Twisted-mass QCD, $O(a)$ improvement and Wilson chiral perturbation theory

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Twisted-mass QCD (tmQCD)

Twisted mass term: \( m' e^{i \omega \gamma_5 \tau_3} = m + i \mu \gamma_5 \tau_3 \)

\( \omega \): twist angle

Advantages of tmQCD with Wilson fermion:

- No zero mode at \( \mu \neq 0 \) \( \Rightarrow \) No exceptional configurations
- Simplified renormalization of weak matrix elements
- **Automatic O(a) improvement** at maximal twist \( \omega = \frac{\pi}{2} \)

Frezzotti, Rossi ’03

But: Definition of maximal twist on the lattice is non-trivial!

Caveat: Definition of Frezzotti/Rossi works only if \( m' \gg a^2 \)
Outline of the talk

• Automatic $O(a)$ improvement at maximal twist
  • Argument a la Frezzotti and Rossi
  • Caveat for small quark masses
  • Alternative proposal for maximal twist
• Pion mass in ChPT at non-zero lattice spacing
  • Brief introduction into ChPT at non-zero lattice spacing
  • Example: $O(a)$ improvement of the pion mass at maximal twist
• Alternative scenario ($c_2 < 0$)
• Summary
Twisted mass term on the lattice

Mass term + Wilson term on the lattice:

\[
\bar{\psi}(x) \left[ \left( -a \frac{r}{2} \sum_{\mu} \nabla_{\mu}^* \nabla_{\mu} + M_{\text{cr}}(r) \right) + m_q \exp(i \omega \gamma_5 \tau_3) \right] \psi(x)
\]

\[
m_q = m_0 - M_{\text{cr}}(r)
\]

Field redefinition:

\[
\psi_{ph} = \exp(i \frac{\omega}{2} \gamma_5 \tau_3) \psi,
\]

\[
\bar{\psi}_{ph} = \bar{\psi} \exp(i \frac{\omega}{2} \gamma_5 \tau_3)
\]

\[
\bar{\psi}_{ph}(x) \left[ \left( -a \frac{r}{2} \sum_{\mu} \nabla_{\mu}^* \nabla_{\mu} + M_{\text{cr}}(r) \right) \exp(-i \omega \gamma_5 \tau_3) + m_q \right] \psi_{ph}(x)
\]
Wilson average and $O(a)$ improvement

The Wilson average

$$\langle O \rangle^{WA}(r, m_q, \omega) \equiv \frac{1}{2} \left[ \langle O \rangle(r, m_q, \omega) + \langle O \rangle(-r, m_q, \omega) \right]$$

can be shown to be $O(a)$ improved:

$$= \langle O \rangle^{\text{cont}}(m_q) + O(a^2)$$

Crucial assumption:

$$M_{cr}(-r) = -M_{cr}(r)$$
Automatic $O(a)$ improvement at maximal twist

Consider the twist average:

$$\langle O \rangle^{TA}(r, m_q, \omega = \frac{\pi}{2}) \equiv \frac{1}{2} \left[ \langle O \rangle(r, m_q, \omega = \frac{\pi}{2}) + \langle O \rangle(r, m_q, \omega = -\frac{\pi}{2}) \right]$$

$$\exp(-i \frac{\pi}{2} \gamma_5 \tau_3) = - \exp(i \frac{\pi}{2} \gamma_5 \tau_3)$$

$$= \frac{1}{2} \left[ \langle O \rangle(r, m_q, \omega = \frac{\pi}{2}) + \langle O \rangle(-r, m_q, \omega = \frac{\pi}{2}) \right]$$

For observables even in $\omega$ (e.g. masses):

$$\langle O \rangle(r, m_q, \omega = \frac{\pi}{2}) = \langle O \rangle^{TA}(r, m_q, \omega = \frac{\pi}{2}) = \langle O \rangle^{\text{cont}}(m_q) + O(a^2)$$

$O(a)$ improvement without taking an average!
Crucial assumption: \[ M_{cr}(-r) = -M_{cr}(r) \]

“Proof”:

1. Defining equation: \[ m_\pi(r, M_{cr}(r)) = 0, \]

2. Symmetries of the lattice theory: \[ m_\pi(r, m_0) = m_\pi(-r, -m_0) \]

1 & 2: \[ m_\pi(r, M_{cr}(r)) = m_\pi(-r, M_{cr}(-r)) = m_\pi(r, -M_{cr}(-r)) = 0 \]

3. “Conclusion”: \[ M_{cr}(r) = -M_{cr}(-r) \]

Only true if the defining equation has a unique solution!
Critical Mass: Phase diagram with Wilson fermion

\begin{align*}
\langle \bar{\psi} i \gamma_5 \tau^3 \psi \rangle &= 0 \\
\langle \bar{\psi} i \gamma_5 \tau^3 \psi \rangle &\neq 0
\end{align*}

At least 2 values for the critical mass!
Critical Mass: ChPT analysis

Sharpe, Singleton ‘98

Two possible scenarios:

Scenario 1: No massless pion at non-zero a

Scenario 2: Spontaneous breaking of flavor and parity

Massless pions

Quantitative result for critical mass:

Width of “fingers” is $\mathcal{O}(a^2)$

$$M_{cr}(r) = M_{\text{odd}}(r) + a^2 c M_{\text{even}}(r) \equiv M_{cr}^{(1)}(r)$$

$$-M_{cr}(-r) = M_{\text{odd}}(r) - a^2 c M_{\text{even}}(r) \equiv M_{cr}^{(2)}(r)$$

$$\Rightarrow M_{cr}(r) \neq -M_{cr}(-r)$$
What happens to $O(a)$ improvement?

Ansatz:

$$M_{cr}(r) = M_{odd}(r) + a^2 c \ M_{even}(r)$$

$$\Rightarrow \quad \langle O \rangle(r, m_q, \omega = \frac{\pi}{2})^{TA} = \frac{1}{2} \left[ \langle O \rangle(r, m_q, \omega = \frac{\pi}{2}) + \langle O \rangle(-r, m'_q, \omega = \frac{\pi}{2} + \omega') \right]$$

$$m'_q = \sqrt{m_q^2 + (2a^2 c M_{even}(r))^2} \quad \tan \omega' = \frac{2a^2 c M_{even}(r)}{m_q}$$

$$\Rightarrow \quad \text{Twist average} = \text{Wilson average only if} \quad m_q \gg a^2$$

$$\Rightarrow \quad \text{Automatic O(a) improvement only if} \quad m_q \gg a^2$$
Alternative definition for the twist angle

Define:

\[
\overline{M}_{\text{cr}}(r) = \frac{M_{\text{cr}}(r) - M_{\text{cr}}(-r)}{2} = -\overline{M}_{\text{cr}}(-r)
\]

\[
\Delta M_{\text{cr}}(r) = \frac{M_{\text{cr}}(r) + M_{\text{cr}}(-r)}{2} = \Delta M_{\text{cr}}(-r)
\]

\[\Rightarrow\quad \text{New definition for the twist angle:}\]

\[
\overline{\psi}_{\text{ph}}(x) \left[ - \left( -a \frac{r}{2} \sum_{\mu} \nabla^*_\mu \nabla_\mu + \overline{M}_{\text{cr}}(r) \right) \exp(-iw\gamma_5\tau_3) + m_q + \Delta M_{\text{cr}}(r) \right] \psi_{\text{ph}}(x)
\]

You can show:

Twist average = Wilson average irrespective of \( m_q \)

Automatic O(a) improvement holds for all \( m_q \)
Sketch of the different definitions

\( \omega \): Frezzotti / Rossi

\( \omega' \): Alternative definition
Part 2: ChPT analysis of the pion mass

- Brief introduction into ChPT at non-zero $a$ (Wilson ChPT)
- Example: Pion mass at maximal twist: Are the linear $a$-effects absent?
ChPT at nonzero $\alpha$: Strategy

Two-step matching to effective theories:

1. Lattice theory $\rightarrow$ Symanzik’s effective theory
   continuum theory making the $\alpha$-dependence explicit

2. Symanzik’s effective theory $\rightarrow$ ChPT
   including the $\alpha$-dependence

$\Rightarrow$ Chiral expressions for $m_\pi$, $f_\pi$ ... with explicit $\alpha$-dependence
Symanzik's action for Lattice QCD with Wilson fermions

Locality and symmetries of the lattice theory

\[ S_{\text{eff}} = S_{QCD} + a c \int \bar{\psi} i \sigma_{\mu \nu} G_{\mu \nu} \psi + O(a^2) \]

- At \( O(a) \) only one additional operator (making use of EOM)
- \( c \): unknown coefficient ("low-energy constant")
- \( O(a^2) \): dim-6 operators: - fermion bilinears
  - 4-fermion operators
- \( \frac{1}{a} \) divergence in quark mass must be subtracted

Sheikholeslami, Wohlert
Reminder: Chiral Lagrangian

Fields:
\[
\Sigma(x) = \exp \left( \frac{2i}{F} \pi^a(x) T^a \right)
\]
\(T^a: \text{Group generators}\)

Lagrangian:
\[
\mathcal{L}_{eff} [\Sigma, M] = \mathcal{L}_{eff} [\Sigma', M']
\]
\(M: \text{Quark mass matrix}\)

\[
\Sigma' = L \Sigma R^\dagger \quad M' = L M R^\dagger
\]
\(L, R: \text{Left, Right transformations}\)

Expand in powers of derivatives and masses:
\[
\mathcal{L}_{eff} = \mathcal{L}_2 + \mathcal{L}_4 + \ldots
\]

\[
\mathcal{L}_2 = \frac{f^2}{4} \text{tr} \left[ \partial_\mu \Sigma \partial_\mu \Sigma^\dagger \right] - \frac{f^2 B}{2} \text{tr} \left[ \Sigma^\dagger M + M^\dagger \Sigma \right]
\]

\(f, B: \text{undetermined low-energy constants}\)
Chiral Lagrangian including $a$

$$S_{\text{eff}} = S_{QCD} + a c \int \overline{\psi} i \sigma_{\mu\nu} G_{\mu\nu} \psi + O(a^2)$$

Pauli term breaks the chiral symmetry exactly like the mass term in $S_{QCD}$

$\Rightarrow$ $a$ enters chiral Lagrangian exactly like the mass term

$$\Rightarrow \quad L_2 = \frac{f^2}{4} \text{tr} \left[ \partial_\mu \Sigma \partial_\mu \Sigma^\dagger \right] - \frac{f^2 B}{2} \text{tr} \left[ \Sigma^\dagger M + M^\dagger \Sigma \right] - \frac{f^2 W_0}{2} a \text{tr} \left[ \Sigma + \Sigma^\dagger \right]$$

$W_0$ : new undetermined low-energy constant

includes $c = c(g_0^2)$ not really a constant (weak $a$ dependence)
\( \mathcal{L}_4 \)-Lagrangian:

\[
\mathcal{L}_4 = \mathcal{L}_4(p^4, p^2m, m^2) + \mathcal{L}_4(p^2a, ma) + \mathcal{L}_4(a^2)
\]

- No O(4) symmetry breaking terms in \( \mathcal{L}_4(a^2) \) (start at \( O(a^2p^4) \))
- Total number of low-energy constants: \( 10L_i + (5 + 3)W_i = 18 \)
Power counting

The power counting is non-trivial because of

1. the additive mass renormalization \( \propto \frac{1}{a} \)

2. two symmetry breaking parameters, \( m_{\text{quark}} \)

\( \Rightarrow \) their relative size matters
Leading order pion mass (degenerate case)

\[ M_\pi^2 = 2Bm + 2aW_0 \quad m = m_u = m_d \]

- Leading \( a \)-effect: Shift in the pion mass
- \( M_\pi^2 \) does not vanish for \( m = 0 \)

Common practice on the lattice:

\[ M_\pi^2 = 0 \quad \text{for} \quad m' = Z_m(m_0 - m_{cr}) = 0 \]

In ChPT this corresponds to

\[ m' = m \left( 1 + \frac{W_0}{B}a \right) \]

\[ \Rightarrow \quad M_\pi^2 = 2Bm' \]

\[ M_\pi^2 < 0 \quad \text{for} \quad m' < 0 \quad \text{Tachyon!} \]

\( a^2 \) effect must be included for small \( m' \)
Different power countings have been discussed:

If $m' \gg a^2 \rightarrow$ continuum like ChPT + small $O(a^n)$ corrections

Rupak, Shoresh, Baer '03

If $m' \approx a^2 \rightarrow$ qualitatively different:

Non-trivial phase diagram

Modification of chiral logs

Sharpe, Singleton '98

Aoki '03
Spontaneous flavor and parity breaking

Potential energy:
\((N_f = 2)\)

\[ V = -c_1 m' \text{tr} \left[ \Sigma + \Sigma^\dagger \right] + c_2 a^2 \left( \text{tr} \left[ \Sigma + \Sigma^\dagger \right] \right)^2 \]

\[ c_1(f, B) \]
\[ c_2(f, B, W_i) \]

A: \(\text{sign } c_2 = +1 \quad \Rightarrow \quad \Sigma_{\text{vacuum}} \neq \pm 1\) \quad \text{flavor and parity are broken}
\quad \text{massless pions at } a \neq 0

B: \(\text{sign } c_2 = -1 \quad \Rightarrow \quad \Sigma_{\text{vacuum}} = \pm 1\) \quad \text{no flavor/parity breaking}
\quad \text{no massless pions}

The realized scenario depends on the details of the underlying lattice theory
\((\text{i.e. the particular Lattice action})\)
ChPT for tmQCD

Symanzik action: \( S_{\text{eff}} = S_{\text{tmQCD}} + a \) Pauli term + \( \mathcal{O}(a^2) \)

\[
\Rightarrow \quad \mathcal{L}_{\text{chiral}} [m, \omega, a, a^2]
\]

\[
\Rightarrow \quad m_{\pi}^2, f_\pi \quad \text{as a function of} \quad m, \omega, a, a^2
\]

Again: Proper parameter matching required!
Here \( m \) and \( \omega \)
Check for $O(a)$ improvement of the pion mass

1. Lagrangian $\mathcal{L}_{\text{chiral}} \rightarrow$ potential Energy $\mathcal{V}_{\text{chiral}}$

2. Find ground state $\Sigma_0 = e^{i\phi \tau_3}$ by $\frac{d\mathcal{V}_{\text{chiral}}}{d\phi} = 0$

3. Expand around $\Sigma_0$ and find $M_\pi^2$ (to LO)

4. Express $M_\pi^2$ in terms of the twist angle $\omega$ corresponding to the lattice theory

5. Go to $\omega = \frac{\pi}{2}$ and check for $O(a)$

$$m_{\pi_a}^2 = \frac{2Bm + 2W_0a}{\cos \phi} - 2c_2a^2$$

$$m = m' \cos w$$

Aoki, Baer
hep-lat/0409006
Definition of Frézotti / Rossi

Definition of $\omega$ : Lattice theory

Effective theory

For $\omega = \pi/2$ ($\mu := m_L \sin \omega_L$)

1. $2B\mu \geq O(a)$  $\Rightarrow$

   $$m_{\pi_3}^2 - m_{\pi_a}^2 = 2c_2a^2$$

2. $2B\mu \ll 2c_2a^2$  $\Rightarrow$

   $$m_{\pi_a}^2 = (c_2a^2)^{1/3}(2B\mu)^{2/3}$$
   $$m_{\pi_3}^2 - m_{\pi_a}^2 = 2(c_2a^2)^{1/3}(2B\mu)^{2/3}$$

$O(a)$ improvement only in case 1
Alternative definition for the twist

Definition of $\omega$ : Lattice theory

$$\left( m_0 - \frac{M_{cr}(r) - M_{cr}(-r)}{2} \right) e^{i\omega \tau_3}$$

Effective theory

$$\tan \omega = \frac{2Bm_L \sin \omega_L}{2Bm_L \cos \omega_L + 2W_0a}$$

For $\omega = \pi/2$ \quad ($\mu := mL \sin \omega_L$)

Without restrictions on $2B\mu \quad \Rightarrow \quad m_{\pi a}^2 = 2B\mu$

$$m_{\pi_3}^2 - m_{\pi a}^2 = 2c_2a^2$$

Automatic $O(a)$ improvement irrespective of the size of $\mu$!

Note: \quad $m_{\pi_3} = \sqrt{2c_2} a = O(a)$ at $\mu = 0$
\( \omega \): Frezzotti / Rossi

\( \omega' \): Alternative definition

2\textsuperscript{nd} order phase transition point
Mean field critical exponent = 2/3
Twist angle from Ward identities

In continuum tmQCD:

\[ \tan \omega_{WT} = \frac{\langle \partial_\mu V_\mu^2 P^1 \rangle}{\langle \partial_\mu A^1_\mu P^1 \rangle} \]

Vector and Axial vector WT identities:

\[ \partial_\mu V_a^a = -2\mu \epsilon^{3ab} P^b \]

\[ \partial_\mu A^a_\mu = 2m P^a + 2i\mu S^0 \delta_{a3} \]

\[ \Rightarrow \tan \omega_{WT} = \frac{\mu}{m} \]
$\omega_{WT}$ in the effective theory

1. Maximal twist of Frezzotti / Rossi:

   For $2B\mu \ll 2c^2a^2$ \implies \tan \omega_{WT} \simeq \left(\frac{2B\mu}{2c^2a^2}\right)^{1/3}$

   \implies \omega_{WT} \neq \pi/2 \quad (\omega_{WT} = 0 \text{ for } \mu = 0)$

2. Alternative definition:

   \tan \omega_{WT} = \infty

   $\omega_{WT} = \pi/2 = \omega$
Part 3: Alternative Scenario ($c_2 < 0$)

$\mu = 0$

- **Vacuum**
  \[
  \cos \phi = \begin{cases} 
  1 & c_1 > 0 \\
  -1 & c_1 < 0 
  \end{cases} \quad c_1 := 2Bm + 2W_0a
  \]

- **Pion mass**
  \[
  m_{\pi_3}^2 = m_{\pi_a}^2 = |c_1| - 2c_2a^2
  \]

- **AWI quark mass**
  \[
  m_{\text{AWI}} \propto \begin{cases} 
  m_{\pi}^2 & c_1 > 0 \\
  -m_{\pi}^2 & c_1 < 0 
  \end{cases}
  \]
  
  No massless point (Minimum: $m_{\pi}^2 = -2c_2a^2$ )  \rightarrow \text{No $M_{cr}$}

No Flavor-Parity breaking phase

No zero modes for Wilson-Dirac operator
\[ \mu \neq 0 \]

Define

\[ M_{cr} \iff c_1 = 2Bm + 2W_0a = 0 \]

Maximal twist

\[ c_1 = 0, \mu \neq 0 \]

- **Vacuum**

\[ \cos \phi = \begin{cases} 
\pm \sqrt{1 - \frac{(2B\mu)^2}{(2c_2a)^2}} & |2B\mu| < -2c_2a^2 \quad c_1 \to 0^\pm \\
0 & |2B\mu| \geq -2c_2a^2 \quad \forall c_1
\end{cases} \]

- **Twist angle from WTI**

\[ \tan w_{WT} = \begin{cases} 
\pm \frac{2B\mu}{-2C_2a^2} & |2B\mu| < -2c_2a^2 \quad c_1 \to 0^\pm \\
\pm \infty & |2B\mu| \geq -2c_2a^2 \quad \pm|\mu|
\end{cases} \]

Twist angle becomes maximal (\( w_{WT} = \pm \pi/2 \)) only for \( |2B\mu| \geq -2c_2a^2 \)
• Pion mass

\[
m^2_{\pi a} = \begin{cases} 
-2c^2a^2 & |2B\mu| < -2c^2a^2 \\
|2B\mu| & |2B\mu| \geq -2c^2a^2 
\end{cases}
\]

\[
m^2_{\pi 3} = \begin{cases} 
-2c^2a^2 \left(1 - \frac{(2B\mu)^2}{(2c^2a^2)^2}\right) & |2B\mu| < -2c^2a^2 \\
0 & |2B\mu| = -2c^2a^2 \\
|2B\mu| + 2c^2a^2 & |2B\mu| > -2c^2a^2 
\end{cases}
\]

\[m^2_{\pi 3} < m^2_{\pi a}\]
Summary

- Twisted mass QCD can be automatically $O(a)$ improved at max. twist.
- Tricky: Definition of the proper twist angle.
- Definition of Frezzotti / Rossi works only for $m \gg a^2$.
- Alternative definition can be given that ensures $O(a)$ improvement without restrictions on the quark mass.
- $O(a)$ improvement can be explicitly demonstrated for the pion mass in ChPT at non-zero lattice spacing.
- Physics at $c_2 < 0$ is different from the one at $c_2 > 0$.
  - No massless pion.
  - No critical quark mass.
  - No zero mode.