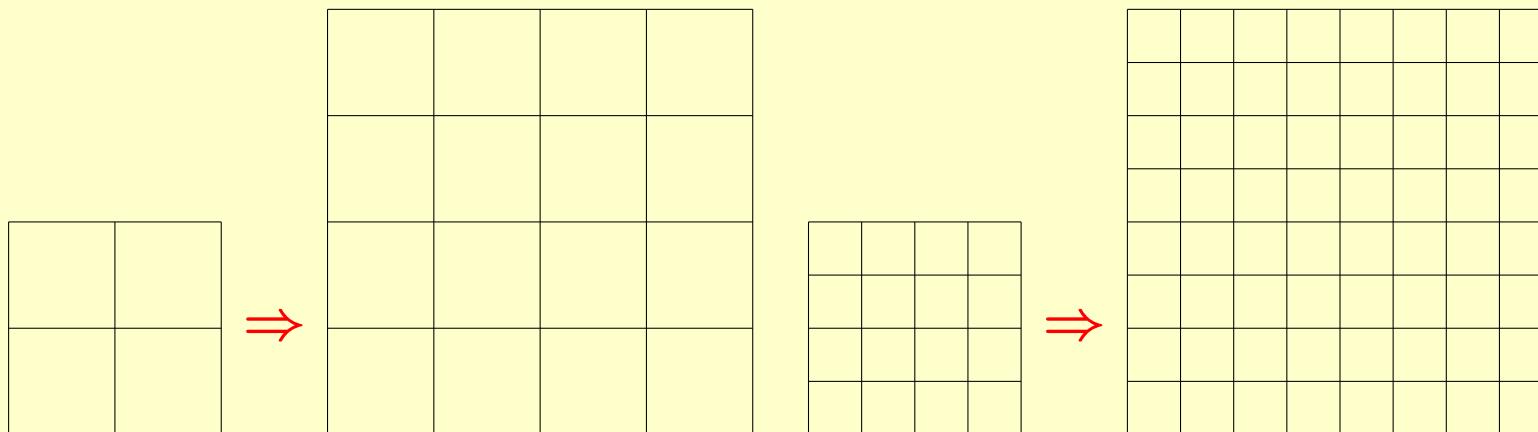
$$S_0 = \frac{1}{g_0^2} F_{\mu\nu}^2 \Rightarrow \Gamma_0[B_\mu] = \Gamma[B_\mu] = \frac{1}{g_R^2(L)} k[B_\mu]$$

- Mass renormalization factor

$$Z_m(L) = \frac{1}{k_{\text{tree}}} \frac{\langle P(t=L/2) \cdot \mathcal{O}_{\text{boundary}} \rangle}{\langle \mathcal{O}_{\text{boundary}} \rangle}$$

Step scaling function

- The point is:
 - To take continuum limit for every step of RG flow.
 - $L \rightarrow 2L, g(L, a) \rightarrow g(2L, a)$



- Obtain the RG flow $g(L) \rightarrow g(2L)$ in the continuum.