Study of bound state in compact scalar QED

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Preliminary result in hep-lat/0510032

4th ILFTN Workshop "Lattice QCD via International Research Network"

@ The Graduate University for Advanced Studies(Sokendai)

March 8-11, 2006

1. Introduction

Motivation :

Understand nuclear force from lattice QCD

We need to study not only scattering state, but bound state on lattice.

nn scattering, Deuteron(np bound state)

However, it is difficult to study bound state like Deuteron,
because binding energy is very small, (-2.22 MeV).

Difficulty on finite volume

In infinite volume;

Binding energy (M: bound state energy, $2m$: threshold)

$$\Delta E = M - 2m < 0 \neq 0$$

Above threshold continuous scattering states exist.
Lowest scattering state energy = threshold ($2m$).

We can identify bound state from negative, non-zero energy shift.

Difficulty on finite volume(cont'd.)

On finite volume;

Binding energy (M: bound state energy, $2m$: threshold)

$$\Delta E = M - 2m < 0 \neq 0$$

Continuous scattering state is discretized due to finite volume.
Lowest scattering state energy $\neq$ threshold ($2m$) due to finite volume effect.

$$\Delta E_0 = E_0 - 2m = \mathcal{O}(1/L^3)$$

S-wave scattering system;
Sign of ΔE_0 is determined by scattering length a_0.

$$\Delta E_0 < 0 \text{ for attractive interaction } (I = 0\ \pi\pi)$$

$$\Delta E_0 > 0 \text{ for repulsive interaction } (I = 2\ \pi\pi)$$

In very small ΔE case, it is hard to identify bound state from negative, non-zero energy shift, because $\Delta E_0 < 0$ in attractive interaction case.

Various methods

Method 1. Finite volume effect of ground state energy

Finite volume effect of bound state differs from one of scattering state.

Lowest scattering state

$$2m + \mathcal{O}(1/L^3)$$

Lüscher Commun. Math. Phys. 105(1989) 153; Nucl. Phys. B354(1991) 531

Bound state ($M = 2m + \Delta E, \Delta E < 0$)

$$2m + \Delta E + \mathcal{O}(\mathrm{e}^{-CL}/L) \quad (C > 0)$$

Beane it et al., Phys. Lett. B585(2004) 106

We can identify bound state from finite volume dependence of energy.

Various methods (cont'd.)

Method 2. Anti-periodic boundary condition (spatial direction)

N. Ishii *et al.* Phys. Rev. D71(2005) 034001

Scattering state

Lowest scattering state energy strongly depends on spatial boundary condition

$$\text{Periodic boundary} \quad E_0 = 2m \qquad p_i = \pi/L \cdot 2n_i$$

$$\text{Anti-periodic boundary} \quad E_0 = 2\sqrt{m^2 + (\pi/L)^2} \qquad p_i = \pi/L \cdot (2n_i + 1)$$

Bound state

Energy is not strongly affected by boundary condition.

We can determine bound state from response of spatial boundary condition.

Other method: Spectral weight   Mathur *et al.* Phys. Rev. D70(2004) 074508

Volume dependence of point-point correlation function

Property of bound state

We focus on two properties of bound state to identify state on finite volume.

A. Scattering phase shift δ_0 of bound state

$$\tan\delta_0(k) + i = 0, \quad k^2 = (M^2 - 4m^2)/4 < 0$$

We investigate that δ_0 satisfies condition or not.

B. Difference of scattering length a_0

$np\ ^1S_0$ channel $a_0 = +23.7$ fm (attractive scattering system)
$np\ ^3S_1$ channel $a_0 = -5.47$ fm (system including bound state)

Lowest scattering state corresponds to first excited state of bound system. (Ground state is bound state.)
Finite volume effect of scattering state is determined by sign of $a_0(\tan\delta_0)$, so that difference of sign would be useful to distinguish bound and scattering system.

Property of bound state

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$np\ {}^3S_1$ channel $a_0 = -5.47$ fm (bound system)

Lowest scattering state corresponds to first excited state of bound system. (Ground state is bound state.)
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Purpose

We apply two methods

Method A. Condition of scattering phase shift $\delta_0(k)$

Method B. Finite volume effect of first excited state

for simple known bound(large binding enegy) and scattering systems to investigate whether we can distinguish bound and scattering states or not.

Simple bound and scattering system

Scalar QED (Abelian-Higgs) with quenched approximation

It is easy to control system by changing parameters.

Outline

1. Introduction
2. Methods
3. Simulation parameters
4. Results
   - Method 1 and 2
   - Method A
   - Method B
5. Summary

Methods

Method A. Condition of scattering phase shift $\delta_0(k)$

$\delta_0(k)$ satisfies condition at bound state energy M

$$\tan\delta_0(k) + i = 0$$

$$k^2 = (M^2 - 4m^2)/4 = -\kappa^2 < 0$$

Using analytic continuation of $\delta_0(k)$

$$\tan\delta_0(i\kappa) = i\tan\sigma_0(\kappa),$$

equation is replaced by

$$i\tan\sigma_0(\kappa) + i = 0$$

Then we obtain condition of bound state

$$\sigma_0(\kappa) = -\pi/4$$

Method A. (cont'd.)

We can determine $\sigma_0(\kappa)$ by using Lüscher's finite volume method.

Lüscher Commun. Nucl. Phys. B354(1991) 531

$\delta_0(k)$ and $\sigma_0(\kappa)$ are evaluated from
two-particle and bound state energies on finite volume $E^2 = 4(m^2 + k^2)$

Usual scattering system $k^2 > 0$;

$$\tan \delta_0(k) = \frac{q\pi^{3/2}}{Z_{00}(1; q^2)}$$

Lowest attractive scattering state and bound state $k^2 = -\kappa^2 < 0$;

$$\tan \sigma_0(\kappa) = \frac{\sqrt{|q^2|}\pi^{3/2}}{Z_{00}(1; q^2)}$$

Zeta function

$$Z_{00}(n; q^2) = \frac{1}{\sqrt{4\pi}} \sum_{\vec{l} \in Z^3} \frac{1}{(l^2 - q^2)^n}, \quad q^2 = (kL/2\pi)^2$$

Method A. (cont'd.)

Bound state: $-\kappa^2 \neq 0$ in $L \to \infty \iff q^2 = -(\kappa L/2\pi)^2 \to -\infty$.

Behavior of zeta function at $q^2 \to -\infty$ was investigated.

Beane it et al., Phys. Lett. B585(2004) 106

$$Z_{00}(1;q^2) = -\pi^{3/2}\sqrt{|q^2|} + \sum_{\vec{n}\in Z^3} \frac{\pi^{1/2}}{2\sqrt{n^2}} \, \mathrm{e}^{-2\pi\sqrt{|q^2|n^2}}$$

at $q^2 < 0$ Elizalde, Commun. Math. Phys. 198(1998) 83

Then at large L

$$\begin{aligned} \tan\sigma_0(\kappa) &= \frac{\sqrt{|q^2|}\pi^{3/2}}{-\pi^{3/2}\sqrt{|q^2|} + \mathcal{O}\left(\mathrm{e}^{-\sqrt{|q^2|}}\right)} \\ &= -1 - \mathcal{O}\left(\mathrm{e}^{-\sqrt{|q^2|}}/\sqrt{|q^2|}\right) \end{aligned}$$

We can obtain $\sigma_0(\kappa) = -\pi/4$ only at $L \to \infty$.

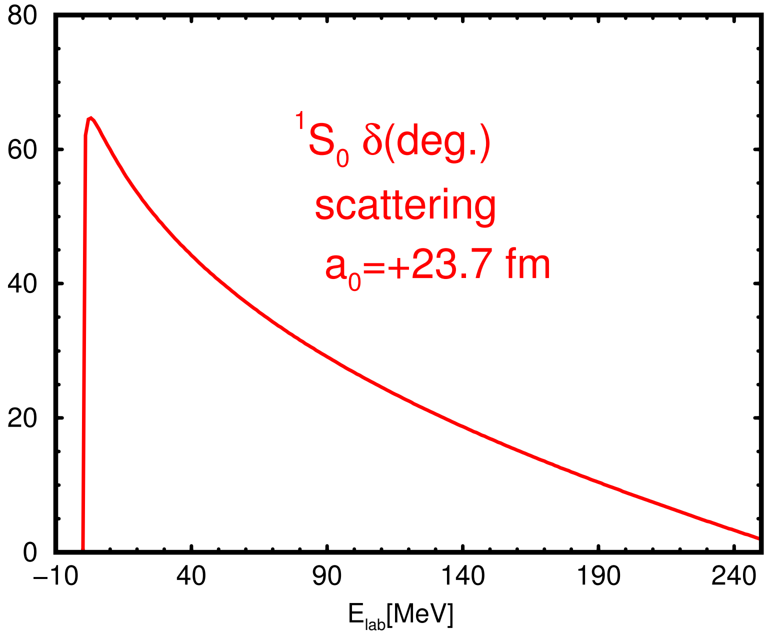
On finite (large) L

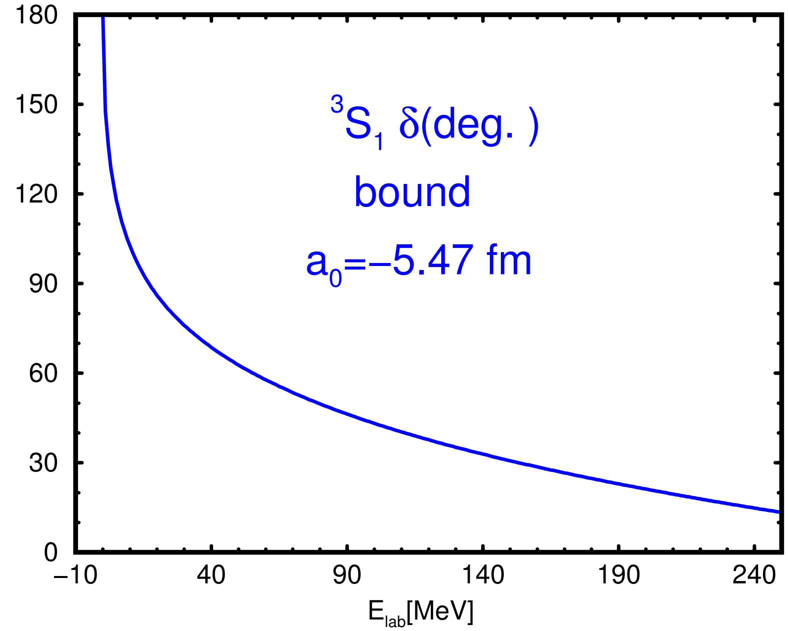
$$\sigma_0(\kappa) = -\pi/4 - \varepsilon \quad (\varepsilon > 0, \varepsilon \to 0 \text{ as } L \to \infty)$$

We investigate σ_0 satisfies the condition.

Method B. Finite volume effect of first excited state

Scattering length $a_0 = \lim_{p \to 0} \tan \delta_0(k)/k$





Plots from NN-OnLine (http://nn-online.org/)

Sign of a_0 in bound system ($np\ ^3S_1$) is different from one in scattering system ($np\ ^1S_0$).

In scattering system, $\pi/2 > \delta_0(k) \geq 0 \iff \tan \delta_0(k) \geq 0$.

Sign of finite volume effect of scattering state is determined by sign of $\tan \delta_0(p)$.

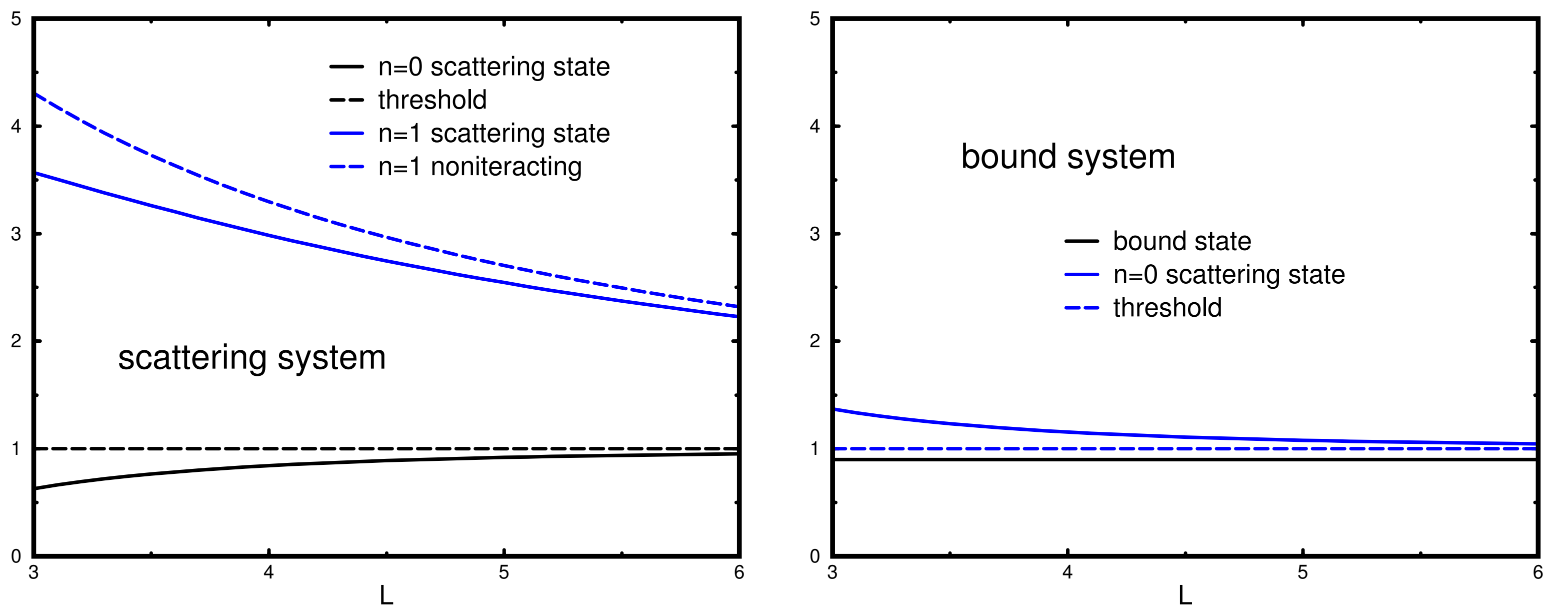
scattering system $\Delta E_n < 0$, but bound system $\Delta E_0 > 0$

Method B. (cont'd.)

scattering system $\Delta E_n < 0$, but bound system $\Delta E_0 > 0$

n state : scattering state with $E_n = 2\sqrt{m^2 + n \cdot (2\pi/L)^2} + \Delta E_n$

ΔE	scattering (state)	bound (state)
ground state	$\Delta E_0 < 0$ $(n = 0)$	$\Delta E < 0$ (bound)
first excited state	$\Delta E_1 < 0$ $(n = 1)$	$\Delta E_0 > 0$ $(n = 0)$
1st state approach	from below	from above



We can identify bound system from energy shift of scattering state, *i.e.*, first excited state in system.

Method A.

$\sigma_0(\kappa)$ from bound state energy E

$$\sigma_0(\kappa) = -\pi/4 - \varepsilon \quad (\varepsilon > 0, \varepsilon \to 0 \text{ as } L \to \infty)$$

Method B.

First excited energy goes to threshold from above as L increases.

Energy shift is positive, corresponding to $a_0 < 0$

Extraction of first excited state

In Method B. extraction of first excited state is important.
However, we cannot obtain first excited state by a naive exponential analysis.

Diagonalization method

Lüsher and Wolff, Nucl. Phys. B339(1990) 222

four point function matrix $G_{ij}(t) = \langle 0|\Omega_i^\dagger(t)\Omega_j(0)|0\rangle$

$$M(t,t_0)w_\nu = \mathrm{e}^{-E_\nu(t-t_0)}w_\nu$$

We extract ground and first excited states energy from eigenvalues of matrix $M(t,t_0)$

$$M(t,t_0) = G^{-1/2}(t_0)G(t)G^{-1/2}(t_0),\ t_0 : \text{reference point}$$

Scalar QED

Important assumption of finite volume method

Short range interaction $V(r) = \mathcal{O}\left(\mathrm{e}^{-r}\right) \approx 0$ in $r > R$, $R < L/2$

Scalar QED (Abelian-Higgs) with $|\Phi_x| = 1$

$$S = -\beta \sum \mathrm{Re}\,(U_{x,\mu\nu}) - h \sum \mathrm{Re}\left(\Phi_x^* U_{x,\mu} \Phi_{x+\mu}\right)$$

In Higgs phase

- Coulomb potential is screened

$$V(r) = \frac{g^2}{4\pi} \frac{\mathrm{e}^{-M_A r}}{r}, \quad M_A = \sqrt{h/\beta} \quad \text{(Tree level)}$$

- Easy to control bound state formation with charge q
  Charge of fermions are controlled by $U_{x,\mu}^q$

$$U_{x,\mu}^q = \Pi_{i=1}^q U_{x,\mu}$$

  in Wilson Dirac operator.

Simulation parameters

- Wilson gauge and Wilson fermion actions
- quenched approximation
- $\beta = 2.0$ and $h = 0.6$ (Higgs phase)
- charge $q = 3$(scattering) and 4(bound)
- fixed fermion mass $m \approx 0.5$
- fixed temporal size $T = 32$
- four spatial volumes

L	16	20	24	28
conf.	640	512	408	312

- Landau gauge fixing
- periodic + anti-periodic boundary for temporal direction

Two-particle operators

$\overline{q}$-q scattering and bound state(positronium) in $^1S_0$ (pion) channel

Two-particle interaction is attractive.

Diagonalization of 3×3 matrix

$$
\begin{aligned}
\text{point } \Omega_P &= \sum_{\vec{x}} \overline{q}(\vec{x})\gamma_5 q(\vec{x}) \\
\text{wall } \Omega_W &= \sum_{\vec{x},\vec{y}} \overline{q}(\vec{x})\gamma_5 q(\vec{y}) \\
\text{mom } \Omega_M &= \sum_{\vec{x},\vec{y}} \overline{q}(\vec{x})\gamma_5 q(\vec{y}) \mathrm{e}^{i\vec{p}\cdot(\vec{x}-\vec{y})} \quad \text{with } |p| = 2\pi/L
\end{aligned}
$$

We expect each operator has better overlap to appropriate states.

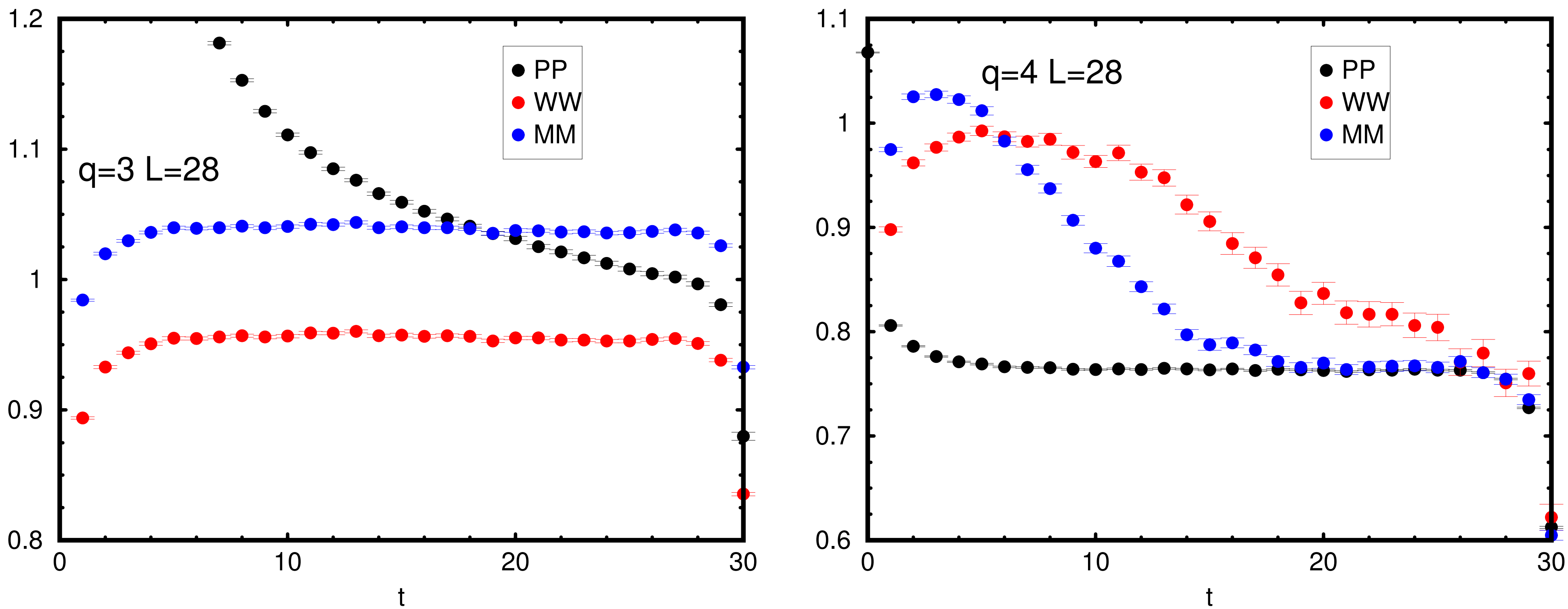
Ω_P	bound state
Ω_W	$n = 0$ scattering state
Ω_M	$n = 1$ scattering state

We analyze lowest two energies in both systems.

Results

Method 1. Finite volume effect of ground state

Effective masses for diagonal parts of G_{ij}

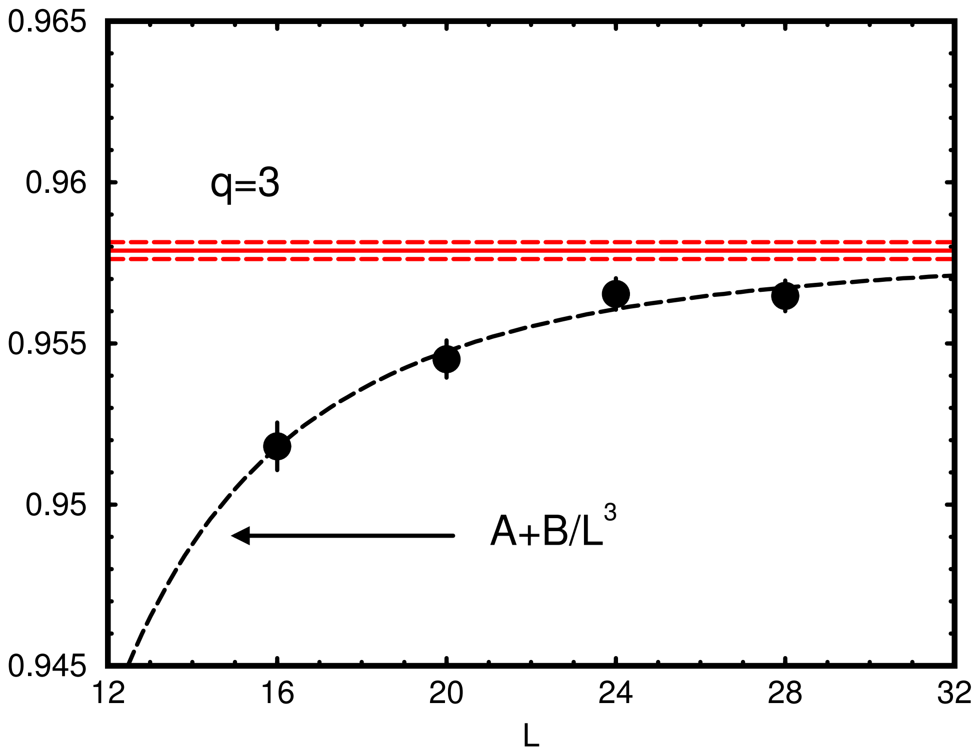


In $q = 3$ case, WW and MM correlators have better overlap to $n = 0$ and $n = 1$ scattering states, respectively.

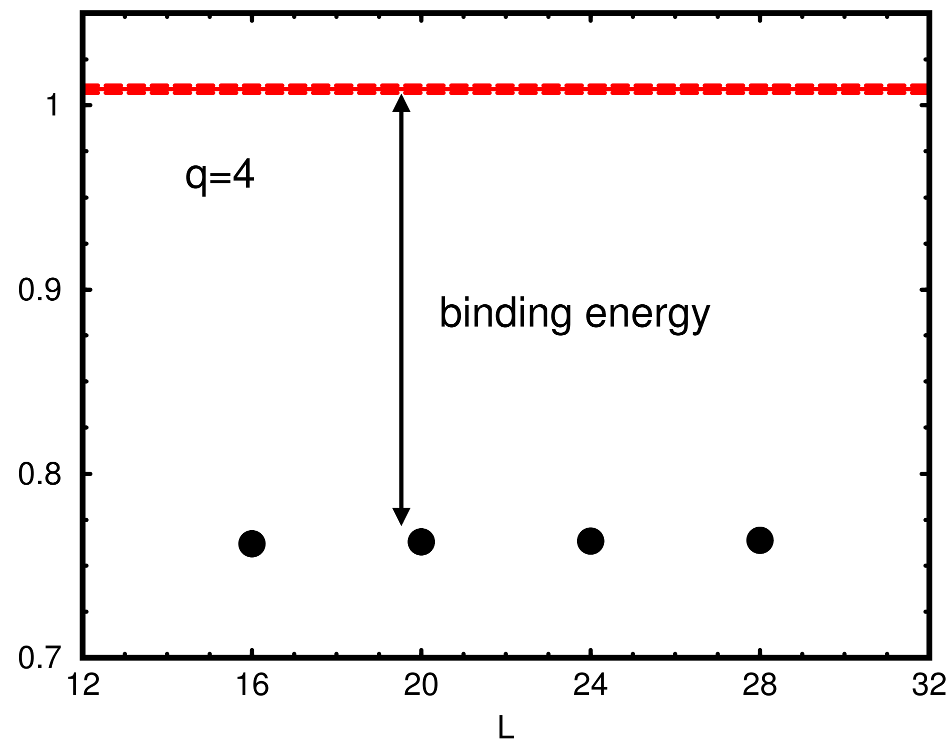
In $q = 4$ case, PP correlator has clear plateau.

Method 1. (cont'd)

Difference of volume dependence for ground state energy



using WW correlators



using PP correlators

$q = 3$ case, energy moves toward threshold as volume increases, and can be reasonably fitted by $A + B/L^3$.
$q = 4$ case, very large energy difference is seen which corresponds to large binding energy, and energy is almost constant as function of volume.

Method 2. Anti-periodic boundary

Anti-periodic boundary conditions in three spatial directions

Momentum is discretized by odd integer $p_i = \pi/L \cdot (2n_i + 1)$

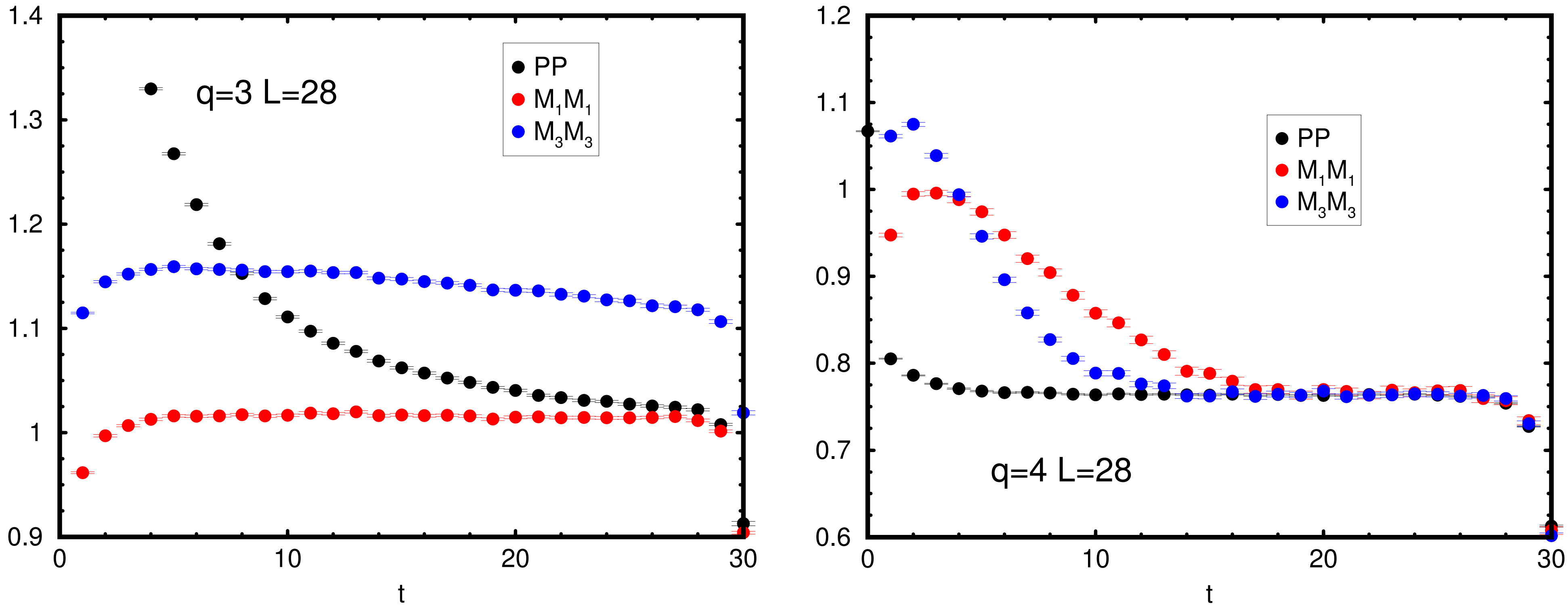
Ground two-particle state has non-zero momentum.

Similar three operators are employed as in periodic case.

$$
\begin{aligned}
\text{point } \Omega_P &= \sum_{\vec{x}} \overline{q}(\vec{x})\gamma_5 q(\vec{x}) \\
\text{mom}_1 \ \Omega_{M_1} &= \sum_{\vec{x},\vec{y}} \overline{q}(\vec{x})\gamma_5 q(\vec{y}) \mathrm{e}^{i\vec{p}_1\cdot(\vec{x}-\vec{y})} \quad \text{with } \vec{p}_1 L/\pi = (1,1,1) \\
\text{mom}_3 \ \Omega_{M_3} &= \sum_{\vec{x},\vec{y}} \overline{q}(\vec{x})\gamma_5 q(\vec{y}) \mathrm{e}^{i\vec{p}_3\cdot(\vec{x}-\vec{y})} \quad \text{with } \vec{p}_3 L/\pi = (3,1,1)
\end{aligned}
$$

Method 2. (cont'd)

Effective masses of each correlators

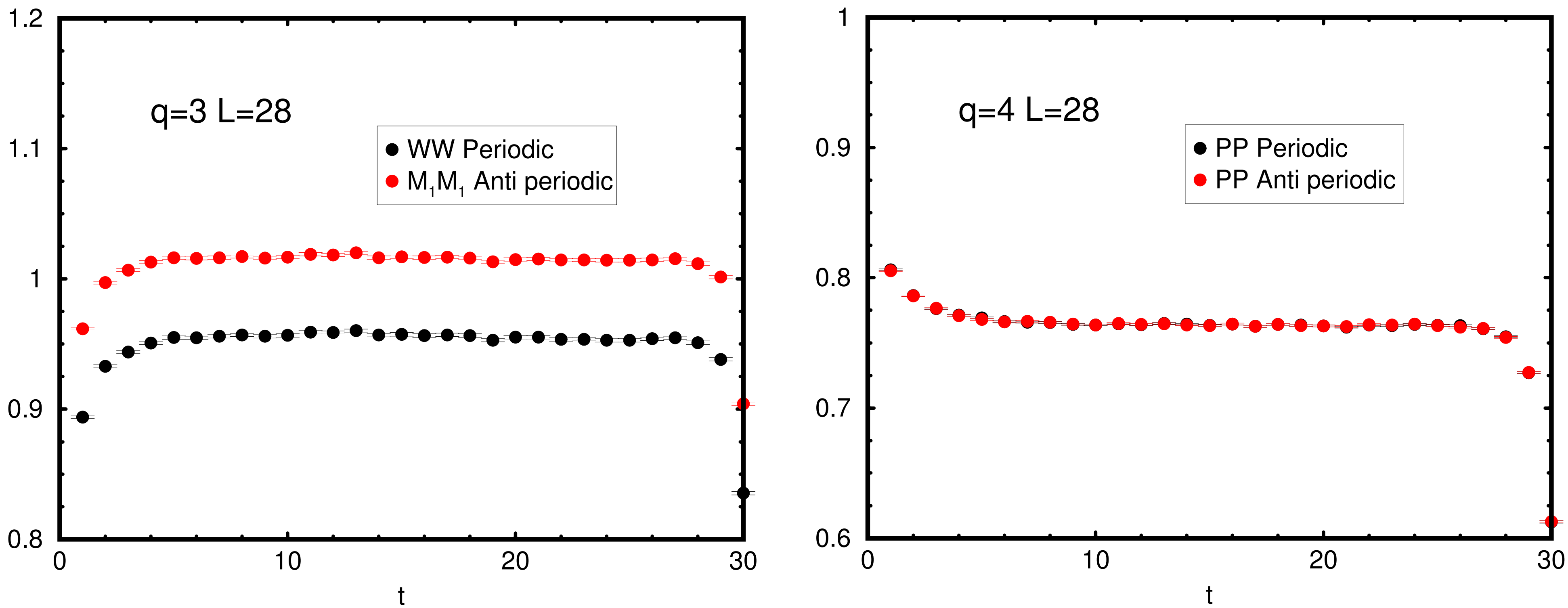


$M_1 M_1$ and PP correlators has better overlap in $q = 3$ and $q = 4$ as expected.
Clear signals of ground state are seen in both $q = 3$ and $q = 4$ cases.

Method 2. (cont'd)

Comparison of periodic and anti-periodic calculations

Effective masses of ground state with both boundary conditions

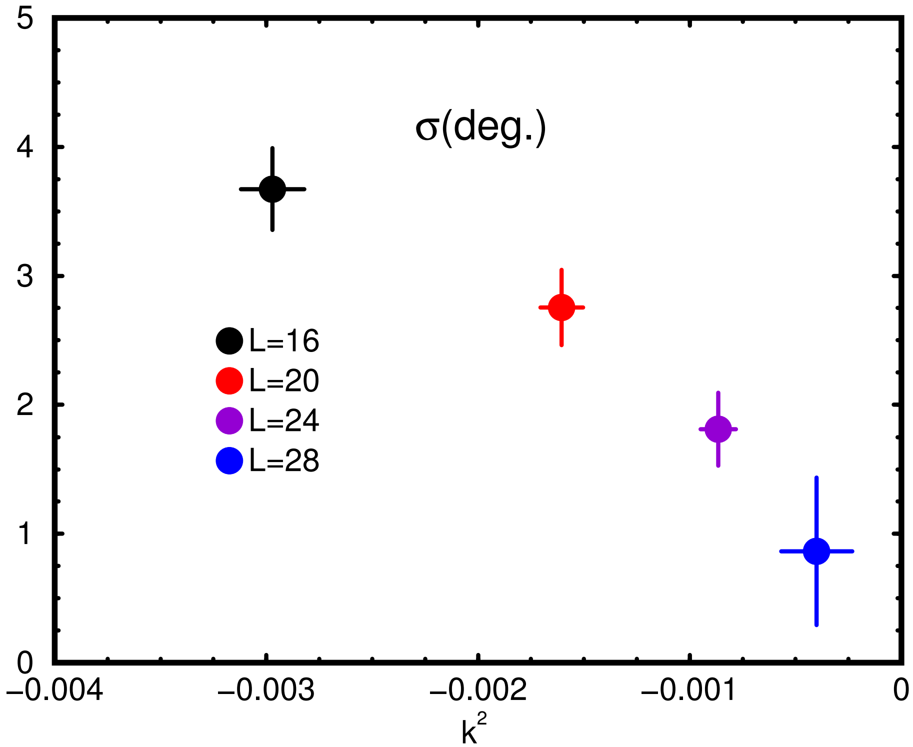


While significant difference is seen in $q = 3$, effective masses agree with each other.

From results of Method 1. and 2.

$q = 3 \rightarrow$ scattering system and $q = 4 \rightarrow$ bound system

Method A. Scattering phase shift $\sigma_0(\kappa)$



Scattering system case

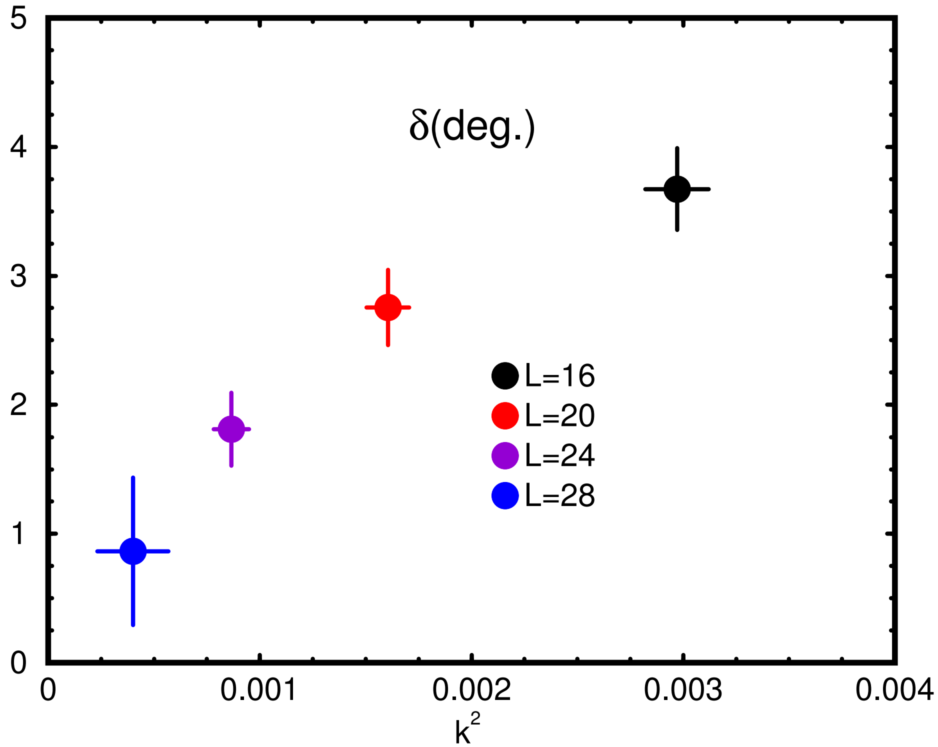
$$\lim_{\kappa \to 0} \frac{\tan \sigma_0(\kappa)}{\kappa} = \lim_{k \to 0} \frac{\tan \delta_0(k)}{k} = a_0$$

where $-\kappa^2 = k^2$

Assumption $\kappa \ll 1$

$$\sigma_0(\kappa) \approx \delta_0(\kappa)$$

Method A. (cont'd)



Scattering system case

$$\lim_{\kappa \to 0} \frac{\tan \sigma_0(\kappa)}{\kappa} = \lim_{k \to 0} \frac{\tan \delta_0(k)}{k}$$
$$= a_0$$

$$\text{where } -\kappa^2 = k^2$$

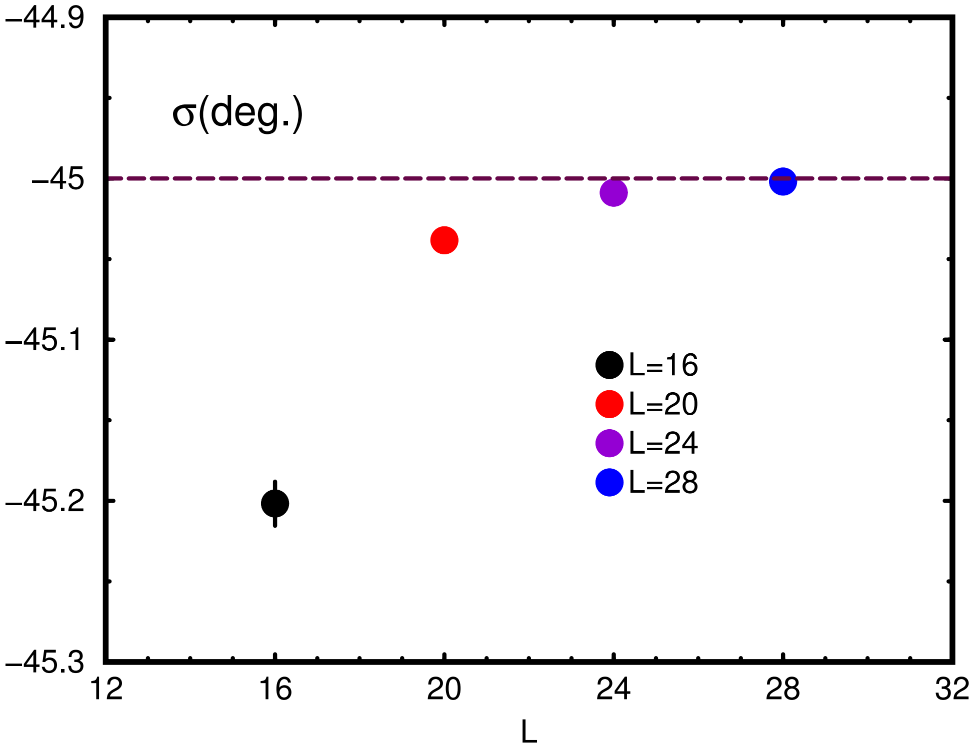
Assumption $\kappa \ll 1$

$$\sigma_0(\kappa) \approx \delta_0(\kappa)$$

$\delta_0(\kappa)$ is positive, and goes to origin as L increases.

a_0 is positive in scattering system.

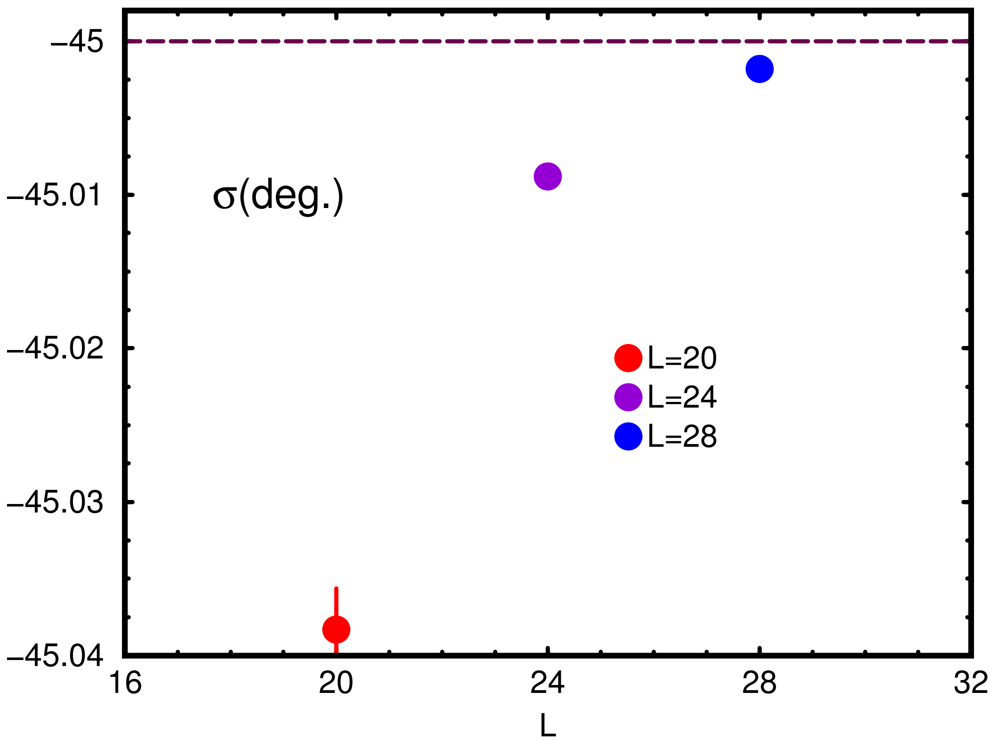
Method A. (cont'd)



Bound system case

$\sigma_0(\kappa)$ is close to $-\pi/4$, and increases with increasing L as expected.

Method A. (cont'd)



Bound system case

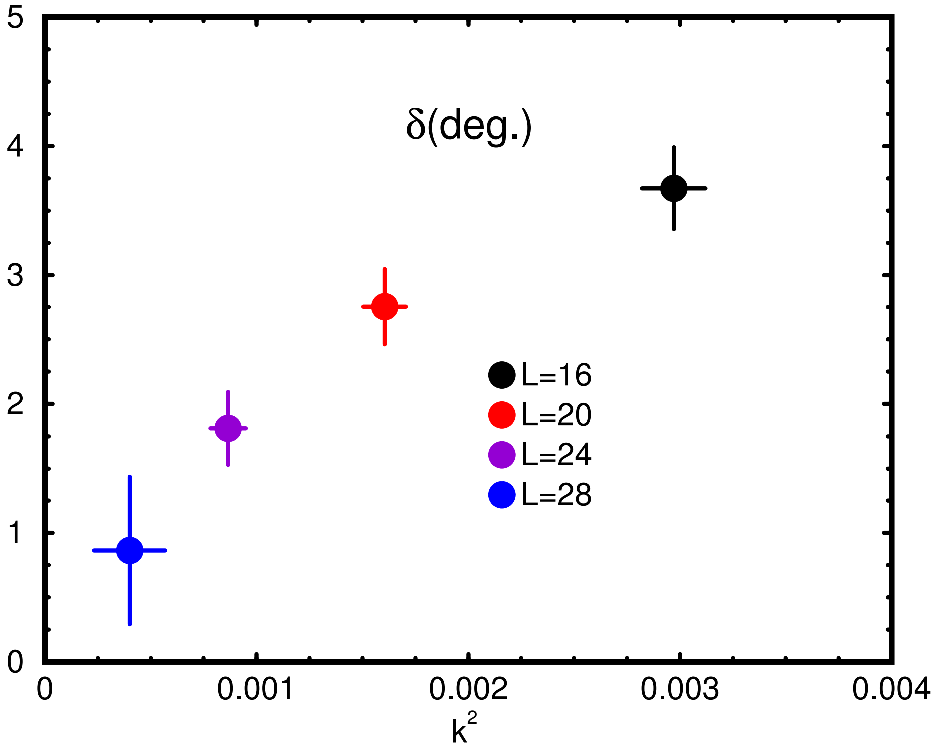
$\sigma_0(\kappa)$ is close to $-\pi/4$, and increases with increasing L as expected.

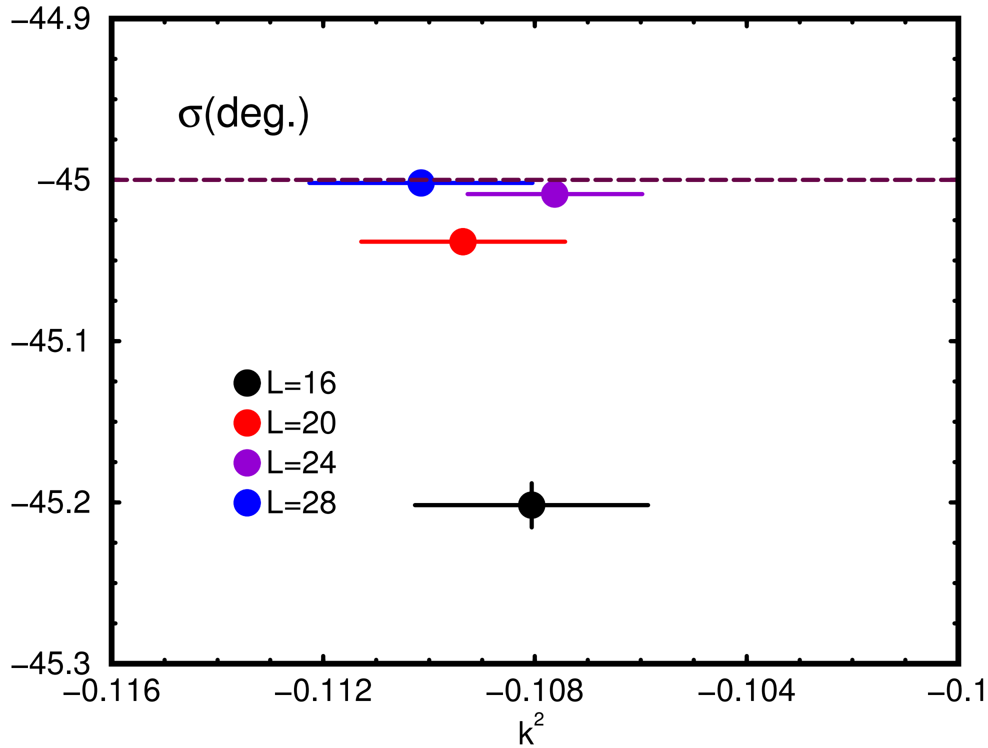
However, $\sigma_0(\kappa) \neq -\pi/4$ even at largest volume.

In bound system

$$\sigma_0(\kappa) = -\pi/4 - \varepsilon, \quad \varepsilon > 0 \text{ and } \varepsilon \to 0 \text{ as } L \to \infty$$

Method A. (summary)





From ground state energy

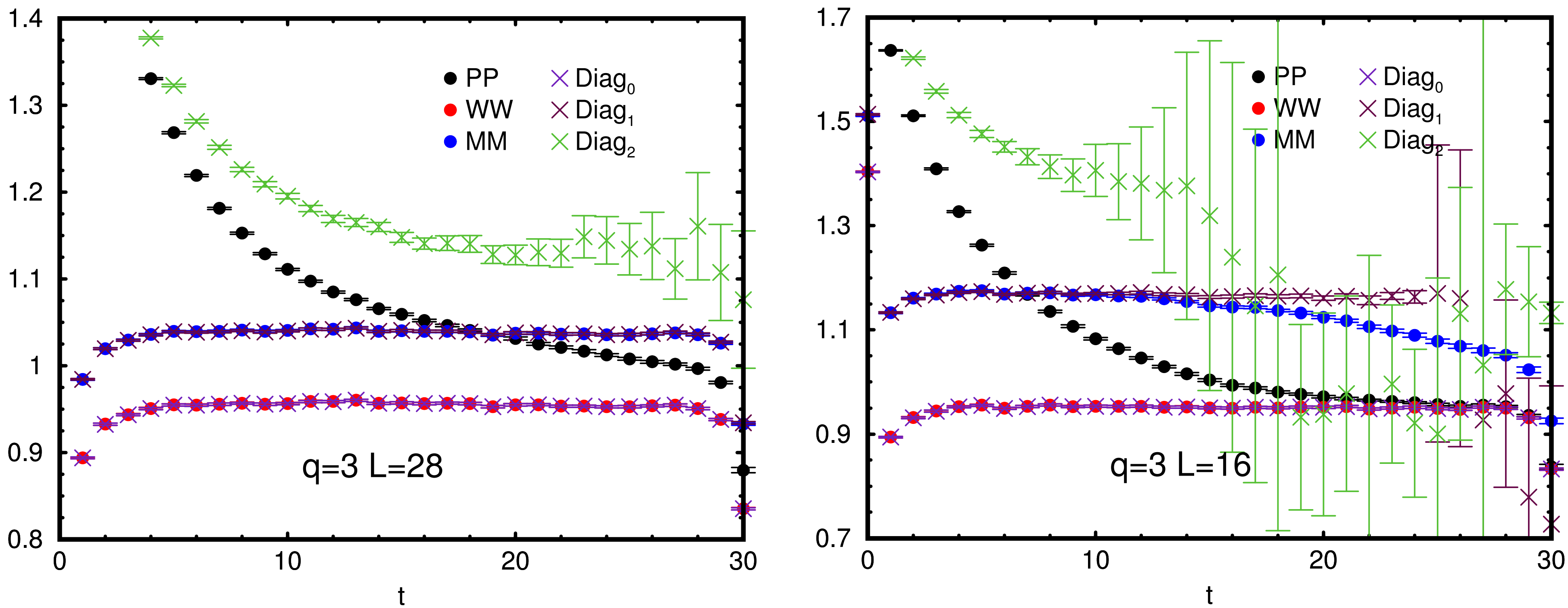
positive a_0 and $\delta_0(k)$ in scattering system

negative $\sigma_0(\kappa) = -\pi/4 - \varepsilon$ in bound system

$\varepsilon > 0$ and $\varepsilon \to 0$ as $L \to \infty$

Diagonalization in scattering system

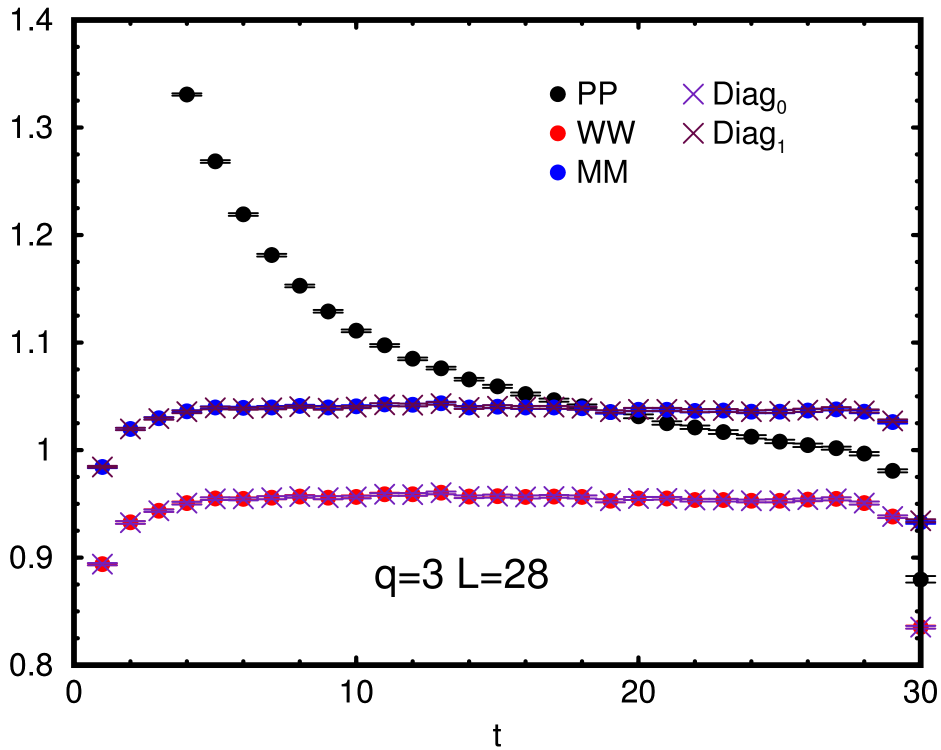
Effective masses for diagonal part of G_{ij} and eigenvalues of M_{ij}

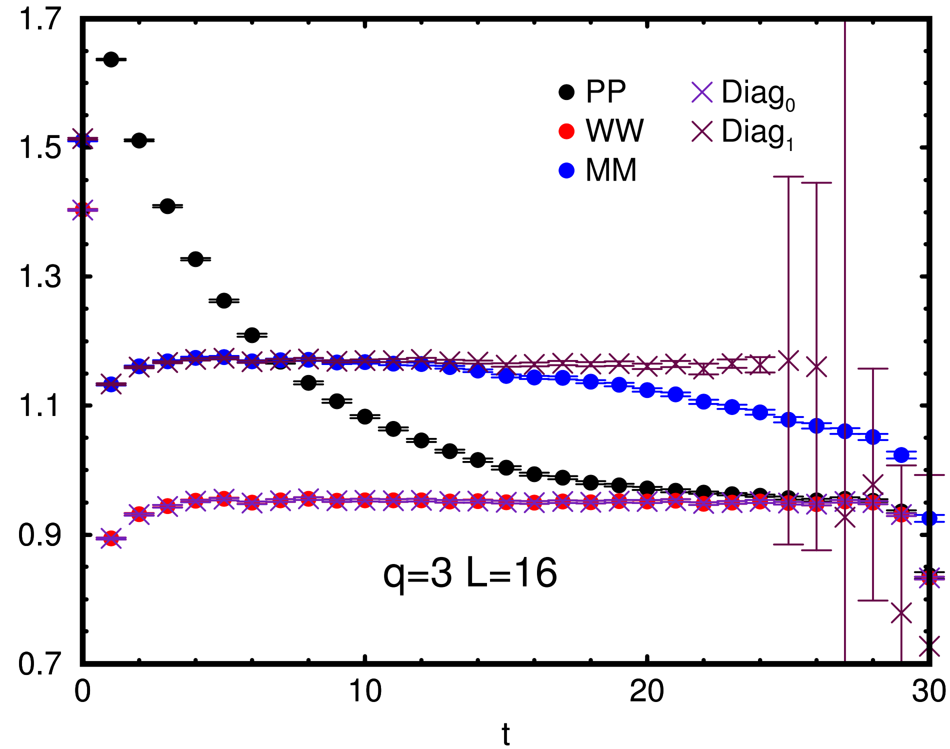


Energy of $\nu = 2$ state has huge error, and $\nu = 2$ state is not used in analysis.

Diagonalization in scattering system (cont'd)

Effective masses for diagonal part of G_{ij} and eigenvalues of M_{ij}





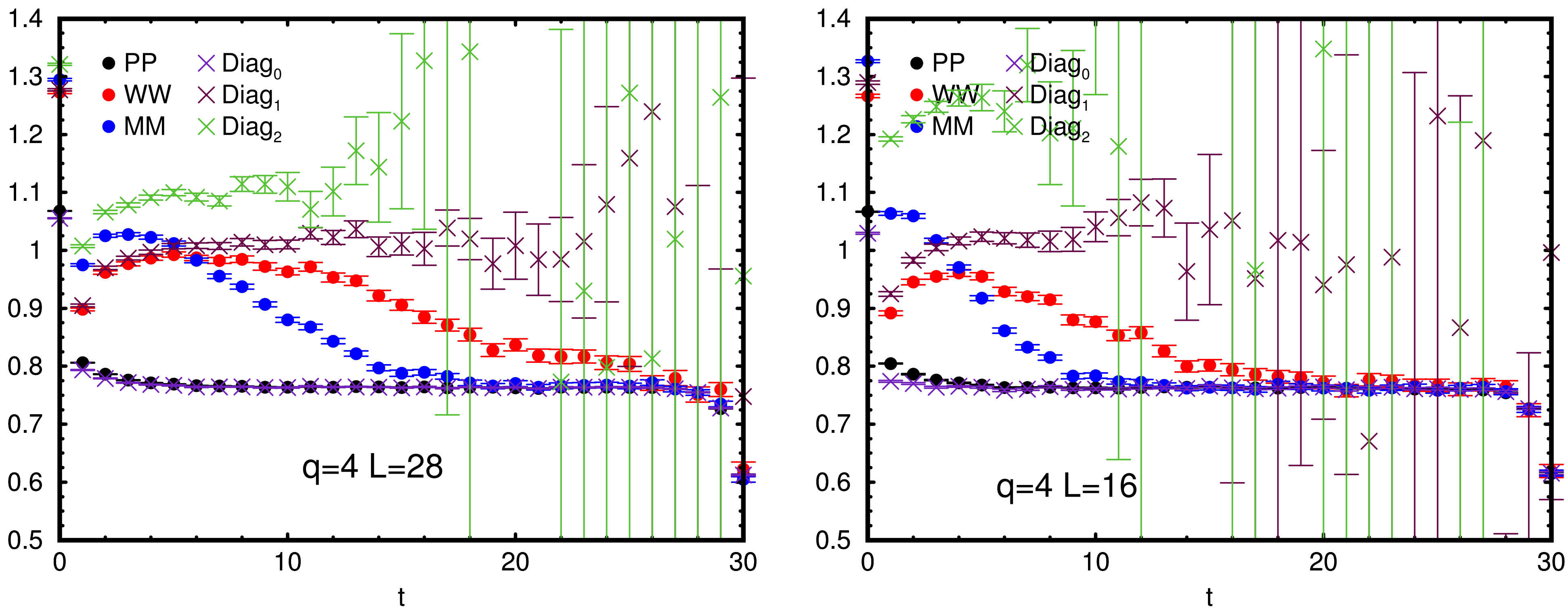
Diagonalization is not effective for ground state in both L, while it is effective for first excited state in smaller L.

In larger L contamination of other state in MM correlator may decrease, so that diagonalization is less effective.

$I = 2$ $\pi\pi$ scattering calculation, CP-PACS, Phys. Rev. D67(2003) 014502

Diagonalization in bound system

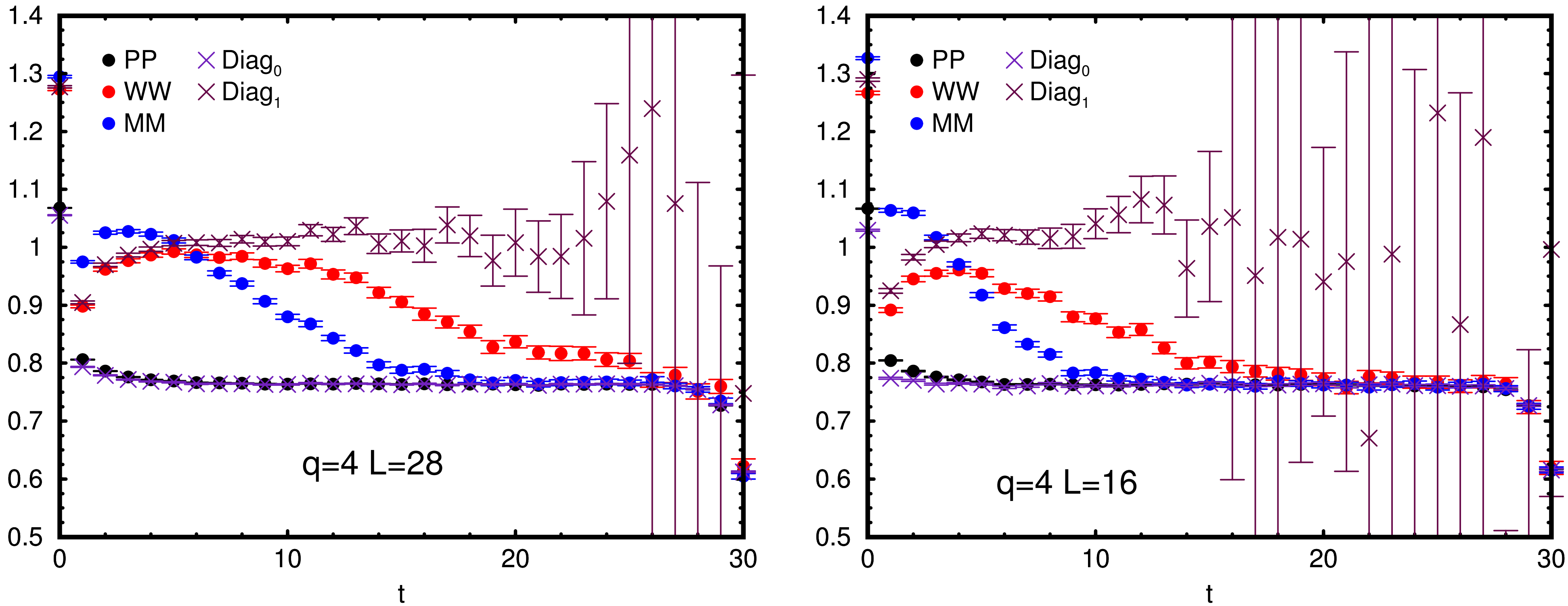
Effective masses for diagonal part of G_{ij} and eigenvalues of M_{ij}



Energy of $\nu = 2$ state has huge error, and $\nu = 2$ state is not used in analysis.

Diagonalization in bound system (cont'd)

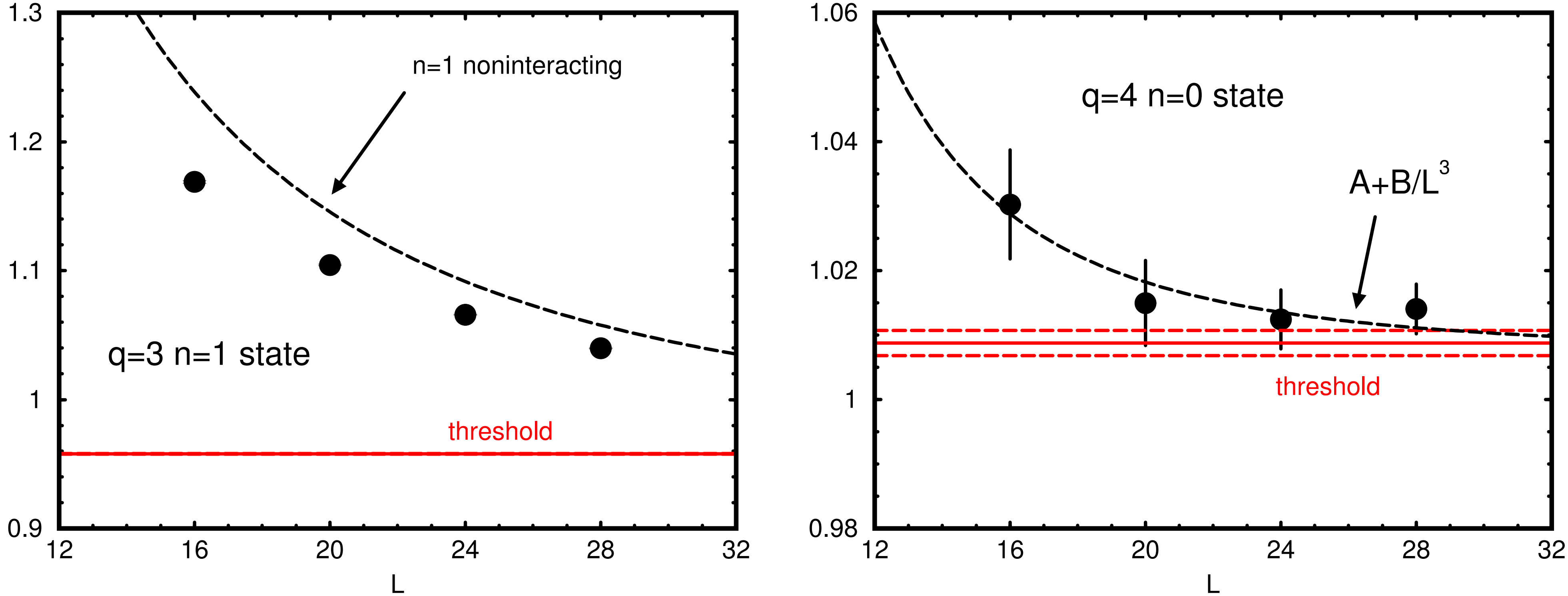
Effective masses for diagonal part of G_{ij} and eigenvalues of M_{ij}



Diagonalization is not effective in ground state as well as $q = 3$ case, while it is effective in first excited state even at largest volume.
Error of first excited state is large in both volumes.

Method B. Finite volume effect of first excited state

Difference of volume dependence for first excited state

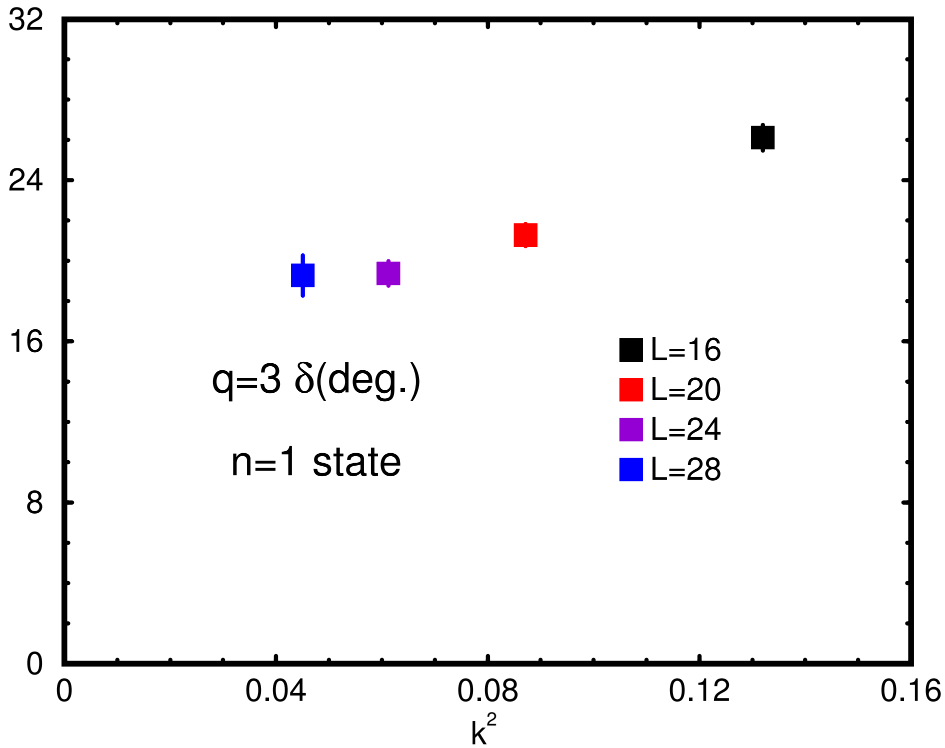


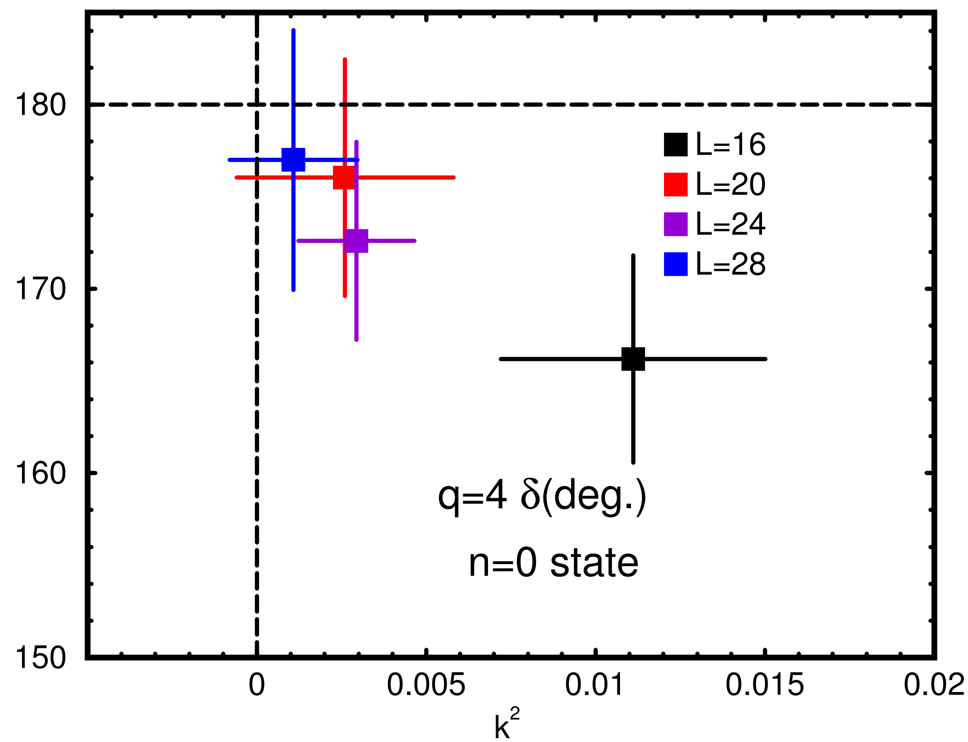
In scattering system energy converges to $n = 1$ noninteracting energy $2\sqrt{m^2 + (2\pi/L)^2}$, while in bound system energy seems to converge to threshold.

Energy shift is positive(negative) in bound(scattering) system as expected.

Method B. (cont'd)

Scattering phase shift $\delta_0(k)$



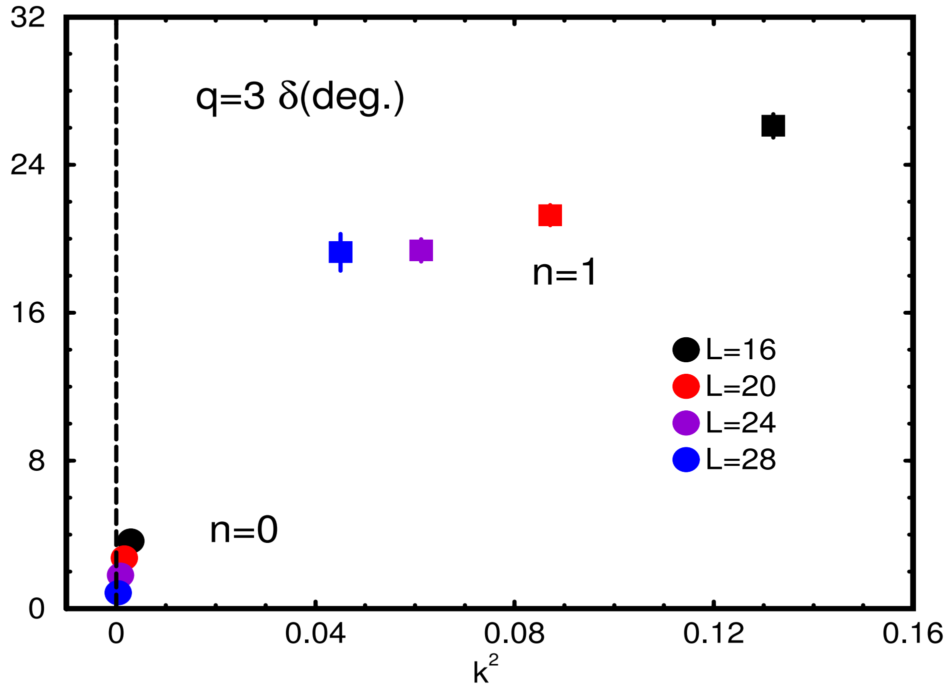


$\delta_0(k)$ decreases(increases) in bound(scattering) system as k increases. This means

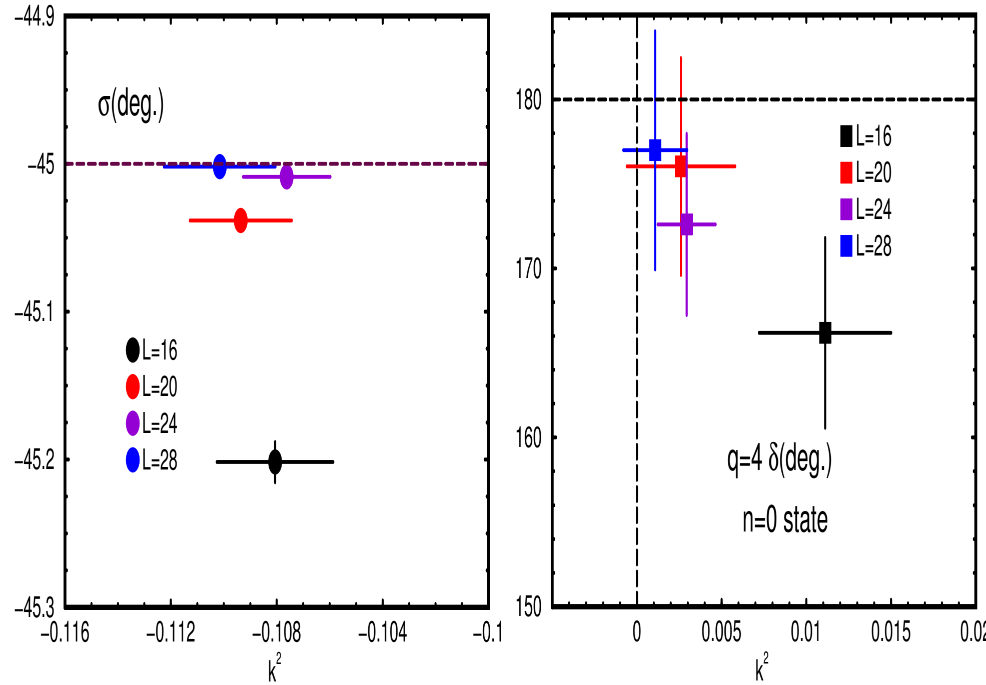
a_0 is negative in bound system.

Summary of Method A. and B.

scattering system



bound system



Positive a_0 and δ_0 in scattering system.

$\sigma_0(\kappa) = -\pi/4 - \varepsilon$ ($\varepsilon > 0$ and $\varepsilon \to 0$ as $L \to \infty$) from ground state, and negative a_0 from first excited state in bound system.

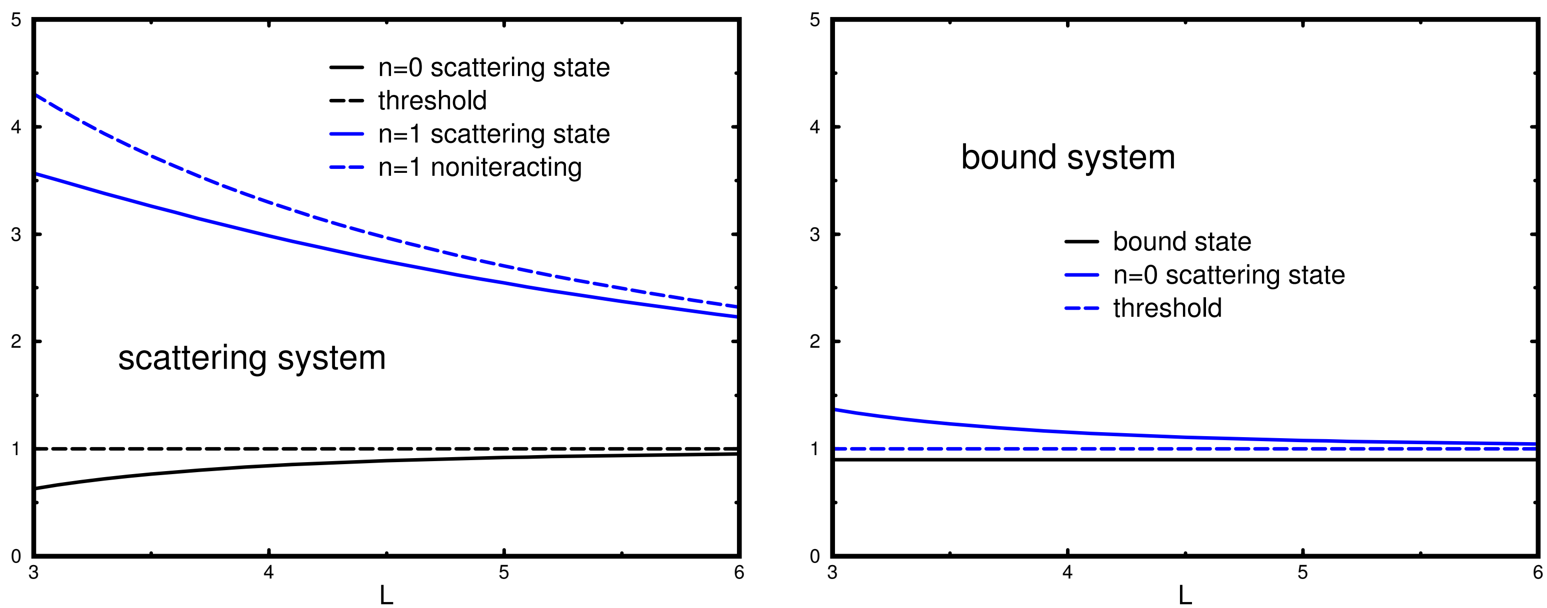
We can distinguish bound and scattering systems by investigating scattering phase shift (or volume effect) for ground and first excited states.

Summary

- We investigate property of system including bound state, and try to identify bound system with two methods.
- From ground state energy we find scattering phase shift $\sigma_0(\kappa)$ is close to $-\pi/4$ and increases as volume increases.
- From first excited energy we find energy shift is positive, and scattering length a_0 is negative.
- These two volume effects are useful to distinguish between bound and scattering system.
- Finite volume effect of first excited state energy seems to be useful in small binding energy case.

Summary (cont'd)

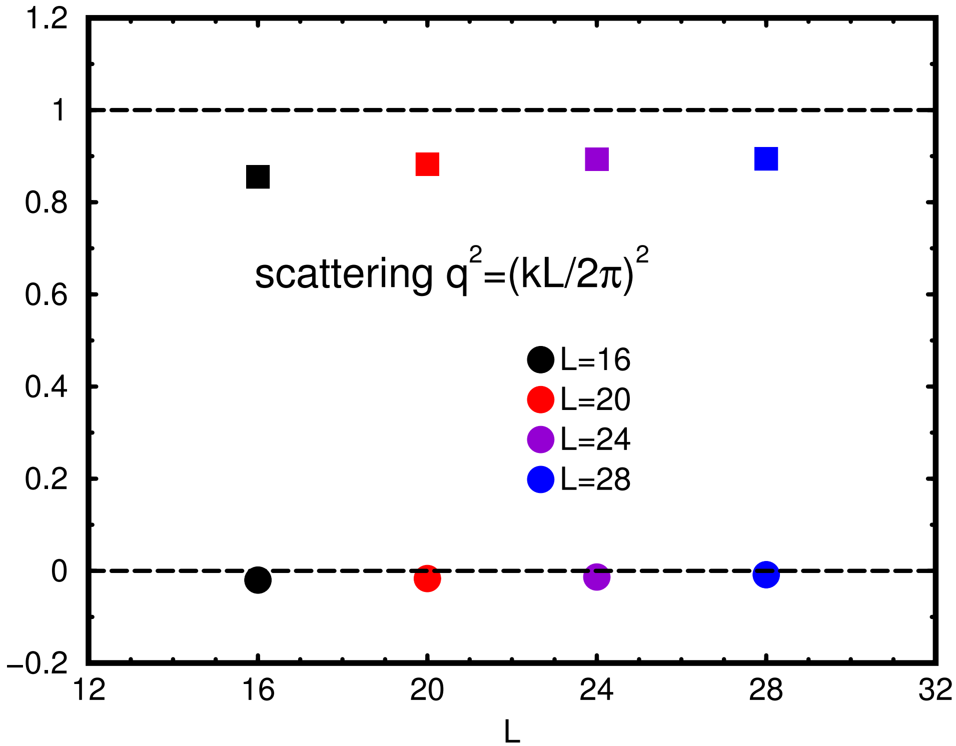
ΔE	scattering (state)	bound (state)
ground state	$\Delta E_0 < 0 \quad (n = 0)$	$\Delta E < 0$ (bound)
first excited state	$\Delta E_1 < 0 \quad (n = 1)$	$\Delta E_0 > 0 \quad (n = 0)$
1st state approach	from below	from above

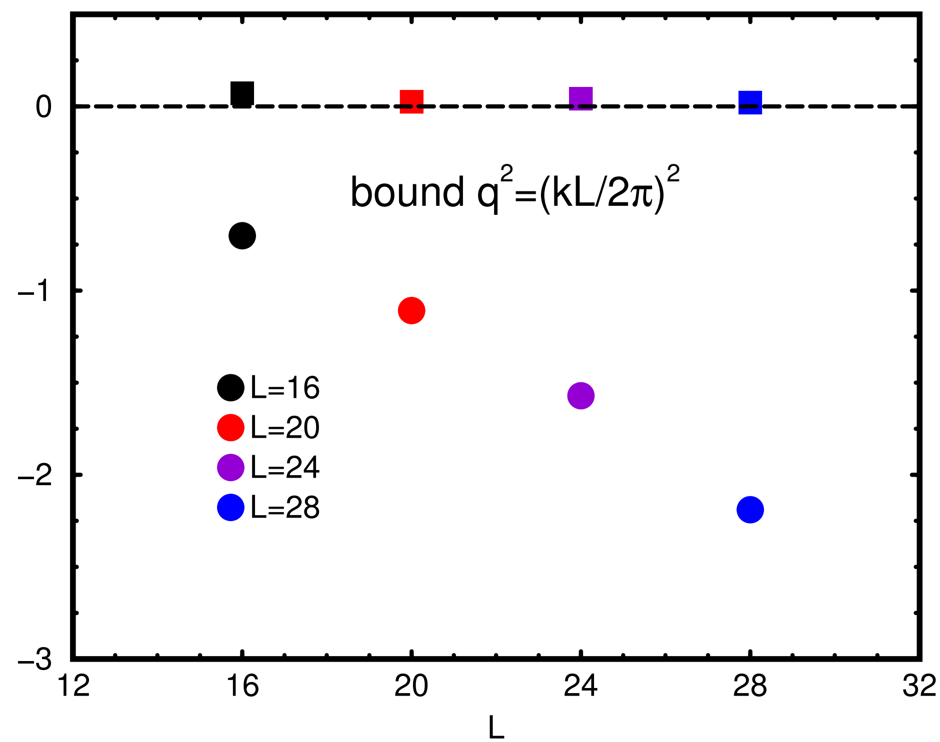


We can identify bound system from energy shift of scattering state, *i.e.*, first excited state in system.

Behavior of $q^2 = (kL/2\pi)^2$ at large L ($\varepsilon > 0$)

q^2	scattering	(state)	bound	(state)
ground state	$0 - \varepsilon$	$(n = 0)$	$-L^2$	(bound)
first excited state	$1 - \varepsilon$	$(n = 1)$	$0 + \varepsilon$	$(n = 0)$





Volume dependence of q^2 is significantly different between two systems.